

COMMUTING CONTRACTIONS の SIMULTANEOUS UNITARY DILATION

山形大学理学部 岡 安 隆 照 (Takateru OKAYASU)

The following matter is really fundamental:

Sz.-Nagy's Unitary Dilation Theorem. Let T be a contraction on a Hilbert space $\mathcal{H}$. Then, there exist an enlarged Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ and a unitary U , called a unitary dilation of T , on $\mathcal{K}$, such that

$$T^m = PU^m|_{\mathcal{H}} \quad \text{for } m = 0, 1, 2, \dots,$$

where P is the projection on $\mathcal{K}$ onto $\mathcal{H}$.

This yields, and, is yielded by, the so-called

von Neumann Inequality. Let T be a contraction on a Hilbert space. Then,

$$\|p(T)\| \leq \|p\| = \sup_{z \in \mathbb{T}} |p(z)|$$

holds for any polynomial p with complex coefficients.

The “logical equivalence” is accompanied by the the following

Theorem [6]. If a set of commuting contractions on a Hilbert space $\mathcal{H}$, $T_1, T_2, \dots, T_n$, admits a simultaneous unitary dilation, namely, there exist a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ and commuting unitaries $U_1, U_2, \dots, U_n$ on $\mathcal{K}$, such that

$$T_1^{m_1} T_2^{m_2} \dots T_n^{m_n} = P U_1^{m_1} U_2^{m_2} \dots U_n^{m_n} |_{\mathcal{H}}$$

for $m_1, m_2, \dots, m_n = 0, 1, 2, \dots$, where P is the projection on $\mathcal{K}$ onto $\mathcal{H}$, then $T_1, T_2, \dots, T_n$ enjoys the von Neumann inequality, namely,

$$\|(p_{ij}(T_1, T_2, \dots, T_n))\| \leq \|(p_{ij})\| = \sup_{z_1, z_2, \dots, z_n \in \mathbb{T}} \|(p_{ij}(z_1, z_2, \dots, z_n))\|$$

holds for any $m \times m$ matrix (p_{ij}) whose entries are polynomials with complex coefficients; and *vice versa*.

On the other hand, the following theorems are known:

Andô's Theorem [1]. Any pair of commuting contractions on a Hilbert space admits a simultaneous unitary dilation.

Andô's Theorem [2]. Any triple of commuting contractions on a Hilbert space, one of which double commutes with others, admits a simultaneous unitary dilation.

We, aside, have examples of triples of commuting contractions which do not admit a simultaneous unitary dilation, [4], [8] and [9].

In [6] we gave the following theorem and corollary:

Theorem. Suppose each of sets of commuting contractions, $S_1, S_2, \dots, S_m$ and $T_1, T_2, \dots, T_n$, on a Hilbert space, admits a simultaneous unitary dilation, and every S_j double commutes with all T_k . If the set $S_1, S_2, \dots, S_m$ generates a nuclear C^* algebra, then the set $S_1, S_2, \dots, S_m, T_1, T_2, \dots, T_n$ admits a simultaneous unitary dilation.

Collorary. Suppose S is a *GCR* contraction, i.e., a contraction which generates a *GCR* (postliminal) algebra, $T_1, T_2, \dots, T_n$ commuting contractions, on a Hilbert space, the set $T_1, T_2, \dots, T_n$ admits a simultaneous unitary

dilation. and S double commutes with all T_k . Then the set $S, T_1, T_2, \dots, T_n$ admits a simultaneous unitary dilation.

The following, furthermore, turned out to be true [7]:

Theorem. Suppose each of sets of commuting contractions, $S_1, S_2, \dots, S_m$ and $T_1, T_2, \dots, T_n$, on a Hilbert space, admits a simultaneous unitary dilation, and every S_j double commutes with all T_k . If the set $S_1, S_2, \dots, S_m$ generates an injective von Neumann algebra, then the set $S_1, S_2, \dots, S_m, T_1, T_2, \dots, T_n$ admits a simultaneous unitary dilation.

Collorary. Suppose S is a type I contraction, i.e., a contraction which generates a type I von Neumann algebra, $T_1, T_2, \dots, T_n$ commuting contractions, on a Hilbert space, the set $T_1, T_2, \dots, T_n$ admits a simultaneous unitary dilation and S double commutes with all T_k . Then, the set $S, T_1, T_2, \dots, T_n$ admits a simultaneous unitary dilation.

We here will improve the theorem, by making the assumption thin as the following

Theorem. Suppose each of sets of commuting contractions, $S_1, S_2, \dots, S_m$ and $T_1, T_2, \dots, T_n$, on a Hilbert space, admits a simultaneous unitary dilation, and every S_j double commutes with all T_k . Then, the set $S_1, S_2, \dots, S_m, T_1, T_2, \dots, T_n$ admits a simultaneous unitary dilation.

This is the aimed theorem of ours. A proof of this is given, on account of the Steinspring representation of completely positive maps, by the preceding theorem and the

Arveson Theorem [3, Theorem 1.3.1]. Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces, V a bounded operator from $\mathcal{H}$ into $\mathcal{K}$, and $\mathcal{B}$ a $*$ -subalgebra of $\mathcal{B}(\mathcal{K})$, the full operator algebra, which satisfies that $[\mathcal{B}V\mathcal{H}] = \mathcal{K}$. Then, for every $T \in (V^*\mathcal{B}V)'$ there exists a unique $\tilde{T} \in \mathcal{B}'$ such that $\tilde{T}V = VT$, and the mapping $(\) \mapsto (V^*\mathcal{B}V)'$ is a σ weakly continuous $*$ -homomorphism.

We have as well

Collorary. Suppose each of pairs of commuting contractions, S_1, S_2 , and T_1, T_2 , on a Hilbert space, admits a simultaneous unitary dilation, and each of S_1, S_2 double commutes with T_1, T_2 . Then, the set S_1, S_2, T_1, T_2 admits a simultaneous unitary dilation.

Our theorem, of course, gives a good understanding to Andô's "triple" assertion; on the Andô's "pair" assertion, the next matter sheds light:

Theorem [5, Theorem 6]. Let T be a contraction on a Hilbert space $\mathcal{H}$, U the minimal unitary dilation of T . Then for every $S \in \{T\}'$ there exists $\tilde{S} \in \{U\}'$ such that $S = P\tilde{S}|_{\mathcal{H}}$ and $\|\tilde{S}\| = \|S\|$.

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