

ENTROPY for CANONICAL SHIFTS

Marie CHODA

Department of Mathematics, Osaka Kyoiku University

§1. Introduction.

The notion of the entropy for $*$ -automorphisms of finite von Neumann algebras is introduced by Connes and Størmer ([3]). In the previous paper [2], we defined the entropy for $*$ -endomorphisms of finite von Neumann algebras as an extended version of it. It is possible to define the entropy for a general completely positive linear map α using results in [4] by a similar method to one for $*$ -endomorphisms. However, the formula of the definition of the entropy for α implies that the entropy is apt to be zero if α^k converges to α when k tends to infinity. The conditional expectation is a trivial example of such a map α . For that reason, the interesting completely positive map α for us to discuss the entropy are those which have the property that α^k goes away from α as k tends to infinity.

In this paper, we shall study such a class of $*$ -endomorphisms of injective finite von Neumann algebras.

In §3, we shall introduce, for a $*$ -endomorphism σ of an injective finite von Neumann algebra A , the notion of an n -shift on the tower $(A_j)_j$ of finite dimensional von Neumann subalgebras of A which generates A and we obtain the formula of the entropy $H(\sigma)$ for an n -shift σ .

In the work [8] on the classification for subfactors of the hyperfinite type II_1 -factor, Ocneanu introduced a special kind of $*$ -endomorphism which is called the canonical shift

on the tower of relative commutants. The $*$ -endomorphism Γ is a generalization of the comultiplication for Hopf algebras and also considered as the canonical shift on the string algebras. In a part, Γ has similar properties to the canonical endomorphism of an inclusion of infinite von Neumann algebras due to Longo [7].

The canonical shift Γ naturally induces a 2-shift for the injective finite von Neumann algebra A which generated by the tower $(A_j)_j$ of relative commutants and the entropy $H(\Gamma)$ is determined by the following :

$$H(\Gamma) = \lim_{k \rightarrow \infty} \frac{H(A_{2k})}{k}.$$

For a $*$ -endomorphism σ of a von Neumann algebra A , the entropy $H(\sigma)$ is a conjugacy invariant, that is, if there is an isomorphism θ of A onto a von Neumann algebra B such that $\theta\sigma = \phi\theta$ for a $*$ -endomorphism ϕ of B , then $H(\sigma) = H(\phi)$. On the other hand, two conjugate $*$ -endomorphisms σ and ϕ of A give two conjugate von Neumann subalgebras $\sigma(A)$ and $\phi(A)$ under automorphisms of A .

In [9], Pimsner and Popa introduced two conjugacy invariants for von Neumann subalgebras. One is the relative entropy $H(A | B)$ for a von Neumann subalgebra B of a finite von Neumann algebra A , which is defined an extended version of one for finite dimensional algebras due to Connes-Størmer [3]. The other is the constant $\lambda(A, B)$, which plays a role like the index for subfactors due to Jones [6]. In fact in the case of factors $B \subset A$, $\lambda(A, B)^{-1}$ is Jones index $[A : B]$.

We shall investigate relations among those invariants.

In §4, we restrict our attention to finite dimensional von Neumann algebras. We need those results later. The Jones index for a subfactor N of a finite factor M is given as $1/\tau(e)$ for the projection e of $L^2(M)$ onto $L^2(N)$ where τ is the trace on the basic extension algebra of $N \subset M$. In the case of finite dimensional von Neumann algebras, we shall show that the constant $\lambda(,)^{-1}$ coincides with Jones index in such a sense.

In §5, it is obtained that in general the following relation holds for an n -shift σ :

$$H(A \mid \sigma(A)) \leq 2H(\sigma).$$

A condition that the equality holds is also given.

In §6, we shall obtain the relation between $H(\sigma)$ and the constant $\lambda(A, \sigma(A))$. We shall define a locally standard tower for an increasing sequence $(A_j)_j$ of finite dimensional von Neumann algebras. The tower $(A_j)_j$ of relative commutants for the inclusion of finite factors $N \subset M$ satisfies this condition. If a $*$ -endomorphism σ of A is an n -shift on a locally standard tower which generates A , then we have the following :

$$H(A \mid \sigma(A)) \leq 2H(\sigma) \leq \log \lambda(A, \sigma(A))^{-1}.$$

In §7, we shall apply the above results to the canonical shift Γ for the tower of relative commutants. Let $N \subset M$ be type II_1 -factors with the finite index. Considering the tower $(M_j)_j$ of factors obtained by iterating Jones basic construction from $N \subset M$, it is obtained the increasing sequence $(A_j)_j$ of finite dimensional von Neumann algebras, where $A_j = M' \cap M_j$. The $*$ -endomorphism Γ is defined on the algebra $\bigcup_j A_j$ as a mapping such that $\Gamma(M'_k \cap M_j) = M'_{k+2} \cap M_{j+2}$ for all $k \leq j$. First, we remark that Γ is extended to the trace preserving $*$ -endomorphism of a finite von Neumann algebra $A = \bigcup_j (A_j)''$. Then Γ has an ergodic property that

$$\bigcap_k \Gamma^k(A) = C1$$

and satisfies the conditions of Definition for a 2-shift, except only one. In order that Γ satisfies all conditions for 2-shifts, a condition for the inclusion $N \subset M$ is necessary. For example, in the case where $N' \cap M = C1$, Γ is a 2-shift and the following relation holds :

$$H(A \mid \Gamma(A)) \leq 2H(\Gamma) \leq 2 \log[M : N].$$

Furthermore, if the inclusion $N \subset M$ has finite depth ([8], [12]), then we have :

$$H(M \mid N) = H(\Gamma) = \log[M : N]^{-1}.$$

In §8, we shall discuss conditions for a $*$ -endomorphism σ of a factor M to be extended to an automorphism θ of a factor containing M so that $H(\sigma) = H(\theta)$. If the inclusion $N \subset M$ has finite depth, then Γ is extended to an ergodic $*$ -automorphism Θ which satisfies the following :

$$H(M | N) = H(\Theta) = H(\Gamma) = \log[M : N]^{-1}.$$

§2. Preliminaries.

In this section, we shall fix the notations and terminologies frequently used in this paper.

Throughout this section, M will be a finite von Neumann algebra with a fixed normal faithful trace $\tau, \tau(1) = 1$. The inner product $\langle x, y \rangle = \tau(xy^*)$ gives M as a vector space the structure of a pre-Hilbert space. Let $\|x\| = \tau(x^*x)^{1/2}$ and $L^2(M, \tau)$ the Hilbert space completion of M . Then M acts on $L^2(M, \tau)$ by the left multiplication. The canonical conjugation on $L^2(M, \tau)$ is denoted by $J = J_M$. It is the conjugate unitary map induced by the involution $*$ on M . For a von Neumann subalgebra N of M , let e_N be the orthogonal projection of $L^2(M, \tau)$ onto $L^2(N, \tau)$. Then the restriction E_N of e_N to M is the faithful normal conditional expectation of M onto N .

The letter η designates the function on $[0, \infty)$ defined by $\eta(t) = -t \log t$. For each k , we let S_k be the set of all families $(x_{i_1, i_2, \dots, i_k})_{i_j \in N}$ of positive elements of M , zero except for a finite number of indices and satisfying

$$\sum_{i_1, \dots, i_j, \dots, i_k} x_{i_1, \dots, i_k} = 1.$$

For $x \in S_k, j \in 1, 2, \dots, k$ and $i_j \in N$, put

$$x_{i_1}^j = \sum_{i_1, \dots, i_{j-1}, i_{j+1}, \dots, i_k} x_{i_1, i_2, \dots, i_k}.$$

Let $N_1, N_2, \dots, N_k$ be finite dimensional von Neumann subalgebras of M . Then

$$H(N_1, \dots, N_k) = \sup_{x \in S_k} \left[\sum_{i_1, \dots, i_k} \eta \tau(x_{i_1, \dots, i_k}) - \sum_j \sum_{i_j} \tau \eta E_{N_j}(x_{i_j}^j) \right].$$

Let σ be a τ -preserving $*$ -endomorphism of M and N a finite dimensional von Neumann subalgebra of M , then

$$H(N, \sigma) = \lim_{k \rightarrow \infty} \frac{1}{k} H(N, \sigma(N), \dots, \sigma^{k-1}(N))$$

exists by [2]. The entropy $H(\sigma)$ for σ is defined as the supremum of $H(N, \sigma)$ for all finite dimensional subalgebras N of M .

If there exists an increasing sequence $(N_j)_j$ of finite dimensional subalgebras which generates M , then by [2]

$$H(\sigma) = \lim_{j \rightarrow \infty} H(N_j, \sigma).$$

The relative entropy $H(M | N)$ for a von Neumann subalgebra N of M is defined ([10]) as an extension form of one ([3]) by

$$H(M | N) = \sup_{x \in S_1} \sum_i [\tau \eta(x_i) - \tau \eta E_N(x_i)].$$

This $H(M | N)$ is a conjugacy invariant for subalgebras of M . Another conjugacy invariant $\lambda(M, N)$ is introduced in [10] as a generalization of Jones index defined by

$$\lambda(M, N) = \max\{\lambda \geq 0; E_N(x) \geq \lambda x, x \in M_+\}.$$

For an inclusion $N \subset M$ of finite von Neumann algebras, the von Neumann algebra on $L^2(M, \tau)$ generated by M and $e = e_N$ is called the standard basic extension (or basic construction) for $N \subset M$ and denoted by $M_1 = \langle M, e \rangle$. Then by the properties of $J = J_M$ and $e = e_N$, we have $M_1 = \langle M, e \rangle = JN'J$ ([6]). If M_1 is finite and if there is a trace τ_1 on M_1 such that $\tau_1(xe) = \lambda \tau(x)$ for all $x \in M$, then the trace τ_1 is called the λ -Markov trace for $N \subset M$. If $M \supset N$ are factors and there is the λ -Markov trace of M_1 for $N \subset M$, then Jones index $[M : N] = \lambda^{-1}$ ([6]).

We shall call an increasing sequence $(M_j)_{j \in \mathbb{N}}$ of von Neumann algebras a *standard tower* (cf. [5], [9], [13]) if $M_{j-1} \subset M_j \subset M_{j+1}$ is the basic construction obtained from $M_{j-1} \subset M_j$ for each j .

Let L be a finite factor containing M . We shall call L the algebraic basic construction for the factors $N \subset M$ if there is a non zero projection $e \in M$ satisfying :

$$(i) \quad exe = E_N(x)e \text{ for } x \in M$$

and

$$(ii) \quad L \text{ is generated by } e \text{ and } M \text{ as a von Neumann algebra.}$$

In this case, there is an isomorphism ϕ of M_1 onto L such that $\phi(e_N) = e$ and $\phi(x) = x$ for all $x \in M$ ([11]).

We shall call such a projection e the *basic projection* for $N \subset M$ and a decreasing sequence $(N_j)_{j \in \mathbb{N}}$ of finite factors a *standard tunnel* (cf. [5], [9], [13]) if $N_{j-1} \supset N_j \supset N_{j+1}$ is the algebraic basic construction for $N_j \supset N_{j+1}$ for each j .

§3. Entropy of n -shift

In this section, we shall give the definition of n -shifts and a formula of the entropy for n -shifts. Let A be an injective finite von Neumann algebra with a fixed faithful normal trace τ , with $\tau(1) = 1$. Let $(A_j)_{j=1,2,\dots}$ be an increasing sequence of finite dimensional von Neumann algebras such that $A = \text{the weak closure of } \bigcup_j A_j = \{A_j : j\}''$. Assume that σ is a τ -preserving $*$ -endomorphism of A . Then σ is a ultra-weakly continuous, one to one mapping with $\sigma(1) = 1$.

Definition 1. Let n be a natural number. A τ -preserving $*$ -endomorphism σ of A is called an n -shift on the tower $(A_j)_j$ for A if the following conditions are satisfied:

(1) For all j and m , the von Neumann algebra $\{A_j, \sigma(A_j), \dots, \sigma^m(A_j)\}''$ generated by $\{\sigma^j(A_j); j = 0, \dots, m\}$ is contained in A_{j+nm} .

(2) There exists a sequence $(k_j)_{j \in \mathbb{N}}$ of integers with the properties:

$$\lim_{j \rightarrow \infty} \frac{nk_j - j}{j} = 0$$

and

$$x\sigma^m(y) = \sigma^m(y)x, \quad \tau(z\sigma^{lk_j}(x)) = \tau(z)\tau(x)$$

for all $l \in \mathbb{N}$, $x, y \in A_j$, $m \in k_j N$ and $z \in \{A_j, \sigma^{k_j}(A_j), \dots, \sigma^{(l-1)k_j}(A_j)\}''$.

(3) Let E_B be the conditional expectation of A onto a von Neumann subalgebra B of A . Then for an $j \geq n$

$$E_{A_j} E_{\sigma(A_j)} = E_{\sigma(A_{j-n})}$$

(4) For each j , there exists a τ -preserving $*$ -automorphism or antiautomorphism β of A_{nj+n} such that $\sigma(A_{nj}) = \beta(A_{nj})$.

Remark 1. The number n of an n -shift depends on the choice of the sequence $(A_j)_j$. Every given n -shift can be 1-shift on a suitable tower for the same von Neumann algebra.

Example 1. Let S be the $*$ -endomorphism corresponding to the translation of 1 in the infinite tensor product $R = \bigotimes_{i \in \mathbb{N}} (M_i, tr_i)$ of the algebra M_i of $m \times m$ matrices with the normalized trace tr_i on M_i for each $i \in \mathbb{N}$. For each j , let $A_j = \bigotimes_{i=1}^j (M_i, tr_i)$. Then for all n , S^n is an n -shift on the tower $(A_j)_j$ for R .

In fact, for an $n \in \mathbb{N}$, let $k_j = [\frac{j}{n}] + 1$. Then $(k_j)_j$ satisfies the following properties (2') which are stronger than (2):

$$\lim_{j \rightarrow \infty} \frac{nk_j - j}{j} = 0$$

and

$$x\sigma^m(y) = \sigma^m(y)x, \quad \tau(z\sigma^{lk_j}(x)) = \tau(z)\tau(x)$$

for all $l \in \mathbb{N}$, $x, y \in A_j$, $k_j \leq k$, $m \in N$ and $z \in \{A_j, \sigma^k(A_j), \dots, \sigma^{k(l-1)}(A_j)\}''$. It is obvious that another conditions are satisfied by S^n .

Example 2. Let $(e_j)_j$ be the sequence of projections with the following properties for some natural number k and $\lambda \in (0, 1/4] \cup \{1/(4\cos^2(\pi/n)); n \geq 3\}$:

- (a) $e_i e_j e_i = \lambda e_i$ if $|i - j| = k$
- (b) $e_i e_j = e_j e_i$ if $|i - j| \neq k$
- (c) $(e_j)_j$ generates the hyperfinite type II_1 -factor R
- (d) $\tau(w e_i) = \lambda \tau(w)$ for the trace τ of R and a reduced word w on $\{1, e_1, \dots, e_{i-1}\}$.

Let A_j be the von Neumann algebra generated by $\{e_1, \dots, e_j\}$. Then, by [6], A_j is finite dimensional. Let σ be the $*$ -endomorphism of R such that $\sigma(e_i) = e_{i+1}([1])$. Then σ^n is an n -shift on the tower $(A_j)_j$ of R for all n . In fact, for an $n \in N$, let $k_j = [\frac{j+k}{n}] + 1$. Then $(k_j)_j$ satisfies properties (2') in Example 1. The condition (3) and (4) are satisfied by using results by [6] and [1].

In §7, we shall show that the canonical shift due to Ocneanu is a 2-shift on the tower of relative commutant algebras.

Theorem 1. If a τ -preserving $*$ -endomorphism σ of A satisfies the condition (1) and (2) in Definition 1 for the tower $(A_j)_j$ of A , then

$$H(\sigma) = \lim_{k \rightarrow \infty} \frac{H(A_{nk})}{k}.$$

§4. Finite dimensional algebras.

In this section, M will be a finite dimensional von Neumann algebra and τ a fixed faithful normal trace of M with $\tau(1) = 1$. Then M is decomposed into the direct summand:

$$M = \sum_{l \in K} \bigoplus M_l$$

where M_l is the algebra of $d(l) \times d(l)$ matrices and $K = K_M$ is a finite set. Then the vector $d_M = d = (d(l))_{l \in K}$ is called the *dimension vector* of M . The column vector

$t_M = t = (t(l))_{l \in K}$ has $t(l)$ as the value of the trace for the minimal projections in M_l , and is called the *trace vector* of τ . Let N be a von Neumann subalgebra of M with $N = \sum_{k \in K_N} \oplus N_k$. The *inclusion matrix* $[N \hookrightarrow M] = (m(k, l))_{k \in K_N, l \in K_M}$ is given by the number $m(k, l)$ of simple components of a simple M_l module viewed as an N_k module. Then

$$d_N[N \hookrightarrow M] = d_M \quad \text{and} \quad [N \hookrightarrow M]t_M = t_N.$$

Here we shall give a simple formula for $\lambda(M, N)$.

By the definition of the basic construction of $N \subset M$, there is a natural isomorphism between the centers of N and $\langle M, e \rangle$ via $x \rightarrow JxJ$. Hence there is a natural identification between the sets of simple summands of N and $\langle M, e \rangle$. We put $K = K_N = K_{\langle M, e \rangle}$.

The following theorem assures that in the case of finite dimensional von Neumann algebras, the constant $\lambda(\cdot)$ plays the same role as the index for finite factors.

Theorem 2. (1) Assume that there is a trace of $\langle M, e \rangle$ which is an extension of τ . Then

$$\lambda(\langle M, e \rangle, M)^{-1} = \max_{k \in K} \frac{t_N(k)}{t_{\langle M, e \rangle}(k)}.$$

(2) If the trace τ of $\langle M, e \rangle$ has the $\tau(e)$ -Markov property, then

$$\lambda(\langle M, e \rangle, M)^{-1} = 1/\tau(e) = \| [N \hookrightarrow M] \|^2.$$

Definition 2. Let $N \subset M \subset L$ be an inclusion of finite dimensional von Neumann algebras. Then L is said to be the *algebraic basic construction* for $N \subset M$ if there is a projection e in L satisfying

- (a) L is generated by M and e ,
- (b) $xe = ex$ for an $x \in N$,

(c) If $x \in N$ satisfies $xe = 0$, then $x = 0$,

(d) $exe = E_N(x)e$ for all $x \in M$.

In this case, there is a $*$ -isomorphism of the basic construction $M_1 = JN'J$ onto L .

We shall call $N \subset M \subset L$ a *locally algebraic basic extension* of $N \subset M$ if there is a projection $p \in L \cap L'$ which satisfies that the inclusion $M \subset Lp$ is the algebraic basic construction for $N \subset M$.

If $L \supset M \supset N$ is a locally standard extension of the inclusion $M \supset N$, we can identify the set K_N with a subset of K_L via the equality $Ne = eLe$. Under such an identification, we have the following:

Proposition 3. Let $L \supset M \supset N$ be a locally standard extension of $M \supset N$. Then

$$\lambda(L, M)^{-1} \geq \max_{k \in K_N} \min_{l \in K_L} \frac{t_N(k)}{t_L(l)}.$$

Let

$$I(M) = \sum_{l \in K} d(l)t(l) \log \frac{d(l)}{t(l)},$$

where $K = K_M$, $d = d_M$ and $t = t_M$.

proposition 4.

(i) $H(M | N) \leq I(M) - I(N)$

(ii) $H(\langle M, e \rangle | M) = I(\langle M, e \rangle) - I(M)$

(iii) $I(M) \leq 2H(M)$ and the equality holds if and only if M is a factor.

§5. $H(\sigma)$ and $H(A | \sigma(A))$

In this section we shall investigate a relation between $H(\sigma)$ and $H(A | \sigma(A))$ for an n -shift σ on the tower $(A_j)_j$ for a finite von Neumann algebra A .

Let $(A_j)_j$ be an increasing sequence of finite dimensional von Neumann algebras. Let $A_j = \sum_{k \in K_j} \oplus A_j(k)$ be such a decomposition as in §4, and d_j the dimension vector of A_j . Then we shall say $(A_j)_j$ satisfies *the bounded growth conditions* ([2]) if the following two conditions are satisfied :

(i)

$$\sup_j \frac{|(K_j)|}{j} < +\infty$$

and

(ii) For some m , $A_{j+1}(l)$ contains at most $d_j(k)$ $A_j(k)$ - components for all $j \geq m$, where $|K_j|$ is the cardinal number of K_j .

For examples, let us consider two towers which are treated in Examples 1 and 2 . Both of them satisfies the bounded growth conditions ([2]). We shall discuss another example in §7.

Theorem 5. Let σ be a τ -preserving $*$ -endomorphism of an injective finite von Neumann algebra A with a faithful normal trace τ , $\tau(1) = 1$. If σ is an n -shift on the tower $(A_j)_j$ for A , then

$$H(A \mid \sigma(A)) \leq 2H(\sigma).$$

Furthermore, if the bounded growth conditions are satisfied, for the tower $(A_{nj})_j$

$$H(A \mid \sigma(A)) = 2H(\sigma).$$

In order to prove Theorem 5, we need the following:

Lemma 6. Let σ be the same as in Theorem 5. If σ satisfies the conditions (1), (3) and (4) in Definition 1 for n , then

$$H(A \mid \sigma(A)) = \lim_{j \rightarrow \infty} H(A_{nj+n} \mid A_{nj}).$$

By considering the standard tower

$$N \subset M \subset M_1 \subset M_2 \subset \dots \subset M_n = \langle M_{n-1}, e_{n-1} \rangle \subset \dots$$

obtained from the pair $N \subset M$ of II_1 -factors with $[M : N] < \infty$ by iterating the basic construction, it is proved in [11] that $H(M_n | N) = \log[M_n : N]$ if $H(M | N) = \log[M : N]$. Since the index has the multiplicative property ([6]), it implies that $H(M_n | N) = nH(M | N)$ if $H(M | N) = \log[M : N]$. Next corollary shows a similar result holds for the pair $\sigma(M) \subset M$.

Corollary 7. Let a $*$ -endomorphism σ satisfy the same condition as in Theorem 5. Then for all n

$$H(A | \sigma^n(A)) = nH(A | \sigma(A)).$$

§6. $H(\sigma)$ and $\lambda(A, \sigma(A))$ for n -shift σ .

In this section, we shall investigate relations between the entropy $H(\sigma)$ and the constant $\lambda(A, \sigma(A))$ for an n -shift σ of the tower $(A_j)_{j \in \mathbb{N}}$ for a finite von Neumann algebra A with a fixed faithful normal trace τ , $\tau(1) = 1$.

Definition 3. We shall call an increasing sequence $(A_j)_j$ of finite dimensional von Neumann subalgebras of a finite von Neumann algebra A with a faithful normal trace τ a *locally standard tower* for α if there exists a natural number k which satisfies the following conditions :

- 1) For a certain central projection $p_{k(j+1)}$ of $A_{k(j+1)}$, the inclusion matrix $[A_{jk} \hookrightarrow A_{k(j+1)}p_{j+1}]$ is the transpose of $[A_{k(j-1)} \hookrightarrow A_{kj}]$, for each j .

2) If $(t_{k(j-1)}(i))_i$ is the trace vector for the restriction of τ to $A_{k(j-1)}$, then the value of τ of the minimal projections for $A_{k(j+1)}p_{k(j+1)}$ are given by $(\alpha t_{k(j-1)}(i))_i$ for each j .

3) There is an $c > 0$ such that $H(A_{2kj}) \leq c - j \log \alpha$ for each j .

We call the number $2k$ a *period* of the locally standard tower.

As the examples of locally standard towers, we have followings:

(i). The tower $(A_j)_j$ in Example 1 is obviously a locally standard tower for $1/m$, because the inclusion matrix in each step are all same.

(ii). The standard tower is a locally standard tower for $\|T^t T\|^{-1}$, because the inclusion matrix in the j -th step is the transpose of one in the $(j-1)$ -th step for all j ([6]). Hence the tower $(A_j)_j$ is also locally standard if A_{j+1} is a locally algebraic basic extension of $A_{j-1} \subset A_j$.

(iii). The tower $(A_j)_j$ in Example 2 is a locally standard tower for λ , because the central support of e_j in A_j satisfies the condition (1) and (2) in Definition 3 and the condition (3) are proved by results in § 4.2 and § 5.1 in [6].

We shall treat another locally standard tower in the next section.

Theorem 8. Let A be a finite von Neumann algebra with a fixed faithful normal trace τ , $\tau(1) = 1$. Let σ be an n -shift on the locally standard tower $(A_j)_j$ for α with a period $2n$, then

$$H(A \mid \sigma(A)) \leq 2H(\sigma) \leq -\log \alpha \leq \log \lambda(A, \sigma(A))^{-1}$$

The author would like to her hearty thanks to F.Hiai for pointing out a mistake in the proof of Theorem 8 in the preliminary version.

Corollary 9. Let A be an injective finite factor with the canonical trace τ and σ an n -shift of a locally standard tower for A with a period $2n$, then

$$H(A \mid \sigma(A)) \leq 2H(\sigma) \leq \log[A : \sigma(A)].$$

In the case of a factor A , it is obtained in [10] equivalent conditions that $H(A \mid \sigma(A)) = \log[A : \sigma(A)]$. In such a case, we have

$$H(A \mid \sigma(A)) = 2H(\sigma) = \log[A : \sigma(A)].$$

For example, the shifts S in Example 1 and σ for $\lambda > (1/4)$ in Example 2 satisfy the equality ([2]). However, the shifts σ in Example 2 have the following relation([2]):

$$H(R \mid \sigma(R)) = 2H(\sigma) < \log[R : \sigma(R)]$$

if $\lambda \leq (1/4)$.

§7 Canonical shift .

In [9], Ocneanu defined a very nice *-endomorphism for the tower of the relative commutant algebras for the inclusion $N \subset M$ of type II_1 -factors with the finite index.

At first, we shall recall from [9] the definition and main properties of the canonical shift on the tower of relative commutants.

Let M be a finite factor with the canonical trace τ and N a subfactor of M such that $[M : N] < +\infty$. Then the basic extension $M_1 = \langle M, e \rangle$ is a II_1 -factor with the $\lambda = [M : N]^{-1}$ - Markov trace ([6]) and there are the family $\{m_i\} \subset M$ which forms an "orthonormal basis" in M with respect to the N valued inner product $E_N(xy^*)(x, y \in M)$, that is, each $x \in M$ is decomposed in the unique form as the following ([9], [10]):

$$x = \sum_i E_N(m_i^* x) m_i.$$

Iterating the basic construction from $N \subset M$, we have the standard tower

$$M_{-1} = N \subset M_0 = M \subset M_1 = \langle M_0, e_0 \rangle \subset M_2 \subset \dots$$

in which, e_j is the projection of $L^2(M_j, \tau_j)$ onto $L^2(M_{j-1}, \tau_{j-1})$, where τ_j is the λ -Markov trace for M_j . Then from the family $(e_j)_j$ the projection $e(n, k)$ is obtained and

$$M_{n-k} \subset M_n \subset M_{n+k} = \langle M_n, e(n, k) \rangle$$

is an algebraic basic extension ([9], [11]). Furthermore it is obtained in [9] that the "orthonormal basis" in M_n with respect to M_{n-k} valued inner product from the family of the basis in $(M_j)_j$.

Let $A_j = M' \cap M_j$ for all j . The antiautomorphism γ_j of $A_{2j} = M' \cap M_{2j}$ defined by

$$\gamma_j(x) = J_j x^* J_j, \quad x \in A_{2j}$$

is called the *mirroring*, where J_j is the conjugate unitary on $L^2(M_j, \tau_j)$. Then for all $x \in M' \cap M_{2j}$, the following expression of the mirrorings is given :

$$\gamma_j(x) = [M_j : M] \sum_i E(em_i^* x) em_i,$$

where E is the conditional expectation of M_j onto M , e is the projection of $L^2(M_j)$ onto $L^2(M)$ and $(m_i)_i$ a module basis of M_j over M . The expression implies that the mirrorings satisfies the following relation:

$$\gamma_{j+1} \cdot \gamma_j = \gamma_j \cdot \gamma_{j-1}$$

for all $j \geq 1$ on A_{2j-1} . In the view of this relation, the endomorphism Γ of $\bigcup_n A_n$ can be defined by

$$\Gamma(x) = \gamma_{j+1}(\gamma_j(x)),$$

for $x \in A_{2j}$. Ocneanu called the endomorphism Γ the *canonical shift* on the tower of the relative commutants. In the case of inclusions of infinite factors, similar *-endomorphisms

are investigated by Longo [8]. The mapping Γ has the following properties ; for any $k, n \geq 0$ with $n \geq k$,

$$\Gamma(M'_k \cap M_n) = M_{k+2}' \cap M_{n+2}.$$

Now, we shall consider the finite von Neumann algebra A generated by the tower $(A_j)_j$ and extend Γ to a trace preserving $*$ -endomorphism of A as follows.

Since $N \subset M$ are II_1 -factors with $[M : N] < +\infty$, there is a faithful normal trace on $\bigcup_j M_j$ which extends the canonical trace τ on M . We denote the trace by the same notation τ .

Although M_{j+1} is defined as a von Neumann algebra on $L^2(M_j, \tau_j)$, each M_j can be considered as von Neumann algebras on the Hilbert space $L^2(M, \tau)$. Hence $\bigcup A_j$ and $\bigcup M_j$ can be considered as von Neumann algebras acting on $L^2(M, \tau)$. Let

$$M_\infty = \{\bigcup_j M_j\}'' , \quad A = \{\bigcup_j A_j\}''.$$

Then M_∞ is a finite factor with the canonical trace which is the extension of τ . We denote it by the same notation τ . Then A is a von Neumann subalgebra of M_∞ . Since Γ is a ultra-weakly continuous endomorphism of $\bigcup_j A_j$, Γ is extended to a $*$ -endomorphism of A .

Although, in the case where discussed by Ocneanu, for all k , the mirroring γ_k is a trace preserving map thanks to the assumption $N' \cap M = \mathbb{C}1$, in general, the mirrorings are not always trace preserving. However the canonical shift is always trace preserving :

Lemma 10. For every k , $\gamma_{k+1} \cdot \gamma_k$ is a τ - preserving isomorphism of $M' \cap M_{2k}$ onto $M'_2 \cap M_{2k+2}$.

Furthermore, if $E_{A_1}(e_1) = \lambda$ (for example $N' \cap M = \mathbb{C}1$), then γ_j is a trace preserving antiautomorphism of A_{2j} for all j .

By Lemma 10, the canonical shift Γ on the tower of the relative commutants $(A_j)_j$ of

M is extended to a τ -preserving $*$ -endomorphism of A . We shall call the $*$ -endomorphism of A the *canonical shift* for the inclusion $M \supset N$ and denote it by the same notation Γ .

We shall show the canonical shift Γ is a 2-shift on the tower $(A_j)_j$ for A .

Lemma 11. Let L be a finite von Neumann algebra with a faithful normal trace τ , $\tau(1) = 1$. If M is a subfactor of L , then

$$\tau(xy) = \tau(x)\tau(y) \quad (x \in M, y \in M' \cap L).$$

Proposition 12. The canonical shift Γ for the inclusion $N \subset M$ satisfies the conditions (1), (2) and (3) for 2-shifts.

If $E_{A_1}(e_1) = [M : N]^{-1}$, then Γ is a 2-shift on the tower $(A_j)_j$ for A .

Next, we shall show the entropy $H(\Gamma)$ of the $*$ -endomorphism Γ of A is always dominated by $\log[M : N]$.

Lemma 13. Let $B = A \cap N$ for von Neumann subalgebras A and N of a finite von Neumann algebra M satisfying the commuting square condition : $E_A E_N = E_N E_A = E_B$. Then,

$$H(M | N) \geq H(A | B), \quad \lambda(M, N) \leq \lambda(A, B).$$

Let B and C be the von Neumann subalgebras of A defined by

$$B = \left(\bigcup_j (M'_1 \cap M_j) \right)'' , \quad C = \left(\bigcup_j (M'_2 \cap M_j) \right)''$$

Theorem 14. Let Γ be the canonical shift for the inclusion $N \subset M$ of type II_1 -factors with $[M : N] < \infty$. Then

$$H(\Gamma) = \lim_{k \rightarrow \infty} \frac{H(M' \cap M_{2k})}{k}.$$

If $E_{A_1}(e_1) = [M : N]^{-1}$, then

$$H(A | C) \leq 2H(\Gamma) \leq \log \lambda(A, C)^{-1} = 2H(M | N) = 2\log[M : N].$$

Corollary 15. Under the same conditions as in Theorem 14, let A be a factor. Then

$$H(A | C) \leq 2H(\Gamma) \leq 2\log[A : B] = 2\log[M : N].$$

Corollary 16. Let Γ be the canonical shift for the inclusion $N \subset M$ of type II_1 -factors with $[M : N] < \infty$. If $N' \cap M = \mathbb{C}1$, then

$$H(\Gamma) \leq H(M | N) = \log[M : N].$$

For a pair $N \subset M$ of hyperfinite type II_1 -factors with $[M : N] < \infty$, Popa says that $N \subset M$ has the *generating property* if there exists a choice of the standard tunnel of subfactors $(N_j)_j$ such that M is generated by the increasing sequence $(N'_j \cap M)_j$.

Corollary 17. Assume that $N \subset M$ has the generating property. If $E_{N' \cap M}(e_0) = [M : N]^{-1}$, then

$$H(M | N) = H(\Gamma) = \log[M : N]^{-1}.$$

As a sufficient condition that satisfies two assumptions in Corollary 17, Ocneanu [9] introduced the following notion for a pair $N \subset M$ with $N' \cap M = \mathbb{C}1$, and Popa [13] extended it to general cases. The inclusion $N \subset M$ of type II_1 -factors with $[M : N] < +\infty$ is said to have the *finite depth* if

$$\sup_j (k_j) < +\infty$$

where k_j is the cardinal number of simple summands of $M' \cap M_j$.

Remark 18. If the inclusion $N \subset M$ of type II_1 -factors with the finite index and finite depth, then the tower $(A_j)_j$ of relative commutants satisfies the bounded growth conditions.

If an inclusion $N \subset M$ has the finite depth, then $E_{N' \cap M}(e_0) = [M : N]^{-1}$ and $N \subset M$ has the generating property ([13]). Hence we have :

Corollary 19. Let $N \subset M$ be type II_1 -factors with the finite index and the finite depth. Let Γ be the canonical shift for $N \subset M$. Then

$$H(M | N) = H(\Gamma) = \log[M : N]^{-1}.$$

Remark 20. In Corollary 18, the shift Γ is considered as an $*$ -endomorphism of the algebra A generated by the tower $(A_j)_j$ of the relative commutants of M . Since $N \subset M$ has the finite depth, the shift Γ induces a trace preserving $*$ -endomorphism of M which transpose M onto such the subfactor P that $P \subset N \subset M$ is the algebraic basis extension for $P \subset N$. Then the $*$ -endomorphism of M has the same property as Γ .

In the last of this section, we shall show that the canonical shift has an ergodic property, which is similar to the canonocal endomorphism in [7]. So that the canonical shift is a shift in the sense due to Powers [14].

Proposition 21. Let $N \subset M$ be type II_1 -factors with the finite index. Then the canonical shift Γ for $N \subset M$ satisfies that

$$\bigcap_k \Gamma^k(A) = C1.$$

§8 Extension of canonical shift.

In this section, we shall show that the canonical shift Γ is extended to an ergodic *-automorphism Θ of a certain big von Neumann algebra such that $H(\Gamma) = H(\Theta)$.

Let $N \subset M$ be type II_1 -factors with $[M : N] < \infty$. Let

$$M_{-1} = N \subset M = M_0 \subset M_1 = \langle M, e \rangle \subset \dots \subset M_j = \langle M_{j-1}, e_{j-1} \rangle \subset \dots$$

be the standard tower obtained from $N \subset M$. Let M_∞ be the finite factor generated by the tower $(M_j)_j$.

Proposition 22. Let $N \subset M$ be type II_1 -factors with the finite index and τ the canonical trace of M . Let σ be a *-isomorphism of M onto N . Then the following statements are equivalent :

- (1) There exists a *-isomorphism σ_1 of M_1 onto M such that for all $x \in M$,

$$\sigma_1(x) = \sigma(x).$$

- (2) There exists a projection $e \in M$ such that

$$\sigma(N) = \{e\}' \cap N \quad \text{and} \quad E_N(e) = \lambda 1 = [M : N]^{-1}.$$

- (3) There exists a projection $e \in M$ such that for all $y \in N$,

$$eye = E_{\sigma(N)}(y)e, \quad \tau(ey) = \lambda \tau(y)$$

and

M is generated by N and e as a von Neumann algebra.

- (4) There exists an automorphism Θ on M_∞ such that for all $x \in M$ and all j ,

$$\Theta(x) = \sigma(x) \quad \text{and} \quad \Theta(e_j) \in M_j.$$

- (5) The decreasing sequence

$$M \supset N \supset \sigma(N) \supset \dots \supset \sigma^j(N) \supset \dots$$

is a standard tunnel.

Definition 4. Let σ be a $*$ -isomorphism of a type II_1 -factor M onto a subfactor N with the finite index. If σ satisfies the equivalent conditions in Proposition 22, then we call σ basic $*$ -endomorphism for the inclusion $N \subset M$.

Let σ be the basic $*$ -endomorphism of the inclusion $N \subset M$ of type II_1 -factors with the finite index. Let $P_j = M \cap \sigma^j(M)'$. Then $(P_j)_j$ is an increasing sequence of finite dimensional von Neumann algebras. Let P be the von Neumann algebra generated by $(P_j)_j$. Then P is a von Neumann subalgebra of M and we have the following :

Proposition 23. Let σ be the basic $*$ -endomorphism for the inclusion $N \subset M$ of type II_1 -factors with the finite index. Then,

$$H(\sigma) = \lim_{k \rightarrow \infty} \frac{H(M \cap \sigma^k(M)')}{k}$$

Assume that $E_{N \cap M}(e) = [M : N]^{-1}$ for a basic projection of $\sigma(N) \subset N$. Then σ^m is a m -shift on the tower $(P_j)_j$ for P for all even number m and satisfies the following relations. For all even m ,

$$H(P \mid \sigma^m(P)) \leq 2mH(\sigma) \leq \log \lambda(P, \sigma^m(P))^{-1} = m \log[M : N]$$

Corollary 24. Let σ be the same as in Proposition 23. Then

$$2H(\sigma) \leq \log[M : N].$$

Fuethermore, if the inclusion $N \subset M$ has finite depth, then

$$H(M \mid N) = 2H_M(\sigma) = 2H(\sigma) = \log[M : N],$$

where $H_M(\sigma)$ is the entropy of σ as a $*$ -endomorphism of M .

As an example of a basic $*$ -endomorphism, we have the $*$ -endomorphism σ in Example 2.

We shall show that another good example of a basic $*$ -endomorphisms is the canonical shift on the tower of relative commutants in §7.

Proposition 25. Let $M \supset N$ be type II_1 -factors with the finite index and finite depth. Then the canonical shift Γ for the inclusion $M \supset N$ is a basic $*$ -endomorphism of $A = (\bigcup_j (M' \cap M_j))''$.

In [2], we proved that some kinds of $*$ -endomorphisms are extended to ergodic $*$ -automorphisms of big algebras with same values as entropies. Here we shall show it also holds for the canonical shifts.

Let R be the von Neumann algebra which is generated by the standard tower obtained from $A \supset \Gamma(A)$. Since Γ is a basic $*$ -endomorphism of A , there exists a $*$ -automorphism of R which is an extension of Γ . We denote it by Θ .

Theorem 26. Let $N \subset M$ be type II_1 -factor with finite index. Then the automorphism Θ induced by the canonical shift Γ for the inclusion $N \subset M$ is ergodic. If $N \subset M$ has finite depth

$$H(M | N) = H(\Theta) = H(\Gamma) = \log[M : N]^{-1}.$$

References.

- [1] M. Choda : Shifts on the hyperfinite II_1 -factor, J. Operator Theory, 17(1987), 223-235.
- [2] M. Choda : Entropy for $*$ -endomorphisms and relative entropy for subalgebras, Preprint.
- [3] A. Connes and E. Størmer : Entropy for automorphism of II_1 von Neumann algebras,

- Acta Math., 134(1975), 288-306.
- [4] A. Connes, H.Narnhofer and W. Thirring : Dynamical entropy of C^* -algebras and von Neumann algebras, Commun. Math. Phys., 112(1987), 691-719.
 - [5] F.Goodman, P.de la Harpe and V. Jones : Coxeter graphs and towers of algebras, Mathematical Sciences Research Institute Publications 14, Springer-Verlag.
 - [6] V. F. R. Jones : Index for subfactors, Invent. Math., 72(1983), 1-25.
 - [7] R. Longo : Simple injective subfactors, Adv. in Math., 63(1987), 152-171.
 - [8] A. Ocneanu : Quantized groups, string algebras and Galois theory for algebras, Preprint.
 - [9] M. Pimsner and S. Popa : Entropy and index for subfactors, Ann. Sci. Ec. Norm. Sup. 19(1986), 57-106.
 - [10] M. Pimsner and S. Popa : Iterating the basic construction, Trans. A. M. S., 310(1988), 127-134.
 - [11] S. Popa : Maximal injective subalgebras in factors associated with free groups, Adv. in Math., 50(1983), 27-48.
 - [12] S. Popa : Classification of subfactors : the Reduction to commuting squares, Preprint.
 - [13] R. T. Powers : An index theory for semigroups of $*$ -endomorphisms of $B(H)$ and type II_1 factors, Canad. J. Math., 40(1988), 86-114.
 - [14] M. Takesaki : Theory of operator algebras I, Springer Verlag 1979.
 - [15] H. Umegaki : Conditional expectations in an operator algebra, Tohoku Math. J., 6(1954), 358-362.
 - [16] H. Wenzl : Representation of Hecke algebras and subfactors, Thesis, Univ. of Pennsylvania.