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POWER VALUES OF BINOMIAL COEFFICIENTS

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It was an old diophantine problem to determine all binomial coefficients which are perfect powers. Recently, this problem has been completely solved. The purpose of this article is to give a brief survey on this problem and some related questions.

The first section is concerned with power values of products of consecutive integers. The second section is devoted to power values of binomial coefficients. In the third and last sections some common generalizations and open problems are discussed.

1. Power values of products of consecutive integers

Consider the equation

$$(1)\quad n(n+1)\dots(n+k-1) = x^{\ell} \qquad \text{in integers } n \geq 1,\ k, x, \ell \geq 2.$$

In 1724, in a letter addressed to D.Bernoulli, Goldbach argued that (1) is not solvable for $k = 3$, $\ell = 2$. It was a **conjecture** from the 1820's that (1) has no solution. There is an extensive literature of the problem. Many special results were established (for references, see [4], [18], [9]). We mention here the most important results only.

In 1857, Liouville settled the case when at least one factor $n, \dots, n+k-1$ is a prime, or $k > n-5$. This implied that $k! \neq x^{\ell}$ if $k > 1$. In the same year Mlle A.D. proved that (1) has no solution for $k = 3$. In 1917, Narumi proved the conjecture for $k \leq 202$, $\ell = 2$, and Szekeres, in the thirties, for $2 \leq k \leq 9$ and $\ell \geq 2$.

In 1939, Erdős [5] and Rigge [24] proved independently of each other the conjecture for the case $\ell = 2$. Their proof

was elementary and ingenious. They developed the method of Narumi and used among others a theorem of Sylvester which asserts that $n>k$ implies $P(n(n+1)\dots(n+k-1))>k$.
Here $P(a)$ denotes the greatest prime factor of an integer $a>1$.

Erdős and Siegel proved in 194o the conjecture for any sufficiently large k. Their proof was never published. In 1955, Erdős [8] gave a different, elementary proof for this theorem.

By using Erdős' method, Erdős and Selfridge proved in 1975 the conjecture in full generality.

THEOREM A (Erdős and Selfridge [9]). Equation (1) has no solution.

2. Power values of binomial coefficients

Consider now the equation

$$(2)\qquad \binom{n+k-1}{k} = x^{\ell} \qquad \text{in integers } k\geq 2,\ n\geq k+1,\ x,\ell\geq 2.$$

The assumption $n\geq k+1$ is not a restriction. Indeed, using the relation

$$\binom{n-1+k}{k} = \binom{k+n-1}{n-1},$$

one can interchange k and $n-1$ if $n\leq k$, and any result for (2) applies automatically to the case $n\leq k$ as well.

For $k=\ell=2$, (2) is equivalent to the systems of equations

$$n = u^2,\quad n+1 = 2v^2$$

and

$$n = 2v^2,\quad n+1 = u^2$$

which lead to the Pell equations

$$u^2 - 2v^2 = \pm 1 \quad \text{in integers } u, v.$$

This implies that in this case (2) has infinitely many (n,x)

solutions.

For $k=3$, $\ell=2$, (2) can be written in the form

$$n(n+1)(n+2)=6x^2.$$

In this case Meyl [17] proved in 1878 that if n is odd then $(n,x)=(1,1)$ is the only solution. For n even, Watson [36] showed in 1919 that the only solutions are $(n,x)=(2,2)$ and $(48,140)$. Hence, for $k=3$, $\ell=2$,

$$\binom{50}{3}=140^2$$

is the only solution of (2).

In 1939 Erdős [6] **conjectured** that for $\ell>2$, equation (2) has no solution. In the same paper he proved this for $\ell=3$ and for $k\geq 2^\ell$.

Obláth [19] settled the cases $\ell=4,5$. In 1951, Erdős proved the following.

THEOREM B (Erdős [7]). For $k\geq 4$, equation (2) has no solution.

His proof was based on his elementary, ingenious method applied to equation (1).

For $k<4$, the method of Erdős does not work. It is interesting to note that in case of equation (1) the cases $k=2,3$ are the easiest ones. For equation (2) and its generalizations just the converse is true, these cases proved to be the most difficult.

Next we deal with the remaining cases $k=2,3$, that is with the equations

$$\binom{n+1}{2}=x^\ell \quad \text{in integers } n\geq 3,\ x\geq 2,\ \ell>2 \tag{2a}$$

and

$$\binom{n+2}{3}=x^\ell \quad \text{in integers } n\geq 4,\ x\geq 2,\ \ell>2. \tag{2b}$$

It follows from some results of Dénes [3] that for certain regular primes ℓ, (2a) and (2b) have no solutions. Later I showd (cf. [11], [13]) that if ℓ is a prime with

$3^{l-1} \not\equiv 1 \pmod{l^2}$ then the solvability of (2b) implies the solvability of (2a).

An important contribution was made by Tijdeman [33] who proved that (2a) and (2b) have only finitely many solutions. Further, he gave effective upper bounds for the solutions. In his proof he used Baker's method concerning linear forms in logarithms. Terai [32] used recent estimates for linear forms in logarithms to show that in (2a) and (2b) , $l < 4250$ must hold.

The following theorem follows immediately from a recent result of Darmon and Merel [2] on the equation

$$x^l + y^l = 2z^l \quad \text{in } x, y, z \in \mathbb{Z} \text{ with } (x, y, z) = 1. \tag{3}$$

THEOREM C (Darmon and Merel). Equation (2a) has no solution.

Using an extension of Wiles' method concerning the Fermat equation, in [2] the authors showed that (3) has the only solutions $xyz = 0$ or ± 1 . Equation (2a) is equivalent to the systems of equations

$$n = y^l \,, \; n+1 = 2z^l$$

and

$$n = 2z^l \,, \; n+1 = y^l$$

with positive integers y, z , whence

$$y^l \pm 1 = 2z^l$$

and the above-formulated theorem on (3) applies.

Concerning equation (2b) , I have recently proved [13] the following theorem which completed the proof of Erdős' conjecture.

THEOREM D (Győry [13]). Equation (2b) has no solution.

I give a **sketch of the proof** of Theorem D.

Equation (2b) can be written in the form

$$n(n+1)(n+2) = 6x^\ell .$$

Here ℓ cannot be even, since for ℓ even $\binom{50}{3} = 140^2$ is the only solution and 140 is not a perfect power. Hence we may assume that ℓ is an odd prime and by Obláth's result $\ell > 5$.

If $3 \mid n$ or $3 \mid n+2$ then $\binom{n+2}{2} = y^\ell$ or $\binom{n}{2} = z^\ell$ with some positive integers y, z and Theorem C applies.

If $3 \mid n+1$ then it follows that

$$n = 2w^\ell , \quad n+1 = 3v^\ell , \quad n+2 = 2^\ell u^\ell$$

or

$$n = 2^\ell u^\ell , \quad n+1 = 3v^\ell , \quad n+2 = 2w^\ell ,$$

whence

$$(4) \quad 2(w^\ell \pm 1) = 3v^\ell \pm 1 = (2u)^\ell \quad \text{in integers } u \geq 1, v, w > 1 .$$

LEMMA 1 (Lubelski [16], (1935), case $3 \mid x-y$; Győry [12] (1966), case $3 \mid x+y$). Let $\ell > 5$ be a prime and $c > 1$ an integer. If

$$x^\ell + y^\ell = c z^\ell$$

for some relatively prime non-zero integers x, y, z (with some technical conditions on x, y, z and c) such that $3 \mid x-y$ or $3 \mid x+y$ then

$$(5) \qquad 3^{\ell-1} \equiv 1 \pmod{\ell^2}.$$

This result was proved by means of Einsenstein reciprocity theorem in cyclotomic fields.

Using Lemma 1, one can show that (4) implies (5) .

LEMMA 2. If $\ell > 5$ is a prime with (5) and $\ell < 2^{30}$ then $\ell = 11$ or $\ell = 1006003$.

For this lemma, see e.g. [22].

LEMMA 3 (Bennett and de Weger [17]). Let a, b, ℓ be integers with $b > a > 1$ and $3 \le \ell < 17$ or $\ell > 347$. Then the equation

$$|ax^\ell - by^\ell| = 1$$

has at most one solution in positive integers x, y .

This was proved by combining Baker's method with some results on hypergeometric functions and some computational methods.

Now (4) implies $|2\omega^\ell - 3\nu^\ell| = 1$ which has solution $\nu = \omega = 1$. Hence, by Lemma 3, (4) has no solution with $\ell < 17$ or $\ell > 347$. Further, by Lemma 2, (4) is not solvable for $17 \le \ell \le 347$ and Theorem D follows.

Theorems B, C and D together confirm the conjecture of Erdős, that is provide the complete solution of equation (2).

THEOREM 1 (Erdős, case $k \ge 4$; Darmon and Merel, case $k = 2$; Győry, case $k = 3$). Apart from the case $k = \ell = 2$, equation (2) has the only solution $\binom{50}{3} = 140^2$.

This theorem which may be regarded as a joint result of Erdős, Darmon, Merel and the present author was published in my paper [13].

Quite recently Filakovszky showed that equation

$$(6) \qquad \binom{n+k-1}{k} = bx^\ell \quad \text{in integers } n, b \ge 1, k, x, \ell \ge 2 \text{ with } n \ge k+1 , P(b) \le k$$

has no solution if $k \ge 800$ or if $\ell = 2$ and $k \ge 9$. Further, by means of Baker's method Terai gave explicit upper bound for ℓ in the following cases: $k = 2$, $b = 3$; $k = 2$, $b = p > 3$ prime ; $k = 3$, $b = p^m$, $p > 3$ prime. Finally, using Nagell's results on the equation $Ax^3 + By^3 = C$, he proved that $(n, x) = (3, 1)$ is the only solution if $k = 2$, $b = 3$, $\ell = 3$, and (6) has no solution if $k = 3$, $b = p^m$ ($p > 3$ prime), $\ell = 3$.

3. Common generalizations of equations (1) and (2)

Equation

$$(7)\quad n(n+1)\dots(n+k-1) = b x^{\ell} \quad \text{in integers } n, b, x \geq 1,\ k, \ell \geq 2 \text{ with } P(b) \leq k,\ b \ \ell\text{-th power free}$$

is a common generalization of equations (1) and (2). For $b = 1$ this is just equation (1), while for $b =$ ℓ-th power free part of $k!$ we get equation (2). It would not be necessary to assume that b is ℓ-th power free. However, under this assumption the right hand side of (7) has a unique representation which will be useful later.

In the last section some further generalizations will also be discussed. Equation (7) and those generalizations were studied by Shorey, Tijdeman, Saradha, Filakovszky, Hajdu, Brindza, Ruzsa and others; see the survey papers [34], [31], [35], [27], [15] and the references given there.

As we have seen before, for $k = b = \ell = 2$ equation (7) has infinitely many solutions.

For **given** k, equation (7) has only finitely many solutions with $P(x) \leq k$ and all them can be easily determined. Indeed, denote by $p^{(k)}$ the least prime with $p^{(k)} > k$. It is a consequence (cf. [14]) of Sylvester's theorem mentioned above that (n, k, b, x, ℓ) is a solution of (7) with $P(x) \leq k$ if and only if $n \in \{1, 2, \dots, p^{(k)} - k\}$. This means that $n = 1$ is always a solution for each k ; $k!$ can be written in the form $b x^{\ell}$ with the properties required.

Example. We have $p^{(2)} = 3$ and $p^{(3)} = 5$, hence all solutions (n, k, b, x, ℓ) of (7) with $k = 2, 3$, $P(x) \leq k$ are as follows:

$$(1, 2, 2, 1, \ell \geq 2),\ (1, 3, 6, 1, \ell \geq 2),\ (2, 3, 24, 1, \ell \geq 4),$$

$$(2, 3, 6, 2, 2),\ (2, 3, 3, 2, 3).$$

Therefore more interesting are those solutions (n, k, b, x, ℓ) for which $P(x) > k$.

Saradha [25] has recently proved the following.

THEOREM E (Saradha [25]). For $k \geq 4$, equation (7) has no solution with $P(x) > k$.

In fact, she proved a more general result which will be presented in the last section. In her proof she combined Erdős method with some results of Shorey and Tijdeman and some computations. It should be remarked that the method of proof of Saradha cannot be applied to the case $k < 4$.

Independently, Filakovszky and Hajdu (unpublished) proved Theorem E for $\ell = 2$.

Recently I have extended in [14] Theorem E to the cases $k = 2$ and 3 .

THEOREM F (Győry [14]). Apart from the case $k = b = \ell = 2$, for $k \leq 3$ equation (7) has the only solution $(n, k, b, x, \ell) = (48, 3, 6, 140, 2)$ with $P(x) > k$.

For $k = 3$, this theorem can be proved by means of the tools used in the proof of Theorem D above. In [14] I gave a different proof which is based on the following

LEMMA 4 (Győry [14]). Let $\ell \geq 3, \alpha \geq 0$ be integers. Equation

$$(8) \qquad u^\ell + 1 = 2^\alpha v^\ell$$

in positive integers u, v is solvable only if $\alpha = 1$, when $(u, v) = (1, 1)$ is the only solution. Further, equation

$$(9) \qquad u^\ell - 1 = 2^\alpha v^\ell$$

in positive integers u, v has no solution.

The case $\alpha = 0$ of Lemma 4 is trivial. For ℓ odd, we have to distinguish two cases. For $\alpha = 1$, the above-presented theorem of Darmon and Merel [2] applies. For $\alpha > 1$, one can apply the following recent theorem of Ribet [23]: if $\ell \geq 3$ is a prime and α an integer with $2 \leq \alpha < \ell$, then equation

$$x^\ell + y^\ell = 2^\alpha \cdot z^\ell$$

has no solution in non-zero relatively prime integers x, y, z .

Ribet proved this theorem by means of an extension of Wiles' method.

For ℓ even, one can proceed by induction on α and Lemma 4 follows.

Now the case $k = 2$ of Theorem F easily follows by observing that (7) implies (8) or (9).

The case $k = 3$ of Theorem F is more complicated, several subcases ($\ell = 2$; $\ell > 2$ and $3 \mid n$, $n+1$ or $n+2$, etc.) had to be distinguished before the application of Lemma 4.

Theorems E and F together give the following final result (cf. [14]) which provides the complete solution of equation (7) under the assumption $P(x) > k$.

THEOREM 2 (Saradha, case $k \geq 4$; Győry, case $k \leq 3$). Apart from the case $k = b = \ell = 2$, equation (7) has only the solution $(n, k, b, x, \ell) = (48, 3, 6, 140, 2)$ with $P(x) > k$.

We show now how to deduce Theorems A and 1 from Theorem 2.

Deduction of Theorem A (Erdős and Selfridge) on equation (1) from Theorem 2 :

For $k = \ell = 2$, equation

$$(1) \qquad n(n+1)\dots(n+k-1) = x^{\ell}, \quad n \geq 1, k, x, \ell \geq 2$$

is not solvable. By Theorem 2, no solution exists with $P(x) > k$. If (n, k, x, ℓ) is a solution of (1) with $P(x) \leq k$, then Sylvester's theorem implies that $n \leq k$, whence $n \leq \frac{n+k}{2}$. By Chebyshev's theorem there exists a prime p with $\frac{n+k}{2} \leq p \leq n+k-1$, hence $p \parallel n(n+1)\dots(n+k-1)$ which is impossible. This proves Theorem A.

Deduction of Theorem 1 (Erdős, Darmon, Merel, Győry) on equation (2) from Theorem 2:

Equation (2) takes the form

$$n(n+1)\dots(n+k-1) = k!\, x^{\ell}.$$

This can be written in the form (7) with the choice $b = \ell$-th free part of $k!$. By assumption $n \geq k+1$, hence Sylvester's theorem implies that $P(x) > k$. Now Theorem 1 follows immediately from Theorem 2.

4. Further generalizations

Finally, we consider the following generalizations of equations (1) and (7), respectively:

(10) $n(n+d)\dots(n+(k-1)d) = x^{\ell}$ in integers $n, d \geq 1, k \geq 3, x, \ell \geq 2$ with $(n,d) = 1$

and

(11) $n(n+d)\dots(n+(k-1)d) = b x^{\ell}$ in integers $n, d, b \geq 1, k \geq 3$, $x, \ell \geq 2$ with $(n,d) = 1, P(b) \leq k$.

In these general situations there are several deep but only partial results. For references, we refer to [4], [34], [31], [35], [27].

It is easy to see that both for $k = 2$ and without the assumption $(n,d) = 1$ these equations have infinitely many solutions. As is easily seen, for $(k,\ell) = (3,2)$ both equations have infinitely many solutions. By a theorem of Euler, (10) has no solution for $(k,\ell) = (4,2)$. The same result was proved by Obláth [20], [21] in 1951 for $(k,\ell) = (5,2), (3,3), (3,4)$ and $(3,5)$.

By a **conjecture** of Erdős, (10) implies that $k = 3$. It has been recently proved by Saradha [25] that for $d \leq 6$, equation (10) has no solution. Further, as was showed by Saradha ([26], case $d \leq 22$) and Filakovszky and Hajdu ([10], case $23 \leq d \leq 30$), for $\ell = 2$ $(n,d,k,x) = (18,7,3,120)$ and $(1,24,3,35)$ are the only solutions of (10) with $d \leq 30$.

Recently I proved [15] that for $k = 3$, $\ell > 2$, equation (10) has no solution.

Tijdeman [34] showed that (11) has infinitely many solutions for $(k,\ell)=(3,3)$ and $(4,2)$. Further, he **conjectured** that (11) has only finitely many solutions with $k+\ell>6$. In a series of papers Shorey and Tijdeman obtained a lot of important results on equation (11). They showed [29], [31] that if $d>1$ and $(n,d,k)\neq(2,7,3)$ then

$$P(n(n+d)\dots(n+(k-1)d))>k.$$

Together with Theorem A this implies that equation (10) has no solution with $P(x)\leq k$. Further, Shorey and Tijdeman proved (cf. [28], [30]) that (11) implies that if ℓ is a prime and $k\geq c_0$ then

$$d_1\geq \max\{c_1 k^{\ell-2},\ k^{c_2 \log\log k}\}$$

and

$$P(d)\geq c_3 \log k \log\log k,\quad \ell^{\omega(d)}\geq c_4 k/\log k,$$

where d_1 is the maximal divisor of d such that all prime factors of d_1 are $\equiv 1 \pmod{\ell}$, $\omega(d)$ denotes the number of distinct prime factors of d, and c_0 to c_4 are effectively computable positive absolute constants.

Saradha [25] has recetly showed that for $k\geq 4$ and $d\leq 6$, equation (11) has no solution provided that $P(x)>k$ if $d=1$. Further, Saradha ([26], case $d\leq 22$) and Filakovszky and Hajdu ([10], case $23\leq d\leq 30$) determined all (finitely many) solutions of (11) for $\ell=2$ and $d\leq 30$.

Furthermore, I have recently proved [15] that for $k=3$, $\ell>2$, equation (11) has no solution with $P(b)\leq 2$.

Finally, it should be remarked that the even more general equation

$$(n+dd_1)\dots(n+dd_t)=bx^{\ell}$$

where $d_1,\dots,d_t$ are distinct integers with $0\leq d_1,\dots,d_t<k$, was also studied by Erdős, Shorey, Tijdeman, Saradha and others (cf. [31], [25], [27]).

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References

[1] M.A.Bennett and B.M.M. de Weger, On the diophantine equation $|ax^n - by^n| = 1$, Math.Comp. 67 (1998),413-438.

[2] H.Darmon and L.Merel, Winding quotients and some variants of Fermat's Last Theorem, J.Reine Angew.Math. 490 (1997), 81-100.

[3] P.Dénes, Über die diophantische Gleichung $x^l + y^l = cz^l$, Acta Math. 88 (1952),241-251.

[4] L.E.Dickson, History of the Theory of Numbers, Vol. II, reprinted by Chelsea, New York, 1971.

[5] P.Erdős, Note on products of consecutive integers, J.London Math.Soc. 14 (1939),194-198.

[6] P.Erdős, Note on the product of consecutive integers (II), J. London Math.Soc. 14 (1939),245-249.

[7] P.Erdős, On a diophantine equation, J.London Math.Soc. 26 (1951),176-178.

[8] P.Erdős, On the product of consecutive integers (III), Indag. Math. 17 (1955),85-90.

[9] P.Erdős and J.L.Selfridge, The product of consecutive integers is never a power, Illinois J.Math. 19 (1975),292-301.

[10] P.Filakovszky and L.Hajdu, The resolution of the diophantine equation $x(x+d)\dots(x+(k-1)d) = by^2$ for fixed d, to appear.

[11] K.Győry, On the diophantine equations $\binom{n}{2} = a^l$ and $\binom{n}{3} = a^l$, Mat.Lapok 14 (1963),322-329 (in Hungarian).

[12] K.Győry, Über die diophantische Gleichung $x^p + y^p = cz^p$, Publ.Math.Debrecen 13 (1966),301-305.

[13] K.Győry, On the diophantine equation $\binom{n}{k} = x^l$, Acta Arith. 80 (1997),289-295.

[14] K.Győry, On the diophantine equation $n(n+1)\dots(n+k-1) = bx^l$, Acta Arith. 83 (1998),87-92.

[15] K.Győry, Power values of products of consecutive integers and binomial coefficients, in preparation.

[16] S.Lubelski, Studien über den grossen Fermatschen Satz, Prace Mat.-Fiz. 42 (1935),11-44.

[17] A.J.J.Meyl, Question 1194, Nouv.Ann.Math. (2) 17 (1878),464-467.

[18] R.Obláth, Über Produkte aufeinanderfolgender Zahlen, Tohoku Math.J. 38 (1933),73-92.

[19] R.Obláth, Note on the binomial coefficients, J.London Math. Soc. 23 (1948),252-253.

[20] R.Obláth, Über das Produkt fünf aufeinander folgender Zahlen in einer arithmetischen Reihe, Publ.Math.Debrecen 1 (1950),222-226.

[21] R.Obláth, Eine Bemerkung über Produkte aufeinander folgender Zahlen, J.Indian Math.Soc. (N.S.) 15 (1951),135-139.

[22] P.Ribenboim, The Little Book of Big Primes, Springer, 1991.

[23] K.A.Ribet, On the equation $a^p + 2^\alpha b^p + c^p = 0$, Acta Arith. 79 (1997),7-16.

[24] O.Rigge, Über ein diophantisches Problem, in: 9th Congress Math.Scand., Helsingfors, 1938, Mercator, 1939,155-160.

[25] N.Saradha, On perfect powers in products with terms from arithmetic progressions, Acta Arith. 82 (1997),147-172.

[26] N.Saradha, Squares in products with terms in an arithmetic progression, Acta Arith., to appear.

[27] T.N.Shorey, Exponential diophantine equations involving products of consecutive integers and related equations, to appear.

[28] T.N.Shorey and R.Tijdeman, Perfect powers in products of terms in an arithmetical progression, Compositio Math. 75 (1990), 307-344.

[29] T.N.Shorey and R.Tijdeman, On the greatest prime factor of an arithmetical progression, in: A Tribute to Paul Erdős (A.Baker, B.Bollobás and A.Hajnal eds.), Cambridge University Press, 1990, 385-389.

[30] T.N.Shorey and R.Tijdeman, Perfect powers in products of terms in an arithmetical progression (II), Compositio Math 82 (1992),119-136.

[31] T.N.Shorey and R.Tijdeman, Some methods of Erdős applied to finite arithmetic progressions, The Mathematics of Paul Erdős, I, 251-267. Springer, 1997.

[32] N.Terai, On a Diophantine equation of Erdős, Proc.Japan Acad., Ser.A 70 (1994),213-217.

[33] R.Tijdeman, Applications of the Gelfond-Baker method to rational number theory, in: Topics in Number Theory, North-Holland, 1976, 399-416.

[34] R.Tijdeman, Diophantine equations and diophantine approximations, in: Number Theory and Applications, (R.A.Mollin ed.), Kluwer Academic Press, 1989, 215-243.

[35] R.Tijdeman, Exponential diophantine equations 1986-1996, in: Number Theory (K.Győry, A.Pethő and V.T.Sós eds.), de Gruyter, 1998, 523-539.

[36] G.N.Watson, The problem of the square pyramid, Messenger Math. 48 (1919),1-22.

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