

Arithmetization of another formulation of a subsystem of Kaneko-Nagashima's GL

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July 11, 1997

Abstract

We propose a system G of game logic related to Kaneko-Nagashima's GL_ω . Our aim is to make the system more constructive than GL_ω . Though G is an infinitary system, formulae and sequents are finitary. We define a Gödel numbering of formulae, sequents and derivations, and we consider some problems concerning undecidable sentences.

1 The Language and the Rules of the System G

Terms, formulae and sequents of the semiformal deductive system G are defined in this section. Derivations (proof figures) are defined in a later section. G has an infinitary inference rule ($\rightarrow C$); all other elements of G are finitary.

Symbols.

Free variables: $a_0, a_1, \dots$

Bound variables: $x_0, x_1, \dots$

Logical symbols: $\neg, \supset, \wedge, \vee, \forall, \exists$.

Modal (epistemic) symbols: K_1, K_2, C .

Predicate symbols: $=$.

Function symbols: $0, S, +, \times$.

*Until March 1998.

Auxiliary symbols: $(,), \rightarrow$.

Remark. Though other functions, predicates may be allowed without difficulty, we confine ourself to this language for the sake of notational simplicity.

Terms, formulae, cedents and sequents are defined as usual. $S(t)$, the successor of t , is abbreviated as t' .

$(A \supset B) \wedge (B \supset A)$ is abbreviated as $A \sim B$.

$\exists x F(x) \wedge \forall y \forall z [F(y) \wedge F(z) \supset y = z]$ is abbreviated as $\exists! x F(x)$.

Sequents are defined as usual.

For any formula A , we define $K_{i,k}A$ ($i = 1, 2; k \in \mathbb{N}$) inductively as follows:

$$K_{i,0}A \text{ is } A, \quad K_{i,k+1}A \text{ is } K_i K_{j,k}A \text{ where } i \neq j.$$

For any formula A , we define $N_k A$ ($k \in \mathbb{N}$) as follows:

$$N_0 A \text{ is } A, \quad N_{2k+1} A \text{ is } K_{1,k+1} A, \quad N_{2k+2} A \text{ is } K_{2,k+1} A$$

Schemata for Initial Sequents:

$$\begin{aligned} A &\longrightarrow A \\ \forall x K_1(F(x)) &\longrightarrow K_1(\forall x F(x)) \\ \forall x K_2(F(x)) &\longrightarrow K_2(\forall x F(x)) \\ \forall x C(F(x)) &\longrightarrow C(\forall x F(x)) \\ &\longrightarrow t = t \\ s = t, F(s) &\longrightarrow F(t) \\ t' = 0 &\longrightarrow \\ s' = t' &\longrightarrow s = t \\ &\longrightarrow t + 0 = t \\ &\longrightarrow (t + s)' = t + s' \\ &\longrightarrow t \times 0 = 0 \\ &\longrightarrow t \times s' = t \times s + t \end{aligned}$$

Inference Rules:

$$\begin{aligned} \frac{\Gamma \longrightarrow \Theta}{A, \Gamma \longrightarrow \Theta} (t \rightarrow) \quad & \frac{\Gamma \longrightarrow \Theta}{\Gamma \longrightarrow \Theta, A} (\rightarrow t) \\ \frac{A, A, \Gamma \longrightarrow \Theta}{A, \Gamma \longrightarrow \Theta} (c \rightarrow) \quad & \frac{\Gamma \longrightarrow \Theta, A, A}{\Gamma \longrightarrow \Theta, A} (\rightarrow c) \\ \frac{\Gamma, A, B, \Delta \longrightarrow \Theta}{\Gamma, B, A, \Delta \longrightarrow \Theta} (i \rightarrow) \quad & \frac{\Gamma \longrightarrow \Theta, A, B, \Lambda}{\Gamma \longrightarrow \Theta, B, A, \Lambda} (\rightarrow i) \\ & \frac{\Gamma \longrightarrow \Theta, A \quad A, \Delta \longrightarrow \Lambda}{\Gamma, \Delta \longrightarrow \Theta, \Lambda} (\text{cut}) \end{aligned}$$

$$\begin{array}{c}
\frac{\Gamma \rightarrow \Theta, A}{\neg A, \Gamma \rightarrow \Theta} (\neg \rightarrow) \quad \frac{A, \Gamma \rightarrow \Theta}{\Gamma \rightarrow \Theta, \neg A} (\rightarrow \neg) \\
\\
\frac{A, \Gamma \rightarrow \Theta}{A \wedge B, \Gamma \rightarrow \Theta} (\wedge \rightarrow 1) \quad \frac{\Gamma \rightarrow \Theta, A \quad \Gamma \rightarrow \Theta, B}{\Gamma \rightarrow \Theta, A \wedge B} (\rightarrow \wedge) \\
\frac{B, \Gamma \rightarrow \Theta}{A \wedge B, \Gamma \rightarrow \Theta} (\wedge \rightarrow 2) \\
\\
\frac{A, \Gamma \rightarrow \Theta \quad B, \Gamma \rightarrow \Theta}{A \vee B, \Gamma \rightarrow \Theta} (\vee \rightarrow) \quad \frac{\Gamma \rightarrow \Theta, A}{\Gamma \rightarrow \Theta, A \vee B} (\rightarrow \vee 1) \\
\frac{\Gamma \rightarrow \Theta, B}{\Gamma \rightarrow \Theta, A \vee B} (\rightarrow \vee 2) \\
\\
\frac{\Gamma \rightarrow \Theta, A \quad B, \Delta \rightarrow \Lambda}{A \supset B, \Gamma, \Delta \rightarrow \Theta, \Lambda} (\supset \rightarrow) \quad \frac{A, \Gamma \rightarrow \Theta, B}{\Gamma \rightarrow \Theta, A \supset B} (\rightarrow \supset) \\
\\
\frac{F(t), \Gamma \rightarrow \Theta}{\forall x F(x), \Gamma \rightarrow \Theta} (\forall \rightarrow) \quad \frac{\Gamma \rightarrow \Theta, F(a)}{\Gamma \rightarrow \Theta, \forall x F(x)} (\rightarrow \forall)^{(1)} \\
\\
\frac{F(a), \Gamma \rightarrow \Theta}{\exists x F(x), \Gamma \rightarrow \Theta} (\exists \rightarrow)^{(1)} \quad \frac{\Gamma \rightarrow \Theta, F(t)}{\Gamma \rightarrow \Theta, \exists x F(x)} (\rightarrow \exists) \\
\\
\frac{\Gamma, K(\Delta) \rightarrow \Theta}{K(\Gamma, \Delta) \rightarrow K\Theta} (K \rightarrow K)^{(3)} \\
\\
\frac{N_k A, \Gamma \rightarrow \Theta}{C(A), \Gamma \rightarrow \Theta} (C \rightarrow) (k \in \mathbb{N}) \quad \frac{\{\Gamma \rightarrow \Theta, N_k A | k \in \mathbb{N}\}}{\Gamma \rightarrow \Theta, C(A)} (\rightarrow C)^{(2)} \\
\\
\frac{F(a), \Gamma \rightarrow \Theta, F(a')}{F(0), \Gamma \rightarrow \Theta, F(t)} (MI)^{(1)}
\end{array}$$

- (1) *Restriction on variable*: The free variable designated by a , the *eigenvariable*, must not occur in the lower sequent.
- (2) *Restriction will be stated later*.
- (3) K is either K_1 or K_2 . Θ consists of at most one formula.

2 Derivations and the Coding

In this section we define *derivations* and the *coding* of derivations simultaneously. Let $(F_0, F_1, F_2, \dots)$ be an effective enumeration of all primitive recursive functions¹. First we introduce some total recursive functions and total recursive predicates needed for Gödel numbering.

$\langle a_0, a_1, \dots, a_k \rangle$ denotes the sequence number $p_0^{a_0'} \cdot p_1^{a_1'} \cdot \dots \cdot p_k^{a_k'}$ where $p_0 = 2$, $p_1 = 3$, $p_2 = 5$, ... is the series of prime numbers. Let $\text{Seqnum}(a)$ be the number

¹Primitive recursiveness is not essential. For some other subrecursive classes, the argument almost parallels.

theoretic predicate denoting that a is a sequence number. Definition is

$$\text{Seqnum}(a) \sim a > 0 \wedge \exists k < a \forall i < a [p_i | a \sim i < k].$$

We define $[a]_i = (\mu x < a \neg (p_i x | a)) \div 1$ and $\text{lh}(a) = \sum_{i < a} \text{sg}([a]_i)$. If $a = \langle a_0, a_1, \dots, a_k \rangle$ then $\text{lh}(a) = k'$ and $[a]_i = a_i$ for any $i < \text{lh}(a)$.

Note. $[a]_i = (a)_i \div 1$.

For any two sequence numbers $a = \langle a_0, \dots, a_k \rangle$ and $b = \langle b_0, \dots, b_l \rangle$, let

$$a * b = \langle a_0, \dots, a_k, b_0, \dots, b_l \rangle.$$

We assign Gödel numbers to the symbols and the names of inference rules. The Gödel number of any symbol $\#$ is denoted as $\ulcorner \# \urcorner$ and similarly for the names of inference rules.

Successive odd numbers (≥ 3) are assigned to the symbols and the names of inference rules: 0, S , $+$, $\times$, $=$, $\neg$, $\supset$, $\wedge$, $\vee$, $\forall$, $\exists$, K_1 , K_2 , C , $\longrightarrow$, $(t \rightarrow)$, $(\rightarrow t)$, $(c \rightarrow)$, $(\rightarrow c)$, $(i \rightarrow)$, $(\rightarrow i)$, (cut) , $(\neg \rightarrow)$, $(\rightarrow \neg)$, $(\wedge \rightarrow 1)$, $(\wedge \rightarrow 2)$, $(\rightarrow \wedge)$, $(\vee \rightarrow)$, $(\rightarrow \vee 1)$, $(\rightarrow \vee 2)$, $(\supset \rightarrow)$, $(\rightarrow \supset)$, $(\forall \rightarrow)$, $(\rightarrow \forall)$, $(\exists \rightarrow)$, $(\rightarrow \exists)$, $(K \rightarrow K)$, $(C \rightarrow)$, $(\rightarrow C)$, (MI) , a_k ($k = 0, 1, \dots$), x_k ($k = 0, 1, \dots$). For example, $\ulcorner + \urcorner = 7$ and $\ulcorner (\text{cut}) \urcorner = 45$.

Gödel numbers of terms and formulae are defined as usual. The Gödel number of a formal expression E is denoted as $\ulcorner E \urcorner$. The Gödel number of a sequent

$$A_1, A_2, \dots, A_k \longrightarrow B_1, B_2, \dots, B_m$$

is

$$\langle \ulcorner \rightarrow \urcorner, \langle \ulcorner A_1 \urcorner, \ulcorner A_2 \urcorner, \dots, \ulcorner A_k \urcorner \rangle, \langle \ulcorner B_m \urcorner, \dots, \ulcorner B_2 \urcorner, \ulcorner B_1 \urcorner \rangle \rangle.$$

We omit "the Gödel number of" if no confusions are likely to occur. For instance, we say " a is a formula" instead of " a is the Gödel number of a formula".

Let $\text{Formula}(a)$ be a number theoretic predicate meaning that a is a formula. Now we define

$$\begin{aligned} N(\ulcorner A \urcorner, k) &= \ulcorner N_k(A) \urcorner, \\ \text{Cedent}(a) &\sim \text{Seqnum}(a) \wedge \forall i < \text{lh}(a) \text{Formula}([a]_i), \\ \text{Sequent}(a) &\sim \text{lh}(a) = 3 \wedge \\ &\quad \wedge [a]_0 = \ulcorner \rightarrow \urcorner \wedge \text{Cedent}([a]_1) \wedge \text{Cedent}([a]_2). \end{aligned}$$

Let $\text{InitialSequent}(a)$ be a number theoretic predicate meaning that a is an initial sequent.

Let $\text{Infer}_1(j, b, a_1)$ or $\text{Infer}_2(j, b, a_1, a_2)$ be the number theoretic predicate meaning that

$$\frac{a_1}{b} (j) \quad \text{or} \quad \frac{a_1 \ a_2}{b} (j)$$

is an instance of a one-premise or two-premise inference rule respectively.

Definition 1 We define derivation and its Gödel number simultaneously by induction.

(1) If S is an initial sequent, then S is a derivation of S with the Gödel number $\langle 0, \ulcorner S \urcorner \rangle$.

(2) If $\mathcal{H}_1$ is a derivation of a sequent S_1 and

$$\frac{S_1}{S} (J)$$

is an instance of a one-premise inference rule, then

$$\frac{\mathcal{H}_1}{S} (J)$$

is a derivation of S with the Gödel number $\langle \ulcorner (J) \urcorner, \ulcorner S \urcorner, \ulcorner \mathcal{H}_1 \urcorner \rangle$.

(3) If $\mathcal{H}_1$ is a derivation of a sequent S_1 , $\mathcal{H}_2$ is a derivation of a sequent S_2 and

$$\frac{S_1 \quad S_2}{S} (J)$$

is an instance of a two-premise inference rule, then

$$\frac{\mathcal{H}_1 \quad \mathcal{H}_2}{S} (J)$$

is a derivation of S with the Gödel number $\langle \ulcorner (J) \urcorner, \ulcorner S \urcorner, \ulcorner \mathcal{H}_1 \urcorner, \ulcorner \mathcal{H}_2 \urcorner \rangle$.

(4) If $\mathcal{H}_k$ is a derivation of a sequent S_k for each $k \in \mathbb{N}$ and if

$$\frac{S_0 \quad S_1 \quad \cdots}{S} (\rightarrow C)$$

is an instance of the rule $(\rightarrow C)$, and if $\ulcorner \mathcal{H}_k \urcorner$ is a primitive recursive function $F_e(k)$ of k , then

$$\frac{\mathcal{H}_0 \quad \mathcal{H}_1 \quad \cdots}{S} (\rightarrow C)$$

is a derivation of S with the Gödel number $\langle \ulcorner (\rightarrow C) \urcorner, \ulcorner S \urcorner, e \rangle$. $\square$

Lemma 1 The nonmodal fragment G_0 of G is the first order arithmetic. $\square$

Theorem 2 G is conservative over G_0 . $\square$

Proof (outline). Let $\mathcal{H}$ be a derivation of a nonmodal sequent S . Delete all modal symbols K_1, K_2, C from $\mathcal{H}$. For every occurrences of $(\rightarrow C)$ in $\mathcal{H}$, delete all premises but the leftmost one. $\square$

Corollary 3 Any undecidable sentence in G_0 is undecidable in G . $\square$

Problem 4 What is the relation between G and Kaneko-Nagashima's GL_ω ? Is primitive recursively restricted GL_ω conservative over G ?

Problem 5 Construct a semantics for G .

3 Undecidability

Let $\text{prov}(a, b)$ be a number theoretic predicate denoting “ a is a derivation of a sequent b ”. This predicate is inductively defined as follows:

$$\begin{aligned} \text{prov}(a, b) &\sim [a = \langle 0, b \rangle \wedge \text{InitialSequent}(b)] \vee \\ &\vee (\exists j, u, x < a)[a = \langle j, b, u \rangle \wedge \text{Infer}_1(j, b, x) \wedge \text{prov}(u, x)] \vee \\ &\vee (\exists j, u, v, x, y < a)[a = \langle j, b, u, v \rangle \wedge \\ &\quad \wedge \text{Infer}_2(j, b, x, y) \wedge \text{prov}(u, x) \wedge \text{prov}(v, y)] \vee \\ &\vee (\exists e, x, u, v < a)[a = \langle \ulcorner \neg \rightarrow C \urcorner, b, e \rangle \wedge b = \langle \ulcorner \neg \rightarrow \urcorner, u, v \rangle \wedge \\ &\quad \wedge \text{Cedent}(u) \wedge \text{Cedent}(v) \wedge \text{Formula}(x) \wedge \text{lh}(v) > 0 \wedge \\ &\quad \wedge [v]_0 = \langle \ulcorner C \urcorner, x \rangle \wedge \\ &\quad \wedge \forall k (\text{prov}(F_e(k), [F_e(k)]_1) \wedge [[F_e(k)]_1]_2]_0 = N(x, k))] \end{aligned}$$

Conjecture 6 *The predicate $\text{prov}(a, b)$ is Π_1 . $\square$*

Let $\text{prov}_F(a, b)$ be a number theoretic predicate denoting “ a is a derivation of a formula b ”:

$$\text{prov}_F(a, b) \sim \text{Formula}(b) \wedge \text{prov}(a, \langle \ulcorner \neg \rightarrow \urcorner, \langle \rangle, \langle b \rangle \rangle).$$

Conjecture 7 *The predicate $\text{prov}_F(a, b)$ is Π_1 . $\square$*

Problem 8 *Is the predicate $\text{prov}_F(a, b)$ proper Π_1 ? $\square$*

Problem 9 *Is the predicate $\text{prov}_F(a, b)$ numeralwise expressible in G ? $\square$*

Theorem 10 *If G is ω -consistent and if prov_F is Π_1 , an undecidable sentence can be constructed from prov_F . $\square$*

Proof. Case 1: The predicate prov_F is numeralwise expressible. The argument is similar to Gödel's. Let $P(u, v)$ be a formula numeralwise expressing prov_F . By diagonalization lemma, there exists a sentence A satisfying

$$\vdash A \sim \neg \exists x P(x, \ulcorner A \urcorner).$$

(i) Proof of $\not\vdash A$. If there is a derivation $\mathcal{H}$ of A , then $\text{prov}_F(\ulcorner \mathcal{H} \urcorner, \ulcorner A \urcorner)$, hence $\vdash P(\ulcorner \mathcal{H} \urcorner, \ulcorner A \urcorner)$, hence $\vdash \exists x P(x, \ulcorner A \urcorner)$.

On the other hand, $\vdash A$ implies $\vdash \neg \exists x P(x, \ulcorner A \urcorner)$. This contradicts the consistency of G .

(ii) Proof of $\not\vdash \neg A$. The result $\not\vdash A$ implies that $\text{prov}_F(m, \ulcorner A \urcorner)$ for no m . By numeralwise expressibility, $\vdash \neg P(\bar{m}, \ulcorner A \urcorner)$ for all m . Since G is ω -consistent, $\not\vdash \neg \forall x \neg P(x, \ulcorner A \urcorner)$, i. e. $\not\vdash \exists x P(x, \ulcorner A \urcorner)$. Hence $\not\vdash \neg A$ by the definition of A .

Case 2: The predicate prov_F is not numeralwise expressible. Because prov_F is Π_1 , there exists a total recursive predicate R such that

$$\text{prov}_F(a, b) \sim \forall x R(a, b, x).$$

Since R is numeralwise expressible, there exists a formula $R(u, v, w)$ numeralwise expressing R . Since prov_F is not numeralwise expressed by $\forall x R(u, v, x)$, there exists a and b satisfying

$$\text{either} \quad \text{prov}_F(a, b) \quad \text{and} \quad \not\vdash \forall x R(\bar{a}, \bar{b}, x)$$

$$\text{or} \quad \neg \text{prov}_F(a, b) \quad \text{and} \quad \not\vdash \neg \forall x R(\bar{a}, \bar{b}, x).$$

If the latter holds, there exists a c such that $\neg R(a, b, c)$. Therefore $\vdash \neg R(\bar{a}, \bar{b}, \bar{c})$ by numeralwise expressibility of R , hence $\vdash \neg \forall x R(\bar{a}, \bar{b}, x)$, a contradiction. Therefore we have

$$\text{prov}_F(a, b) \quad \text{and} \quad \not\vdash \forall x R(\bar{a}, \bar{b}, x)$$

for some a and b . Hence $R(a, b, c)$ for every c , hence $\vdash R(\bar{a}, \bar{b}, \bar{c})$ for every c . By the ω -consistency of G , this implies

$$\not\vdash \neg \forall x R(\bar{a}, \bar{b}, x). \quad \square$$

References

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