

On homology 3-spheres
which bound contractible 4-manifolds

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§1. Introduction.

In this note, we consider the question which homology 3-spheres bound contractible (or acyclic) 4-manifolds. This question is important in the problem of classifying homology 3-spheres. Some families of homology 3-spheres which do bound contractible 4-manifolds are given by Akbulut and Kirby and others in [A-K], [C-H] and [M]. All known examples are the Breiskorn homology 3-spheres.

We deal with the homology 3-spheres of plumbing type. A closed 3-manifold is of plumbing type if it is the boundary of a regular neighbourhood of smooth, normally immersed surface in an oriented 4-manifold. The Breiskorn homology 3-spheres are of course of plumbing type, but all of homology 3-spheres of plumbing type is not a Breiskorn homology 3-sphere.

Throughout this note we use the terminology of Kirby's calculus on framed links. Since Kirby's calculus is widely known, we omit the minute details. We refer to [K] for the fundamentals of Kirby's calculus. Every time we draw a framed link in S^3 , we mean the corresponding 4-manifold obtained by attaching 2-handles on the boundary of B^4 along the framed link. In this note

$\approx$ means a diffeomorphism and $\cong$ means a diffeomorphism between the boundaries of manifolds. "Blowing up" means taking connected sum with CP^2 or $-CP^2$. "Blowing down" means replacing the neighbourhood of an embedded 2-sphere ($=$ the D^2 -bundle over S^2) with B^4 when the Euler characteristic of the bundle is ± 1 .

In §2 we define a contractible 4-manifold by generalizing so called Mazur manifold. Applying Kirby's calculus on framed links in §3, we sketch the proof of the following result:

Theorem 1. ([M1], [M2])

A. A Brieskorn homology 3-spheres $\Sigma(2n+1, 2n+2, 2n+3)$ bounds a contractible 4-manifold.

B. A homology 3-sphere $\Sigma(n)$ of plumbing type given by the following framed link $\mathcal{Y}(n)$ ($n \geq 1$) bounds a contractible 4-manifold where

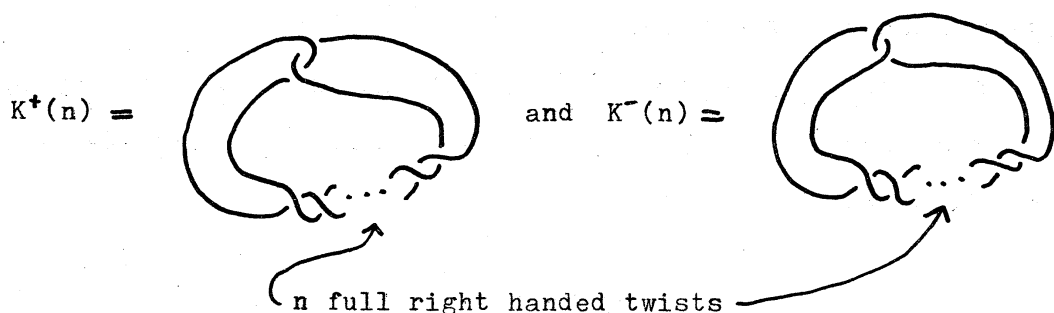
$$\Sigma(n) = \partial \left(\mathcal{Y}(n): \begin{array}{c} \begin{array}{c} \text{Top row: } \text{link with framing } -2, -2, \dots, -2, -2 \end{array} \\ \begin{array}{c} \text{Bottom row: } \text{link with framing } -2, -2, \dots, -2, -2 \end{array} \\ \begin{array}{c} \text{Middle: } \text{link with framing } -1, -2, -4, -2 \end{array} \\ \text{Labels: } -2n-3, n, n-1 \end{array} \right).$$

The statement A is stronger than the result of Martin in [M]: $\Sigma(2n+1, 2n+2, 2n+3)$ bounds an acyclic 4-manifold. But it is the special case of the results of Casson and Harer [C-H]. ~~We note about~~

the family of homology 3-spheres in the statement B of Theorem 1 that $\Sigma(1)$ is the Breiskorn homology 3-sphere $\Sigma(2, 5, 7)$ and that $\Sigma(n)$ ($n \geq 2$) is not a Breiskorn homology 3-sphere [M2].

Another method to give an example of homology 3-sphere which bounds a contractible 4-manifold is to perform ± 1 -surgery on S^3 along a slice knot in S^3 [G]. By Kirby's calculus we can show

Theorem 2. ([M1]) A Breiskorn homology 3-sphere $\Sigma(2, 3, 6n \pm 1)$ is obtained by a ± 1 -surgery on a knot $K^\pm(n)$ ($n \geq 1$) where



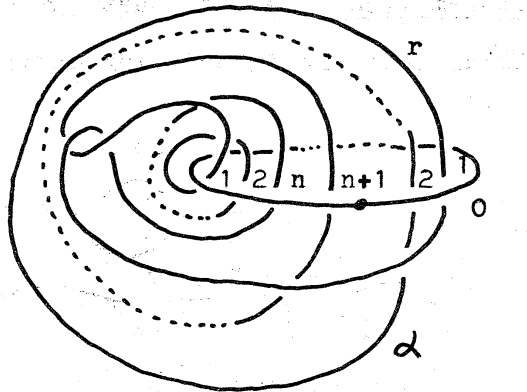
Since we can show only $K^+(2)$ is a slice knot among $K^\pm(n)$'s ($n \geq 1$), we restate that

Corollary. $\Sigma(2, 3, 13)$ bounds a contractible 4-manifold.

§2. Construction of a contractible 4-manifold.

$W^4 = W(n+1, n; r)$ is defined as a 4-manifold obtained by adding two 2-handles to B^4 along the following framed link of two components in $S^3 = \partial B^4$ and surgering S^2 corresponding to the 0-framed unknotted circle with a dot:

$W(n+1, n; r):$



where the r -framed circle α links the 0 -framed circle with a dot algebraically once.

By the definition W is a 4 -manifold with its handle decomposition consisting one 1 -handle and one 2 -handle attached along the r -framed circle α lies in $S^1 \times S^2 = \partial(B^4 \times H^1) = \partial(S^1 \times B^3)$ being homotopic but not isotopic to the standard embedding $S^1 \times (*) \subset S^1 \times S^2$. That is the 2 -handle cancels the 1 -handle homotopically. Therefore W can be seen as a generalized contractible Mazur 4 -manifold with boundary.

The following theorem gives a description of the boundaries of some of these contractible 4 -manifolds of Mazur types.

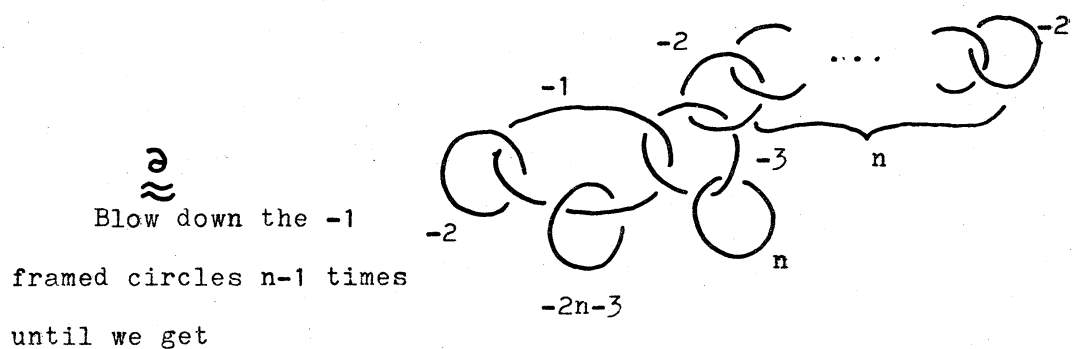
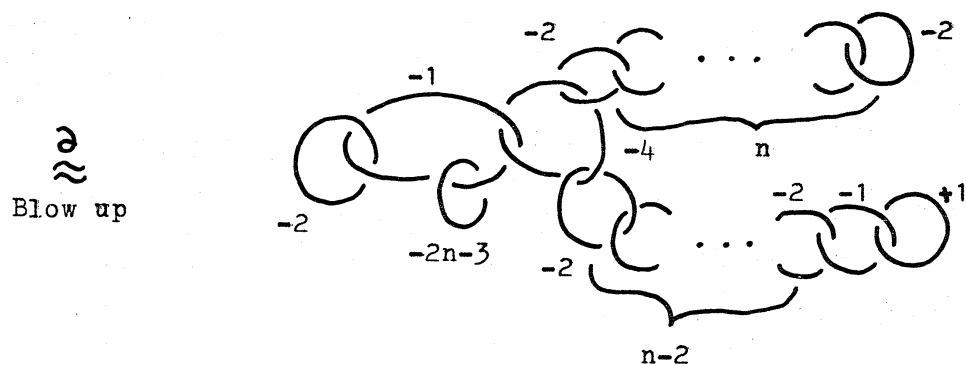
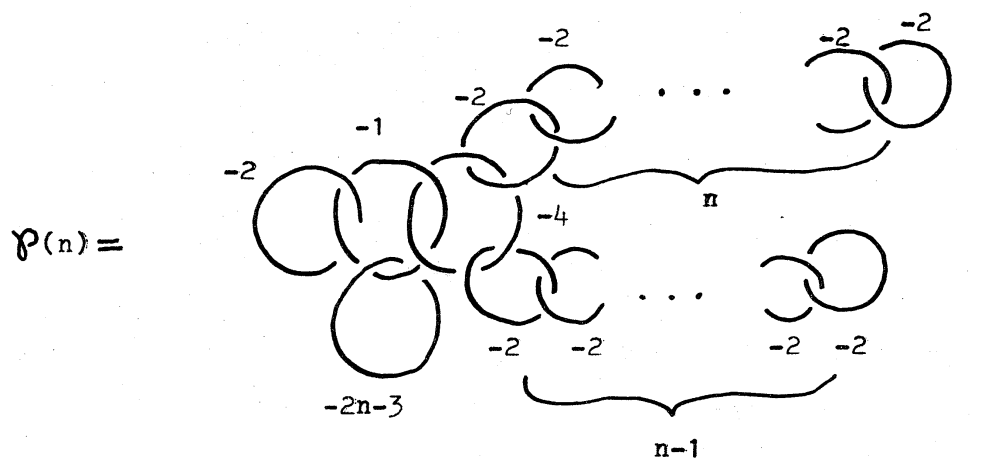
Theorem 1.

- A. $\Sigma(2n+1, 2n+2, 2n+3) \approx \partial W(n+1, n; 2n+2) \quad (n \geq 1).$
- B. $\Sigma(n) \approx \partial W(n+1, n; 2n+1) \quad (n \geq 1).$

§3. Proof.

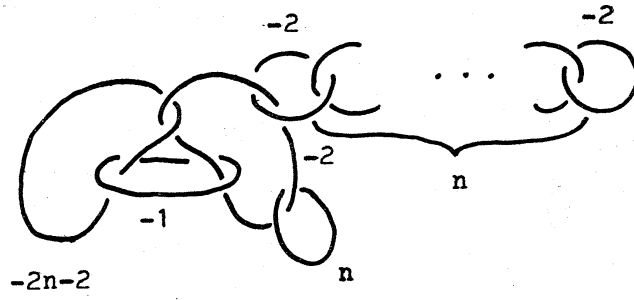
For want of space we prove only the statement of Theorem 1.B. We refer to [M1] and [M2] for the explicit proofs of the results.

Proof of Theorem 1.B. We show that $\mathcal{V}(n)$ can be deformed by Kirby's calculus into the framed link defining $W(n+1, n; 2n+1)$.

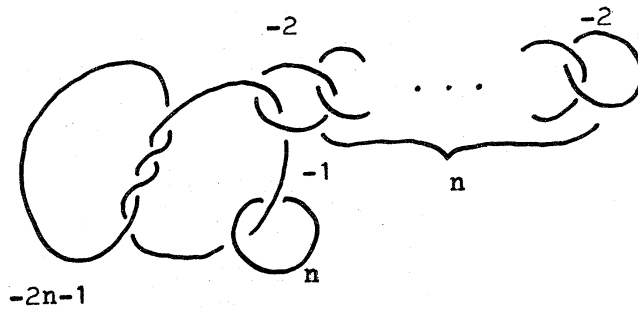


$\approx$

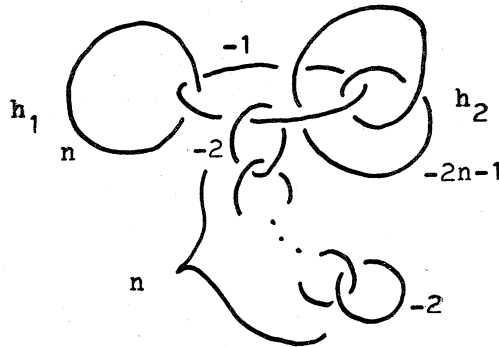
Blow down


 $\approx$

Blow down



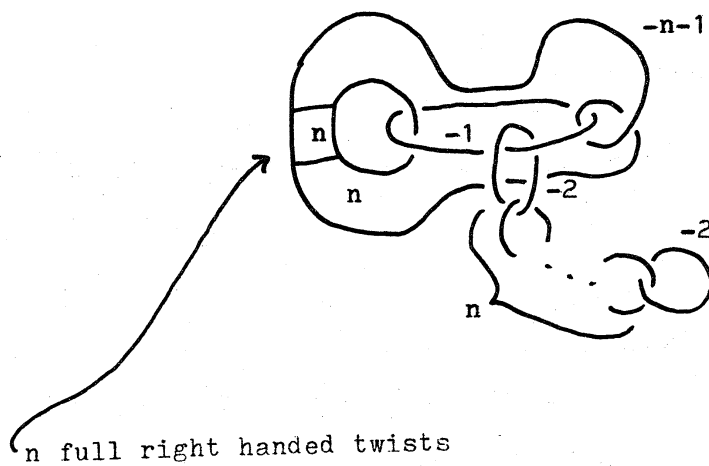
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 $\approx$

Handle adding

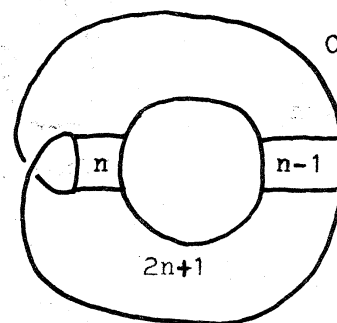
corresponding to

$$\begin{cases} h_1 \rightarrow h_1 \\ h_2 \rightarrow h_2 + h_1 \end{cases}$$

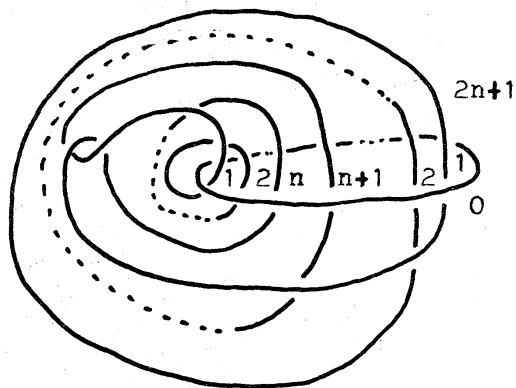


$\approx$

Blow down the -1
framed circles $n+1$ times
one after the other



=



$\approx$

$W(n+1, n; 2n+1).$

Surger S^2
corresponding to O_0

This completes the proof of Theorem 1. B.

Concluding remarks

(i) From a different point of view we can show that the Breiskorn homology 3-sphere $\Sigma(2, 7, 19)$ bounds a contractible 4-manifold, applying Kirby's calculus.

(ii) The proof of (i) and related topics will be described in another paper.

References

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- [M] N. Martin, Some homology 3-spheres which bound acyclic 4-manifolds, Springer Lecture Notes, No 722, 1979, 85-92.
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