

Singular Solutions of Nonlinear Differential Equations — an application of Fuchsian differential equations

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1 Introduction

We consider nonlinear partial differential equations of Kovalevskian type

$$\partial_t^m u = f\left(t, x; (\partial_t^j \partial_x^\alpha u)_{\substack{j \leq m-1 \\ j+|\alpha| \leq m}}\right) \quad (1)$$

where $t \in \mathbb{C}$, $x \in \mathbb{C}^d$ and the coefficients are holomorphic in a neighborhood Ω of the origin in $\mathbb{C}^{d+1}$.

We give a simple example to explain the motivation, before introducing complicated notations.

Example 1 (Burgers equation).

$$u_{tt} + 2uu_t - u_x = 0 \quad (2)$$

has a formal Laurent series solution

$$u = t^{-1} + gt + \left(\frac{1}{10}g_x - \frac{1}{5}g^2\right)t^3 + \cdots, \quad (3)$$

where $g = g(x)$ is an arbitrary holomorphic function.

It is easy to obtain such a formal solution (3): First, assume u is of the form

$$u = t^\sigma \sum_{n=0}^{\infty} u_n(x) t^n \quad (u_0 \neq 0), \quad (4)$$

substitute (4) into the equations and then equate the coefficients of the power of t to 0. We have $\sigma = -1$, $2u_0(1 - u_0) = 0$ and

$$(n+1)(n-2)u_n = -2 \sum_{\substack{i+j=n \\ 1 \leq i, j \leq n-1}} (j-1)u_i u_j + u_{n-2,x} \quad (n \geq 1). \quad (5)$$

It is natural to ask whether the formal series (3) converges or not. Of course, it converges to define exact solutions, which are **singular** on $t = 0$. We have four proofs of its convergence.

(I) Linearization:

By setting

$$u = (\ln w)_t = \frac{w_t}{w},$$

Burgers equation is equivalent to the linear equation

$$w_{tt} - w_x = 0. \quad (6)$$

Singular solution (3) is given by the initial condition

$$w|_{t=0} = 0, \quad w_t|_{t=0} = h(x)$$

with a suitably chosen $h(x)$.

(II) Direct estimates:

Using recurrent equation (5), Ishii [2] and Ōuchi [4] estimated the u_n 's directly.

(III) Leray-Volevich system:

Let $u = \frac{1}{\lambda}$, then λ satisfies the equation

$$\lambda \lambda_{tt} + 2\lambda_t - 2\lambda_t^2 - \lambda \lambda_x = 0. \quad (7)$$

After some calculation, Equation (7) reduces to the following Leray-Volevich system

$$\begin{cases} \lambda_t = 1 - \lambda \lambda_x + \lambda^2 \mu, \\ \mu_t = \lambda \mu_x + \lambda_x \mu - \lambda_{xx}. \end{cases} \quad (8)$$

The solution (3) is given by the initial condition

$$\begin{cases} \lambda|_{t=0} = 0, \\ \mu|_{t=0} = h(x) \end{cases}$$

with suitably chosen $h(x)$.

(IV) Fuchsian differential equations:

Let $\sigma = -1$ and put

$$a_N = \sum_{n=0}^N u_n t^{n+\sigma}, \quad w_N = \sum_{n=0}^{\infty} u_{N+1+n} t^{n+1}. \quad (9)$$

Then $u = a_N + t^{N+\sigma} w_N$ and

$$a_{N,tt} + 2a_N a_{N,t} - a_x = t^{\sigma-2+N+1} \times H_N \quad (10)$$

where H_N is a holomorphic function. Substituting (9) into (2), we obtain from (10)

$$\begin{aligned} (t\partial_t + N + 1)(t\partial_t + N - 2)w_N = \\ tA_N + tB_N w_N + tC_N(t\partial_t + N + \sigma)w_N - t^2 v_{N,x} + 2t^N w_N(t\partial_t + N + \sigma)w_N, \end{aligned} \quad (11)$$

where A_N , B_N and C_N are some holomorphic functions. Now we can apply a theorem by Gérard-Tahara [1].

Consider the following nonlinear differential equation:

$$(t\partial_t)^m w = F\left(t, x; ((t\partial_t)^j \partial_x^\alpha w)_{(j,\alpha) \in \Lambda}\right), \quad (12)$$

where $F(t, x; Z)$ is holomorphic in a neighborhood of $(t, x; Z) = (0, 0; 0)$ and satisfies

$$F(0, x; 0) \equiv 0, \quad (13)$$

$$\frac{\partial F}{\partial Z_{j,\alpha}}(0, x; 0) \equiv 0 \quad \text{if } |\alpha| > 0. \quad (14)$$

The characteristic polynomial of (12) is

$$C(\rho, x) := \rho^m - \sum_{j=0}^{m-1} \frac{\partial F}{\partial Z_{j,0}}(0, x; 0) \rho^j. \quad (15)$$

Theorem 1 (Gérard-Tahara). *If $C(n, 0) \neq 0$ for all positive integers n , then (12) has a unique formal solution $w = \sum_{n=1}^{\infty} w_n(x) t^n$ with $w(0, x) \equiv 0$, where $w_n(x)$ are holomorphic on a common neighborhood of the origin in $\mathbb{C}^d$. Moreover this power series is convergent and holomorphic near the origin in $\mathbb{C}_t \times \mathbb{C}_x^d$.*

2 Characteristic Exponent

We put

$$\Lambda := \{(j, \alpha) \in \mathbb{N} \times \mathbb{N}^d : j < m, j + |\alpha| \leq m\}$$

and write (1) as

$$\partial_t^m u = f(t, x; \partial^\Lambda u), \quad (16)$$

where $f(t, x; Z)$ is holomorphic in $\Omega \times \mathbb{C}^{\#\Lambda}$.

We expand f in Z

$$f(t, x; Z) = \sum_{\mu \in \mathcal{M}} f_\mu(t, x) Z^\mu, \quad (17)$$

where $\mathcal{M}$ is a subset of $\mathbb{N}^{\#\Lambda}$.

Let $k_\mu \in \mathbb{N}$ be the valuation of $f_\mu(t, x)$ in t ,

$$f_\mu(t, x) = t^{k_\mu} \sum_{k=0}^{\infty} f_{\mu,k}(x) t^k. \quad (18)$$

Definition 1. The characteristic exponent σ_c of (16) with respect to the surface $t = 0$ is

$$\sigma_c := \sup_{\substack{\mu \in \mathcal{M} \\ |\mu| \geq 2}} \frac{\gamma_t(\mu) - m - k_\mu}{|\mu| - 1}, \quad (19)$$

where

$$|\mu| := \sum_{(j,\alpha) \in \Lambda} \mu_{j,\alpha}, \quad \gamma_t(\mu) := \sum_{(j,\alpha) \in \Lambda} j \cdot \mu_{j,\alpha}.$$

We assign weights as follows:

$$u \rightarrow \sigma \quad \partial_t \rightarrow -1 \quad t \rightarrow 1.$$

Then the total weight of the right hand side of (16) is $m - \sigma$ and that of the term $f_\mu(\partial^\Lambda u)^\mu$ is $|\mu|\sigma - \gamma_t(\mu) + k_\mu$.

Burgers equation (2):

$$\sigma - 2 = 2\sigma - 1 + 0 \Rightarrow \sigma_c = -1.$$

Example 2 (KdV equation).

$$u_{ttt} - 6uu_t + u_x = 0 \quad (20)$$

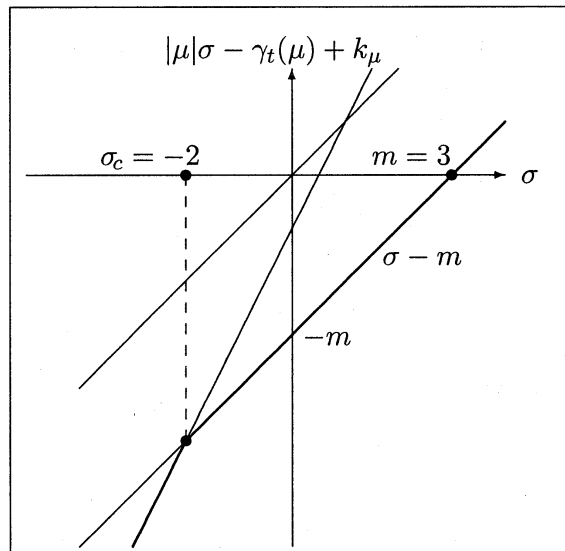
has

$$\sigma - 3 = 2\sigma - 1 + 0 \Rightarrow \sigma_c = -2,$$

and Laurent series solutions

$$u = 2t^{-2} + gt^2 + ht^4 - \frac{1}{24}g_xt^5 + \dots,$$

where $g = g(x)$ and $h = h(x)$ are arbitrary holomorphic functions.



Characteristic exponent of KdV equation (20)

Lemma 1.

- (i) σ_c is invariant with respect to coordinate change which keeps the variable t .
- (ii) $\sigma_c \leq m_0 \leq m - 1$, where m_0 is the order of differentiation with respect to t in $f(t, u; \partial^\Lambda u)$.

3 Singular Solutions

We assume

(A-1) $f(t, x; Z)$ is a polynomial in Z of degree greater than or equal to 2.

Under (A-1), the characteristic exponent

$$\sigma_c = \max_{\substack{\mu \in \mathcal{M} \\ |\mu| \geq 2}} \frac{\gamma_t(\mu) - m - k_\mu}{|\mu| - 1}, \quad (21)$$

is a rational number strictly less than m_0 , and the subset

$$\mathcal{M}^* := \{\mu \in \mathcal{M} : |\mu|\sigma_c - \gamma_t(\mu) + k_\mu = \sigma_c - m\}. \quad (22)$$

is not empty. We call the nonlinear term corresponding to μ in $\mathcal{M}^*$ **principal nonlinear term**.

(A-2) If $\mu \in \mathcal{M}^*$ then $\mu_{j,\alpha} = 0$ for $|\alpha| \geq 1$

We construct a solution to (16) in the form:

$$u(t, x) := t^{\sigma_c} \sum_{n=0}^{\infty} u_n(x) t^{n/p}, \quad (23)$$

where p is the denominator of the reduced fraction σ_c .

Substitute (23) into (16), we obtain recursion equations:

$$\begin{cases} P_c(x; u_0) \cdot u_0 = 0, \\ Q_c\left(x; u_0; \frac{n}{p}\right) \cdot u_n = R_n(x; \partial_x^\alpha u_0, \dots, \partial_x^\alpha u_{n-1})_{|\alpha| \leq m}, \end{cases} \quad (24)$$

where

$$P_c(x; \eta) := [\sigma_c; m] - \sum_{\mu \in \mathcal{M}^*} f_{\mu,0}(x) \left(\prod_{(j,\alpha) \in \Lambda} [\sigma_c; j]^{\mu_{j,\alpha}} \right) \eta^{|\mu|-1}, \quad (25)$$

and

$$Q_c(x; \eta; \rho) := [\rho + \sigma_c; m] - \sum_{\mu \in \mathcal{M}^*} f_{\mu,0}(x) \times \left(\prod_{(j,\alpha) \in \Lambda} [\sigma_c; j]^{\mu_{j,\alpha}} \right) \left(\sum_{(j,\alpha) \in \Lambda} \mu_{j,\alpha} \frac{[\rho + \sigma_c; j]}{[\sigma_c; j]} \right) \eta^{|\mu|-1}. \quad (26)$$

Here we have set for $\rho \in \mathbb{R}$ and $j \in \mathbb{N}$,

$$[\rho; j] := \rho(\rho - 1) \cdots (\rho - j + 1). \quad (27)$$

$P_c(x; \eta)$ and $Q_c(x; \eta; \rho)$ are polynomials in η and ρ and depend only on principal nonlinear terms. The order in η is $\max_{\mu \in \mathcal{M}^*} |\mu| - 1$ and m in ρ .

Burgers equation (2):

$$\begin{aligned} \sigma_c &= -1, & P_c(x; \eta) &= 2 - 2\eta, \\ Q_c(x; \eta = 1; \rho) &= (\rho + 1)(\rho - 2). \end{aligned}$$

KdV equation (20):

$$\begin{aligned} \sigma_c &= -2, & P_c(x; \eta) &= -24 + 12\eta, \\ Q_c(x; \eta = 2; \rho) &= (\rho + 1)(\rho - 4)(\rho - 6). \end{aligned}$$

Remark. If $k_\mu = 0$ for all $\mu \in \mathcal{M}^*$, then we have

$$Q_c(x; \eta; \rho = -1) = \frac{\sigma_c - m}{\sigma_c} P_c(x, \eta). \quad (28)$$

(A-3) The equation $P_c(x; \eta) = 0$ in η has at least one solution $\eta = u_0(x)$ which is holomorphic in a neighborhood of $x = 0$ and $u_0(x) \not\equiv 0$.

(A-4) One of the following holds for each $n \geq 1$

$$Q_c\left(0; u_0(0); \frac{n}{p}\right) \neq 0 \quad (a)$$

$$Q_c\left(x; u_0(x); \frac{n}{p}\right) \equiv 0, \quad R_n(x, \dots, \partial_x^\alpha u_0, \dots, \partial_x^\alpha u_{n-1}) \equiv 0 \quad (b)$$

$$\begin{cases} Q_c\left(0; u_0(0); \frac{n}{p}\right) = 0, & Q_c\left(x; u_0(x); \frac{n}{p}\right) \neq 0 \\ Q_c\left(x; u_0(x); \frac{n}{p}\right) \text{ divides } & R_n(x, \dots, \partial_x^\alpha u_0, \dots, \partial_x^\alpha u_{n-1}) \end{cases} \quad (c)$$

Remark. In case of (a) or (c), u_n is determined uniquely, and in case of (b), $u_n(x)$ may be any holomorphic function.

Remark.

$$Q_c(0; u_0(0); \rho) = 0$$

has at most m distinct roots.

Theorem 2. Suppose (A-1), (A-2), (A-3), (A-4) are satisfied. Then we can construct a solution to (16) in the form (23). Moreover all formal solutions (23) converge near the origin in $\mathbb{C}_t \times \mathbb{C}_x^d$.

We apply a theorem by Gérard–Tahara [1] to prove the convergence of formal solutions. For a positive integer N , we put

$$w_N(t, x) := \sum_{n=0}^{\infty} u_{N+n+1}(x) t^{\frac{n+1}{p}}.$$

Proposition 1. If the formal series (23) satisfies the equation (16), then w_N satisfies the following differential equation:

$$Q_c\left(x; u_0(x); t\partial_t + \frac{N}{p}\right)w_N = t^{1/p} \cdot G\left(t^{1/p}, x; ((t\partial_t)^j \partial_x^\alpha w_N)_{(j, \alpha) \in \Lambda}\right), \quad (29)$$

where $G(\tau, x; Z)$ is a polynomial in Z with coefficients holomorphic near the origin in $\mathbb{C}_{\tau, x}^{d+1}$.

Next put $\tau = t^{1/p}$ and

$$\tilde{w}_N(\tau, x) = \sum_{n=0}^{\infty} u_{N+n+1}(x) \tau^{n+1}. \quad (30)$$

Then $\tilde{w}_N(0, x) \equiv 0$, and by using the relation $t\partial_t = \frac{1}{p}\tau\partial_\tau$, we obtain $\tilde{w}_N$ satisfies

$$Q_c\left(x; u_0(x); \frac{1}{p}\tau\partial_\tau + \frac{N}{p}\right)\tilde{w}_N = \tau \cdot G\left(\tau, x; \left(\left(\frac{1}{p}\tau\partial_\tau\right)^j \partial_x^\alpha \tilde{w}_N\right)_{(j, \alpha) \in \Lambda}\right). \quad (31)$$

Equation (31) satisfies the conditions (13) and (14), and its characteristic polynomial is

$$C(\rho, x) = Q_c\left(x; u_0(x); \frac{1}{p}(\rho + N)\right). \quad (32)$$

If we take N sufficiently large, then $C(n, 0) \neq 0$ for all positive integers.

4 Prolongation of Solutions

We need to define a modified version of characteristic exponent.

Definition 2. For (16), we define σ_c^* by

$$\sigma_c^* = \sup_{\substack{\mu \in \mathcal{M} \\ \nu \leq \mu, |\nu| \geq 2}} \frac{\gamma_t(\nu) - m - k_\mu}{|\nu| - 1}. \quad (33)$$

Example 3.

$$u_{tt} + 6uu_t^3 + xu_t^2 + uu_x = 0,$$

has a singular solution with exponent $\sigma_c = \frac{1}{3}$:

$$u = t^{1/3} - \frac{x}{12}t^{2/3} + \frac{x^2}{240}t^{3/3} + \frac{x^3}{5184}t^{4/3} - \dots,$$

and ones with exponent $\sigma_c^* = \frac{1}{2}$:

$$u = \frac{1}{3}g^2 + \frac{1}{g}t^{1/2} + \left(-\frac{1}{2g^4} - \frac{x}{6g^2}\right)t + \dots,$$

where $g = g(x)$ is an arbitrary holomorphic function with $g(0) \neq 0$.

Lemma 2.

- (i) $\sigma_c \leq \sigma_c^* \leq m_0 (\leq m - 1)$.
- (ii) If $\sigma_c \leq 0$, then $\sigma_c = \sigma_c^*$.

Definition 3. For $\sigma \in \mathbb{R}$, we define $\delta_c(\sigma)$ by

$$\begin{aligned} \delta_c(\sigma) &:= \inf_{\substack{\mu \in \mathcal{M} \\ \nu \leq \mu, |\nu| \geq 2}} (|\nu| - 1)\sigma - \gamma_t(\nu) + m + k_\mu \\ &= \inf_{\substack{\mu \in \mathcal{M} \\ \nu \leq \mu, |\nu| \geq 2}} (|\nu|\sigma - \gamma_t(\nu) + k_\mu) - (\sigma - m). \end{aligned}$$

Lemma 3.

- (i) $\delta_c(\sigma) \geq 0$ if and only if $\sigma \geq \sigma_c^*$.
- (ii) If $\sigma > \sigma_c^*$, then $\delta_c(\sigma) > 0$.
- (iii) $\delta_c(m_0) > 0$.
- (iv) If $\delta_c(\sigma) > 0$ and $\sigma \leq m_0$, then there is a constant $\delta > 0$ such that

$$|\nu|\sigma - \gamma_t(\nu) + k_\mu \geq \sigma - m + \delta$$

for all $\mu \in \mathcal{M}$, $\nu \leq \mu$.

Definition 4. $u \in \mathcal{O}(\Omega_-)$ is bounded of order σ in Ω_- means that $\exists M > 0$ such that for all $(t, x) \in \Omega_-$, if $\sigma \leq 0$

$$|u(t, x)| \leq M |\Re t|^\sigma$$

or if $\sigma > 0$,

$$|\partial_t^j u(t, x)| \leq \begin{cases} M & \text{for } j = 0, 1, \dots, \lfloor \sigma \rfloor, \\ M |\Re t|^{\sigma-j} & \text{for } j = \lfloor \sigma \rfloor + 1, \end{cases}$$

Example 4. $t^\sigma \cdot h(t, x)$ with a holomorphic function $h(t, x)$ is bounded of order σ , and $\log t \cdot h(t, x)$ is bounded of order $-\epsilon$ for any $\epsilon > 0$.

Theorem 3. If $u \in \mathcal{O}(\Omega_-)$ satisfies Equation (16) and is bounded of order σ in Ω_- with $\delta_c(\sigma) > 0$, then u is holomorphic in a neighborhood of the origin. Especially if $\sigma > \sigma_c^*$ or $\sigma = m_0$, then u is holomorphic near the origin.

Corollary 1. If $u \in \mathcal{O}(\Omega_-)$ satisfies Equation (16) and the derivatives of u up to order m_0 are bounded in Ω_- , then u is holomorphic near the origin.

Remark 1. Examples 1 and 2 give singular solutions which are bounded of order $\sigma_c = \sigma_c^*$, and Example 3 gives ones of order σ_c^* with $\sigma_c^* > \sigma_c$.

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