

Hypergeometric type generating functions of several variables associated with the Lerch zeta-function (summarized version) *

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Abstract

This is a summarized version of the forthcoming paper [10].

Let s, z and $(z_0, \mathbf{z}) = (z_0, z_1, \dots, z_n)$ be complex variables, and $\zeta(s, z, \lambda)$ denote the Lerch zeta-function defined by (1.1) below. We introduce in the present article a class of generating functions and their confluent analogues, denoted by $\mathcal{Z}_{a,\lambda}^{(n)}(s, \beta; z_0, \mathbf{z})$ and $\widehat{\mathcal{Z}}_{a,\lambda}^{(n)}(s, \beta_{n-1}; z_0, \mathbf{z})$ respectively (see (2.1) and (2.3)), in the forms of the fourth Lauricella hypergeometric type (of several variables) associated with $\zeta(s, z, \lambda)$. It is shown that complete asymptotic expansions of $\mathcal{Z}_{a,\lambda}^{(n)}(s, \beta; z_0, \mathbf{z})$ exit when $z_0 \rightarrow 0$ (Theorem 1) as well as when $z_0 \rightarrow \infty$ (Theorem 2) through the sectorial region $|\arg z - \theta_0| < \pi/2$ with any fixed angle $\theta_0 \in [-\pi/2, \pi/2]$, while other z_j 's move through the same sector satisfying the conditions $z_j \ll z_0$ ($j = 1, \dots, n$). Similar asymptotic results also hold for $\widehat{\mathcal{Z}}_{a,\lambda}^{(n)}(s, \beta; z_0, \mathbf{z})$ (Theorems 3 and 4) through the confluence operation in (2.3). Our main formulae (3.1) and (3.4) (resp. (3.7) and (3.10)) first assert that $\mathcal{Z}_{a,\lambda}^{(n)}(s, \beta; z_0, \mathbf{z})$ (resp. $\widehat{\mathcal{Z}}_{a,\lambda}^{(n)}(s, \beta_{n-1}; z_0, \mathbf{z})$) can be continued to a meromorphic function of s over the whole s -plane, to the whole poly-sector $|\arg z_j| < \pi$ ($j = 0, 1, \dots, n$), and for all $(\beta, \gamma) \in \mathbb{C}^n \times (\mathbb{C} \setminus \{0, -1, \dots\})$ (resp. for all $(\beta_{n-1}, \gamma) \in \mathbb{C}^{n-1} \times (\mathbb{C} \setminus \{0, -1, \dots\})$). We can further apply (3.1) and (3.4) to deduce complete asymptotic expansions of $(\partial/\partial s)^m \mathcal{Z}_{a,\lambda}^{(n)}(s, \beta; z_0, \mathbf{z})$ ($m = 1, 2, \dots$) at any integer arguments $s = l \in \mathbb{Z}$ when $(z_0, \mathbf{z})$ becomes small (Corollary 6) and large (Corollary 8) under the same settings as in Theorems 1 and 2. Furthermore, several applications of Theorems 1-4 in the cases of $n = 1$ and 2 are finally presented.

Introduction

Throughout this article, $s = \sigma + \sqrt{-1}t$, z and $(z_0, \mathbf{z}) = (z_0, z_1, \dots, z_n)$ are complex variables with $|\arg z| < \pi$ and $|\arg z_j| < \pi$ ($j = 0, 1, \dots, n$), and a and λ real parameters with $a > 0$. We hereafter set $e(\lambda) = e^{2\pi\lambda\sqrt{-1}}$, use the vectorial notation $\mathbf{x} = (x_1, \dots, x_m)$ with the abbreviation

$$\langle \mathbf{x} \rangle = x_1 + \dots + x_m$$

for any $m \geq 1$ and any complex x_i ($i = 1, \dots, m$), and further write $\mathbf{x}_{m-1} = (x_1, \dots, x_{m-1})$ and

$$\frac{\mathbf{x}}{y} = \left(\frac{x_1}{y}, \dots, \frac{x_m}{y} \right)$$

for any $y \neq 0$. The Lerch zeta-function $\zeta(s, z, \lambda)$ is defined by the Dirichlet series

$$(1.1) \quad \zeta(s, z, \lambda) = \sum_{l=0}^{\infty} e(\lambda l) (l+z)^{-s} \quad (\sigma = \operatorname{Re} s > 1),$$

and its meromorphic continuation over the whole s -plane; this is an entire function when $\lambda \in \mathbb{R} \setminus \mathbb{Z}$, while if $\lambda \in \mathbb{Z}$ it reduces to the Hurwitz zeta-function $\zeta(s, a)$, and so $\zeta(s) =$

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$\zeta(s, 1)$ is the Riemann zeta-function. We remark here that the notation (1.1) differs from the original $\phi(z, \lambda, s)$ due to Lerch [13], in order to retain notational consistency with other terminology.

It is the principal aim of the present article to introduce a class of generating functions and their confluent analogues, denoted by $\mathcal{Z}_{a,\lambda}^{(n)}(s, \beta; z_0, \mathbf{z})$ and $\widehat{\mathcal{Z}}_{a,\lambda}^{(n)}(s, \beta_{\gamma}^{n-1}; z_0, \mathbf{z})$ respectively (see (2.1) and (2.4) below), in the forms of the (fourth) Lauricella hypergeometric type (of several variables) associated with $\zeta(s, z, \lambda)$. We shall first show that complete asymptotic expansions of $\mathcal{Z}_{a,\lambda}^{(n)}(s, \beta; z_0, \mathbf{z})$ and $\widehat{\mathcal{Z}}_{a,\lambda}^{(n)}(s, \beta; z_0, \mathbf{z})$ exist when $(z_0, \mathbf{z})$ becomes small (Theorems 1 and 3) and large (Theorems 2 and 4) under certain settings on the movement of $(z_0, \mathbf{z})$. Several applications of Theorems 1–4 will further be presented. Before stating our main results, some necessary notations and terminology will be prepared.

Let $\Gamma(s)$ be the gamma function, $(s)_k = \Gamma(s+k)/\Gamma(s)$ for any $k \in \mathbb{Z}$ the shifted factorial of s , and write

$$\Gamma\left(\begin{smallmatrix} \boldsymbol{\mu} \\ \boldsymbol{\nu} \end{smallmatrix}\right) = \Gamma\left(\begin{smallmatrix} \mu_1, \dots, \mu_h \\ \nu_1, \dots, \nu_k \end{smallmatrix}\right) = \frac{\prod_{i=1}^h \Gamma(\mu_i)}{\prod_{j=1}^k \Gamma(\nu_j)}$$

for complex vectors $\boldsymbol{\mu} = (\mu_1, \dots, \mu_h)$ and $\boldsymbol{\nu} = (\nu_1, \dots, \nu_k)$. In the sequel the sets of non-negative and non-positive integers are respectively denoted by $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $-\mathbb{N}_0 = \{-k \mid k \in \mathbb{N}_0\}$. The (fourth) Lauricella hypergeometric function of m -variables x_i ($i = 1, \dots, m$) is defined by the m -ple power series

$$(1.2) \quad \begin{aligned} & F_D^{(m)}\left(\alpha, \beta_1, \dots, \beta_m; \gamma; x_1, \dots, x_m\right) \\ &= \sum_{k_1, \dots, k_m=0}^{\infty} \frac{(\alpha)_{k_1+\dots+k_m} (\beta_1)_{k_1} \dots (\beta_m)_{k_m}}{(\gamma)_{k_1+\dots+k_m} k_1! \dots k_m!} x_1^{k_1} \dots x_m^{k_m} \end{aligned}$$

for complex parameters α, β_i ($i = 1, \dots, m$) and $\gamma \neq -k$ ($k \in \mathbb{N}_0$), where the series converges absolutely in the poly-disk $|x_i| < 1$ ($i = 1, \dots, m$); this is continued to a one-valued holomorphic function of $(\alpha, \boldsymbol{\beta}, \gamma, \mathbf{x})$ for all $(\alpha, \boldsymbol{\beta}, \gamma) \in \mathbb{C}^{m+1} \times \{\mathbb{C} \setminus (-\mathbb{N}_0)\}$, and $\mathbf{x}$ in the poly-sector $|\arg(1 - x_i) - \varphi_0| < \pi/2$ ($i = 1, \dots, m$) for any angle fixed with $\varphi_0 \in [-\pi/2, \pi/2]$ (cf. [1]). Note that (1.2) reduces when $m = 1$ to Gauss' hypergeometric function ${}_2F_1(\alpha, \beta; \gamma; x)$, and when $m = 2$ to (the first) Appell's hypergeometric function $F_1(\alpha, \beta_1, \beta_2; \gamma; x_1, x_2)$. The abbreviations

$$\begin{aligned} (\boldsymbol{\beta})_{\mathbf{k}} &= (\beta_1)_{k_1} \dots (\beta_m)_{k_m}, & \mathbf{k}! &= k_1! \dots k_m!, \\ \mathbf{x}^{\mathbf{k}} &= x_1^{k_1} \dots x_m^{k_m} \end{aligned}$$

for $\mathbf{k} = (k_1, \dots, k_m)$ and $\mathbf{x} = (x_1, \dots, x_m)$ allow to rewrite (1.2) in a more concise form

$$F_D^{(m)}\left(\alpha, \boldsymbol{\beta}; \gamma; \mathbf{x}\right) = \sum_{\mathbf{k} \geq \mathbf{0}} \frac{(\alpha)_{\langle \mathbf{k} \rangle} (\boldsymbol{\beta})_{\mathbf{k}}}{(\gamma)_{\langle \mathbf{k} \rangle} \mathbf{k}!} \mathbf{x}^{\mathbf{k}},$$

where (and hereafter) the summation condition $\mathbf{k} \geq \mathbf{h}$ means that the sum runs over all indices $\mathbf{k}$ with $k_j \geq h_j$ ($j = 1, \dots, n$). Furthermore, a new class of m -variable hypergeometric functions $\widehat{F}_D^{(m)}(\alpha, \boldsymbol{\beta}_{\gamma}^{m-1}; \mathbf{x})$ is obtained from $F_D^{(m)}(\alpha, \boldsymbol{\beta}; \mathbf{x})$ through the confluence operation

$$(1.3) \quad \begin{aligned} & F_D^{(m)}\left(\alpha, \boldsymbol{\beta}_{\gamma}^{m-1}, \beta_n; \mathbf{x}_{m-1}, \frac{x_m}{\beta_m}\right) \xrightarrow{(\beta_m \rightarrow +\infty)} \widehat{F}_D^{(m)}\left(\alpha, \boldsymbol{\beta}_{\gamma}^{m-1}; \mathbf{x}\right) \\ &= \sum_{\mathbf{k} \geq \mathbf{0}} \frac{(\alpha)_{\langle \mathbf{k} \rangle} (\boldsymbol{\beta}_{\gamma}^{m-1})_{\mathbf{k}_{m-1}}}{(\gamma)_{\langle \mathbf{k} \rangle} \mathbf{k}!} \mathbf{x}^{\mathbf{k}}. \end{aligned}$$

Note that the case $m = 1$ of (1.3) gives Kummer's hypergeometric function

$$\widehat{F}_D^{(1)}\left(\frac{\alpha}{\gamma}; x\right) = {}_1F_1\left(\frac{\alpha}{\gamma}; x\right) = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{(\gamma)_k k!} x^k$$

for $|x| < +\infty$, while $m = 2$ the confluent form of $F_1(\frac{\alpha, \beta_1, \beta_2}{\gamma}; x_1, x_2)$, defined by

$$\widehat{F}_D^{(2)}\left(\frac{\alpha, \beta_1}{\gamma}; x_1, x_2\right) = \Phi_1\left(\frac{\alpha, \beta_1}{\gamma}; x_1, x_2\right) = \sum_{k_1, k_2=0}^{\infty} \frac{(\alpha)_{k_1+k_2} (\beta_1)_{k_1}}{(\gamma)_{k_1+k_2} k_1! k_2!} x_1^{k_1} x_2^{k_2}$$

for $|x_1| < 1$ and $|x_2| < +\infty$ (cf. [4]).

Main objects

We can now introduce the hypergeometric type generating function $\mathcal{Z}_{a,\lambda}^{(n)}(s, \beta; z_0, \mathbf{z})$ of $(n+1)$ -variables $(z_0, \mathbf{z}) = (z_0, z_1, \dots, z_n)$ associated with $\zeta(s, a + z_0, \lambda)$, defined by the n -ple power series

$$\begin{aligned} (2.1) \quad \mathcal{Z}_{a,\lambda}^{(n)}\left(s, \beta; z_0, \mathbf{z}\right) &= \sum_{k_1, \dots, k_n=0}^{\infty} \frac{(s)_{\langle \mathbf{k} \rangle} (\beta)_{\mathbf{k}}}{(\gamma)_{\langle \mathbf{k} \rangle} \mathbf{k}!} \zeta(s + \langle \mathbf{k} \rangle, a + z_0, \lambda) (-\mathbf{z})^{\mathbf{k}} \\ &= \sum_{k_1, \dots, k_n=0}^{\infty} \frac{(s)_{k_1+\dots+k_n} (\beta_1)_{k_1} \cdots (\beta_n)_{k_n}}{(\gamma)_{k_1+\dots+k_n} k_1! \cdots k_n!} \\ &\quad \times \zeta(s + k_1 + \dots + k_n, a + z_0, \lambda) (-z_1)^{k_1} \cdots (-z_n)^{k_n}, \end{aligned}$$

which converges absolutely in the domain $|z_j| < |\operatorname{Im} z_0|$ ($j = 1, \dots, n$). The change of the order of summations in (2.1) readily implies that

$$(2.2) \quad \mathcal{Z}_{a,\lambda}^{(n)}\left(s, \beta; z_0, \mathbf{z}\right) = \sum_{l=0}^{\infty} e(\lambda l) (a + l + z_0)^{-s} F_D^{(n)}\left(s, \beta; -\frac{\mathbf{z}}{a + l + z_0}\right)$$

for $\sigma > 1$; the cases $\beta = \mathbf{0}$ and $\mathbf{z} = \mathbf{0}$ of (2.2) both reduce to

$$\mathcal{Z}_{a,\lambda}^{(n)}\left(s, \mathbf{0}; \mathbf{z}, z_0\right) = \mathcal{Z}_{a,\lambda}^{(n)}\left(s, \beta; z_0, \mathbf{0}\right) = \zeta(s, a + z_0, \lambda),$$

while the cases $n = 1$ and $n = 2$ respectively to

$$\mathcal{Z}_{a,\lambda}^{(1)}\left(s, \beta; z_0, z_1\right) = \sum_{l=0}^{\infty} e(\lambda l) (a + l + z_0)^{-s} {}_2F_1\left(s, \beta; -\frac{z_1}{a + l + z_0}\right),$$

$$\begin{aligned} \mathcal{Z}_{a,\lambda}^{(2)}\left(s, \beta_1, \beta_2; z_0, z_1, z_2\right) &= \sum_{l=0}^{\infty} e(\lambda l) (a + l + z_0)^{-s} \\ &\quad \times F_1\left(s, \beta_1, \beta_2; -\frac{z_1}{a + l + z_0}, -\frac{z_2}{a + l + z_0}\right) \end{aligned}$$

for $\sigma > 1$. It is further possible to obtain a new class of generating functions, denoted by $\widehat{\mathcal{Z}}_{a,\lambda}^{(n)}(s, \beta_{n-1}; z_0, \mathbf{z})$, from $\mathcal{Z}_{a,\lambda}^{(n)}(s, \beta; z_0, \mathbf{z})$ through the confluence operation

$$\begin{aligned} (2.3) \quad \mathcal{Z}_{a,\lambda}^{(n)}\left(s, \beta_{n-1}, \beta_n; z_0, z_{n-1}, \frac{z_n}{\beta_n}\right) &\xrightarrow{(\beta_n \rightarrow +\infty)} \widehat{\mathcal{Z}}_{a,\lambda}^{(n)}\left(s, \beta_{n-1}; z_0, \mathbf{z}\right) \\ &= \sum_{l=0}^{\infty} e(\lambda l) (a + l + z_0)^{-s} \widehat{F}_D^{(n)}\left(s, \beta_{n-1}; -\frac{\mathbf{z}}{a + l + z_0}\right) \end{aligned}$$

for $\sigma > 1$, where the change of the order of summations in the last expression gives

$$(2.4) \quad \widehat{\mathcal{Z}}_{a,\lambda}^{(n)}\left(s, \beta_{\gamma}^{n-1}; z_0, \mathbf{z}\right) = \sum_{\mathbf{k} \geq 0} \frac{(s)_{\langle \mathbf{k} \rangle} (\beta_{\gamma}^{n-1})_{\mathbf{k}_{n-1}}}{(\gamma)_{\langle \mathbf{k} \rangle} \mathbf{k}!} \zeta(s + \langle \mathbf{k} \rangle, a + z_0, \lambda) (-\mathbf{z})^{\mathbf{k}}$$

in the domain $|z_j| < |\operatorname{Im} z|$ ($j = 1, \dots, n$).

We shall give in the remaining of this section a brief overview of the history of research related to various generating functions associated with specific values of zeta-functions[‡]. Several power series involving the particular values of $\zeta(s, a)$ were first studied by Srivastava [18][19][20], while Klusch [11] treated the Taylor series for $\zeta(s, a + z, \lambda)$ in the variable $z \in \mathbb{C}$, and gave its many interesting applications. Hypergeometric type generating functions of $\zeta(s)$ were first introduced and studied by Raina-Srivastava [17] and the author [6][7], independently of each other; we refer the reader to the comprehensive account [21] into this direction. Hikami-Kirillov [5] more recently investigated hypergeometric generating functions of various L -function values in connection with q -hypergeometric series and quantum invariants. Hypergeometric type generating functions associated with $\zeta_{\nu}(s, a, w)$ (a weighted extension of $\zeta(s, a, \lambda)$) were first introduced and studied by Bin-Saad and Al-Gonah [3] and further by Bin-Saad [2]. Li-Kanemitsu-Tsukada [14] made Maijer's G -function theoretic interpretation of the results in [6][7], while similar G -function theoretic study on the results in [8] was made by Kuzumaki [12]. We next mention several relevant asymptotic aspects into this direction. Complete asymptotic expansions of $\zeta(s, a + z, \lambda)$ for small and large $z \in \mathbb{C}$ in the sector $|\arg z| < \pi$ was established by the author [8]. Matsumoto [15] investigated complete asymptotic expansions of $\zeta_2(s, a | (1, w))$ (Barnes' double zeta-function) for small and large basis parameter $w \in \mathbb{C}$ in $|\arg w| < \pi$. Onodera [16] more recently studied complete asymptotic expansions of $\zeta_m(s, a + x | \omega)$ (Barnes' multiple zeta-function) for small and large $x \in \mathbb{R}_+$ and one of ω_i 's in the basis parameters $\omega = (\omega_1, \dots, \omega_m) \in \mathbb{R}_+^m$, where $\mathbb{R}_+$ denotes the set of positive real numbers.

Asymptotic expansions for small and large $(z_0, \mathbf{z})$

To describe our results we introduce the generalized Bernoulli polynomials $B_k(x, y)$ ($k \in \mathbb{N}_0$) for any parameters $x, y \in \mathbb{C}$ by the power series

$$\frac{ze^{xz}}{ye^z - 1} = \sum_{k=0}^{\infty} \frac{B_k(x, y)}{k!} z^k,$$

centered at $z = 0$; this in particular gives

$$B_0(x, y) = \begin{cases} 1 & \text{if } y = 1; \\ 0 & \text{if } y \neq 1. \end{cases}$$

Note that $B_k(x) = B_k(x, 1)$ are the usual Bernoulli polynomials, and so $B_k = B_k(0)$ are the usual Bernoulli numbers. The vertical straight path from $u - i\infty$ to $u + i\infty$ (with $u \in \mathbb{R}$) is hereafter denoted by (u) .

We first state the asymptotic expansion of $\mathcal{Z}_{a,\lambda}^{(n)}(s, \beta_{\gamma}; z_0, \mathbf{z})$ when $(z_0, \mathbf{z})$ becomes small.

Theorem 1. *Let θ_0 be any angle fixed with $\theta_0 \in [-\pi/2, \pi/2]$. Then for any integer $K \geq 0$, in the region $\sigma > 1 - K$ except at $s = 1$ the formula*

$$(3.1) \quad \mathcal{Z}_{a,\lambda}^{(n)}\left(s, \beta_{\gamma}; z_0, \mathbf{z}\right) = S_{a,\lambda,K}^+\left(s, \beta_{\gamma}; z_0, \mathbf{z}\right) + R_{a,\lambda,K}^+\left(s, \beta_{\gamma}; z_0, \mathbf{z}\right)$$

[‡]The author would like to make apology for insufficiency (in many respects) of the present survey.

holds for all $(z_0, \mathbf{z})$ in the poly-sector $|\arg z_j - \theta_0| < \pi/2$ ($j = 0, 1, \dots, n$) and for all $(\beta, \gamma) \in \mathbb{C}^n \times \{\mathbb{C} \setminus (-\mathbb{N}_0)\}$. Here

$$(3.2) \quad S_{a,\lambda,K}^+ \left(\begin{smallmatrix} s, \beta \\ \gamma \end{smallmatrix}; z_0, \mathbf{z} \right) = \sum_{k=0}^{K-1} \frac{(-1)^k (s)_k}{k!} F_D^{(n)} \left(\begin{smallmatrix} -k, \beta \\ \gamma \end{smallmatrix}; -\frac{\mathbf{z}}{z_0} \right) \zeta(s+k, a, \lambda) z_0^k,$$

and $R_{a,\lambda,K}^+$ is the remainder term expressed as

$$(3.3) \quad R_{a,\lambda,K}^+ \left(\begin{smallmatrix} s, \beta \\ \gamma \end{smallmatrix}; z_0, \mathbf{z} \right) = \frac{1}{2\pi\sqrt{-1}} \int_{(u_K^+)} \Gamma \left(\begin{smallmatrix} s+w \\ s \end{smallmatrix}, -w \right) F_D^{(n)} \left(\begin{smallmatrix} -w, \beta \\ \gamma \end{smallmatrix}; -\frac{\mathbf{z}}{z_0} \right) \\ \times \zeta(s+w, a, \lambda) z_0^w dw,$$

where u_K^+ is a constant satisfying $\max(1-\sigma, K-1) < u_K^+ < K$. Formula (3.1) further provides the analytic continuation of $\mathcal{Z}_{a,\lambda}^{(n)} \left(\begin{smallmatrix} s, \beta \\ \gamma \end{smallmatrix}; z_0, \mathbf{z} \right)$ over the whole s -plane except at $s = 1$, to the poly-sector $|\arg z_j - \theta_0| < \pi/2$ ($j = 0, 1, \dots, n$), and for all $(\beta, \gamma) \in \mathbb{C}^n \times \{\mathbb{C} \setminus (-\mathbb{N}_0)\}$. Moreover if $(z_0, \mathbf{z})$ is in $|\arg z_j - \theta_0| \leq \pi/2 - \delta$ with any small $\delta > 0$ ($j = 0, 1, \dots, n$), and satisfies

$$|z_j| \leq c|z_0| \quad (j = 1, \dots, n)$$

for some constant $c > 0$, then the estimates

$$F_D^{(n)} \left(\begin{smallmatrix} -k, \beta \\ \gamma \end{smallmatrix}; -\frac{\mathbf{z}}{z_0} \right) = O(1) \quad \text{and} \quad R_{a,\lambda,K}^+ \left(\begin{smallmatrix} s, \beta \\ \gamma \end{smallmatrix}; z_0, \mathbf{z} \right) = O(|z_0|^K)$$

follow for all $K > k \geq 0$ as $z_0 \rightarrow 0$ through $|\arg z_0 - \theta_0| \leq \pi/2 - \delta$, in the same region of (s, β, γ) above, where the constants implied in the O -symbols may depend on $a, K, c, s, \beta, \gamma$ and δ ; this shows that (3.1) with (3.2) and (3.3) gives a complete asymptotic expansion in the ascending order of z_0 as $z_0 \rightarrow 0$ through the sector $|\arg z_0 - \theta_0| < \pi/2$.

It can be seen that $\lim_{K \rightarrow +\infty} R_{a,\lambda,K}^+ \left(\begin{smallmatrix} s, \beta \\ \gamma \end{smallmatrix}; z_0, \mathbf{z} \right) = 0$ for $|z_j| < a$ ($j = 0, 1, \dots, n$); this yields the following corollary.

Corollary 1. Let (s, β, γ) be as in Theorem 1. Then the infinite series

$$\mathcal{Z}_{a,\lambda}^{(n)} \left(\begin{smallmatrix} s, \beta \\ \gamma \end{smallmatrix}; z_0, \mathbf{z} \right) = \sum_{k=0}^{\infty} \frac{(-1)^k (s)_k}{k!} F_D^{(n)} \left(\begin{smallmatrix} -k, \beta \\ \gamma \end{smallmatrix}; -\frac{\mathbf{z}}{z_0} \right) \zeta(s+k, a, \lambda) z_0^k$$

holds for all $(z_0, \mathbf{z})$ in the poly-disk $|z_j| < a$ ($j = 0, 1, \dots, n$).

Corollary 2. Function $\mathcal{Z}_{a,\lambda}^{(n)} \left(\begin{smallmatrix} s, \beta \\ \gamma \end{smallmatrix}; z_0, \mathbf{z} \right)$ is continued to a one-valued meromorphic function of s over the whole s -plane, to the whole poly-sector $|\arg z_j| < \pi$ ($j = 0, 1, \dots, n$), and for all $(\beta, \gamma) \in \mathbb{C}^n \times \{\mathbb{C} \setminus (-\mathbb{N}_0)\}$; its only singularity, as a function of s , is a (possible) simple pole at $s = 1$ with the residue $B_0(a, e(\lambda))$.

We next state the asymptotic expansion of $\mathcal{Z}_{a,\lambda}^{(n)} \left(\begin{smallmatrix} s, \beta \\ \gamma \end{smallmatrix}; z_0, \mathbf{z} \right)$ when $(z_0, \mathbf{z})$ becomes large.

Theorem 2. Let θ_0 be any angle fixed with $\theta_0 \in [-\pi/2, \pi/2]$. Then for any integer $K \geq 0$, in the region $\sigma > -K$ except the point at $s = 1$ the formula

$$(3.4) \quad \mathcal{Z}_{a,\lambda}^{(n)} \left(\begin{smallmatrix} s, \beta \\ \gamma \end{smallmatrix}; z_0, \mathbf{z} \right) = S_{a,\lambda,K}^- \left(\begin{smallmatrix} s, \beta \\ \gamma \end{smallmatrix}; z_0, \mathbf{z} \right) + R_{a,\lambda,K}^- \left(\begin{smallmatrix} s, \beta \\ \gamma \end{smallmatrix}; z_0, \mathbf{z} \right)$$

holds for all $(z_0, \mathbf{z})$ in the poly-sector $|\arg z_j - \theta_0| < \pi/2$ ($j = 0, 1, \dots, n$) and for all $(\beta, \gamma) \in \mathbb{C}^n \times \{\mathbb{C} \setminus (-\mathbb{N}_0)\}$. Here

$$(3.5) \quad S_{a,\lambda,K}^{-}\left(s, \beta; \gamma; z_0, \mathbf{z}\right) = \sum_{k=-1}^{K-1} \frac{(-1)^{k+1}(s)_k}{(k+1)!} F_D^{(n)}\left(s+k, \beta; \gamma; -\frac{\mathbf{z}}{z_0}\right) B_{k+1}(a, e(\lambda)) z_0^{-s-k},$$

and $R_{a,\lambda,K}^{-}$ is the remainder term expressed as

$$(3.6) \quad R_{a,\lambda,K}^{-}\left(s, \beta; \gamma; z_0, \mathbf{z}\right) = \frac{1}{2\pi\sqrt{-1}} \int_{(u_K^-)} \Gamma\left(s+w, -w\right) F_D^{(n)}\left(-w, \beta; \gamma; -\frac{\mathbf{z}}{z_0}\right) \times \zeta(s+w, a, \lambda) z_0^w dw,$$

where u_K^- is a constant satisfying $-\sigma - K < u_K^- < \min(-\sigma - K + 1, 0)$. Formula (3.4) further provides the analytic continuation of $\mathcal{Z}_{a,\lambda}^{(n)}(s, \beta; \gamma; z_0, \mathbf{z})$ over the whole s -plane except at $s = 1$, to the poly-sector $|\arg z_j - \theta_0| < \pi/2$ ($j = 0, 1, \dots, n$), and for all $(\beta, \gamma) \in \mathbb{C}^n \times \{\mathbb{C} \setminus (-\mathbb{N}_0)\}$. Moreover if $(z_0, \mathbf{z})$ is in $|\arg z_j - \theta_0| \leq \pi/2 - \delta$ with any small $\delta > 0$ ($j = 0, 1, \dots, n$), and satisfies

$$|z_j| \leq c|z_0| \quad (j = 1, \dots, n)$$

for some constant $c > 0$, then the estimates

$$F_D^{(n)}\left(s+k, \beta; \gamma; -\frac{\mathbf{z}}{z_0}\right) = O(1) \quad \text{and} \quad R_{a,\lambda,K}^{-}\left(s, \beta; \gamma; z_0, \mathbf{z}\right) = O(|z_0|^{-\sigma-K})$$

follow for all $K > k \geq 0$ as $z_0 \rightarrow \infty$ through $|\arg z_0 - \theta_0| \leq \pi/2 - \delta$, in the same region of (s, β, γ) as above, where the constants implied in the O -symbols may depend on $a, K, c, s, \beta, \gamma$ and δ ; this shows that (3.4) with (3.5) and (3.6) gives a complete asymptotic expansion in the descending order of z_0 as $z_0 \rightarrow \infty$ through the sector $|\arg z_0 - \theta_0| < \pi/2$.

The cases $s = -l$ ($l \in \mathbb{N}_0$) of Theorem 2 reduce to the evaluations in finite closed form of $\mathcal{Z}_{a,\lambda}^{(n)}(s, \beta; \gamma; z_0, \mathbf{z})$.

Corollary 3. Let (β, γ) be as in Theorem 2, and $(z_0, \mathbf{z})$ in the poly-sector $|\arg z_j - \theta_0| < \pi$ ($j = 0, 1, \dots, n$) with any angle fixed with $\theta_0 \in [-\pi/2, \pi/2]$. Then for any $l \in \mathbb{N}_0$ we have

$$\mathcal{Z}_{a,\lambda}^{(n)}\left(-l, \beta; \gamma; z_0, \mathbf{z}\right) = -\frac{1}{l+1} \sum_{k=-1}^l \binom{l+1}{k+1} F_D^{(n)}\left(k-l, \beta; \gamma; -\frac{\mathbf{z}}{z_0}\right) B_{k+1}(a, e(\lambda)) z_0^{l-k}.$$

The asymptotic expansions of $\widehat{\mathcal{Z}}_{a,\lambda}^{(n)}(s, \beta_{n-1}; \gamma; z_0, \mathbf{z})$ can be derived from our main formulae (3.1) and (3.4) through the confluence operation in (2.3); this asserts the following Theorems 3 and 4.

Theorem 3. Let θ_0 be any angle fixed with $\theta_0 \in [-\pi/2, \pi/2]$. Then for any integer $K \geq 0$, in the region $\sigma > 1 - K$ except at $s = 1$ the formula

$$(3.7) \quad \widehat{\mathcal{Z}}_{a,\lambda}^{(n)}\left(s, \beta_{n-1}; \gamma; z_0, \mathbf{z}\right) = \widehat{S}_{a,\lambda,K}^{+}\left(s, \beta_{n-1}; \gamma; z_0, \mathbf{z}\right) + \widehat{R}_{a,\lambda,K}^{+}\left(s, \beta_{n-1}; \gamma; z_0, \mathbf{z}\right)$$

holds for all $(z_0, \mathbf{z})$ in the poly-sector $|\arg z_j - \theta_0| < \pi/2$ ($j = 0, 1, \dots, n$) and for all $(\beta_{n-1}, \gamma) \in \mathbb{C}^{n-1} \times \{\mathbb{C} \setminus (-\mathbb{N}_0)\}$. Here

$$(3.8) \quad \widehat{S}_{a,\lambda,K}^{+}\left(s, \beta_{n-1}; \gamma; z_0, \mathbf{z}\right) = \sum_{k=0}^{K-1} \frac{(-1)^k (s)_k}{k!} \widehat{F}_D^{(n)}\left(-k, \beta_{n-1}; \gamma; -\frac{\mathbf{z}}{z_0}\right) \zeta(s+k, a, \lambda) z_0^k,$$

and $\widehat{R}_{a,\lambda,K}^+$ is the remainder term expressed as

$$(3.9) \quad \widehat{R}_{a,\lambda,K}^+(s, \beta_{n-1}; z_0, \mathbf{z}) = \frac{1}{2\pi\sqrt{-1}} \int_{(u_K^+)} \Gamma\left(s + \frac{w}{s}, -w\right) F_D^{(n)}\left(-w, \beta_{n-1}; -\frac{\mathbf{z}}{z_0}\right) \times \zeta(s + w, a, \lambda) z_0^w dw,$$

where u_K^+ is a constant satisfying $\max(1 - \sigma, K - 1) < u_K^+ < K$. Formula (3.7) further provides the analytic continuation of $\widehat{Z}_{a,\lambda}^{(n)}(s, \beta_{n-1}; z_0, \mathbf{z})$ over the whole s -plane except at $s = 1$, to the poly-sector $|\arg z_j - \theta_0| < \pi/2$ ($j = 0, 1, \dots, n$), and for all $(\beta_{n-1}, \gamma) \in \mathbb{C}^{n-1} \times \{\mathbb{C} \setminus (-\mathbb{N}_0)\}$. Moreover if $(z_0, \mathbf{z})$ is in $|\arg z_j - \theta_0| \leq \pi/2 - \delta$ with any small $\delta > 0$ ($j = 0, 1, \dots, n$), and satisfies

$$|z_j| \leq c|z_0| \quad (j = 1, \dots, n)$$

for some constant $c > 0$, then the estimates

$$\widehat{F}_D^{(n)}\left(-k, \beta_{n-1}; -\frac{\mathbf{z}}{z_0}\right) = O(1) \quad \text{and} \quad \widehat{R}_{a,\lambda,K}^+(s, \beta_{n-1}; z_0, \mathbf{z}) = O(|z_0|^K)$$

follow for all $K > k \geq 0$ as $z_0 \rightarrow 0$ through $|\arg z_0 - \theta_0| \leq \pi/2 - \delta$, in the same region of (s, β_{n-1}, γ) above, where the constants implied in the O -symbols may depend on $a, K, c, s, \beta_{n-1}, \gamma$ and δ ; this shows that (3.7) with (3.8) and (3.9) gives a complete asymptotic expansion in the ascending order of z_0 as $z_0 \rightarrow 0$ through the sector $|\arg z_0 - \theta_0| < \pi/2$.

Corollary 4. Let (s, β_{n-1}, γ) be as in Theorem 3. Then the infinite series

$$\widehat{Z}_{a,\lambda}^{(n)}\left(s, \beta_{n-1}; z_0, \mathbf{z}\right) = \sum_{k=0}^{\infty} \frac{(-1)^k (s)_k}{k!} \widehat{F}_D^{(n)}\left(-k, \beta_{n-1}; -\frac{\mathbf{z}}{z_0}\right) \zeta(s + k, a, \lambda) z_0^k$$

holds for all $(z_0, \mathbf{z})$ in the poly-disk $|z_j| < a$ ($j = 0, 1, \dots, n$).

Theorem 4. Let θ_0 be any angle fixed with $\theta_0 \in [-\pi/2, \pi/2]$. Then for any integer $K \geq 0$, in the region $\sigma > -K$ except at $s = 1$ the formula

$$(3.10) \quad \widehat{Z}_{a,\lambda}^{(n)}\left(s, \beta_{n-1}; z_0, \mathbf{z}\right) = \widehat{S}_{a,\lambda,K}^-(s, \beta_{n-1}; z_0, \mathbf{z}) + \widehat{R}_{a,\lambda,K}^-(s, \beta_{n-1}; z_0, \mathbf{z})$$

holds for all $(z_0, \mathbf{z})$ in the poly-sector $|\arg z_j - \theta_0| < \pi/2$ ($j = 0, 1, \dots, n$) and for all $(\beta_{n-1}, \gamma) \in \mathbb{C}^{n-1} \times \{\mathbb{C} \setminus (-\mathbb{N}_0)\}$. Here

$$(3.11) \quad \widehat{S}_{a,\lambda,K}^-(s, \beta_{n-1}; z_0, \mathbf{z}) = \sum_{k=-1}^{K-1} \frac{(-1)^k (s)_k}{k!} \widehat{F}_D^{(n)}\left(s + k, \beta_{n-1}; -\frac{\mathbf{z}}{z_0}\right) \times B_{k+1}(a, e(\lambda)) z_0^{-s-k},$$

and $\widehat{R}_{a,\lambda,K}^-$ is the remainder term expressed as

$$(3.12) \quad \widehat{R}_{a,\lambda,K}^-(s, \beta_{n-1}; z_0, \mathbf{z}) = \frac{1}{2\pi\sqrt{-1}} \int_{(u_K^-)} \Gamma\left(s + \frac{w}{s}, -w\right) \widehat{F}_D^{(n)}\left(-w, \beta_{n-1}; -\frac{\mathbf{z}}{z_0}\right) \times \zeta(s + w, a, \lambda) z_0^w dw,$$

where u_K^- is a constant satisfying $-\sigma - K < u_K^- < \min(-\sigma - K + 1, 0)$. Formula (3.10) further provides the analytic continuation of $\widehat{Z}_{a,\lambda}^{(n)}(s, \beta_{n-1}; z_0, \mathbf{z})$ over the whole s -plane

except at $s = 1$, to the poly-sector $|\arg z_j - \theta_0| < \pi/2$ ($j = 0, 1, \dots, n$), and for all $(\beta_{n-1}, \gamma) \in \mathbb{C}^{n-1} \times \{\mathbb{C} \setminus (-\mathbb{N}_0)\}$. Moreover if $(z_0, \mathbf{z})$ is in $|\arg z_j - \theta_0| \leq \pi/2 - \delta$ with any small $\delta > 0$ ($j = 0, 1, \dots, n$), and satisfies

$$|z_j| \leq c|z_0| \quad (j = 1, \dots, n)$$

for some constant $c > 0$, then the estimates

$$\widehat{F}_D^{(n)}\left(s + k, \beta_{n-1}; -\frac{\mathbf{z}}{z_0}\right) = O(1) \quad \text{and} \quad \widehat{R}_{a,\lambda,K}^-\left(s, \beta_{n-1}; z_0, \mathbf{z}\right) = O(|z_0|^{-\sigma-K})$$

follow for all $K > k \geq 0$ as $z_0 \rightarrow \infty$ through $|\arg z_0 - \theta_0| \leq \pi/2 - \delta$ in the same region of (s, β_{n-1}, γ) above, where the constants implied in the O -symbols may depend on $a, K, c, s, \beta_{n-1}, \gamma$ and δ ; this shows that (3.10) with (3.11) and (3.12) gives a complete asymptotic expansion in the descending order of z_0 as $z_0 \rightarrow \infty$ through the sector $|\arg z_0 - \theta_0| < \pi/2$.

Corollary 5. Let (β, γ) be as in Theorem 4, and $(z_0, \mathbf{z})$ in the poly-sector $|\arg z_j - \theta_0| < \pi$ ($j = 0, 1, \dots, n$) with any angle fixed with $\theta_0 \in [-\pi/2, \pi/2]$. Then for any $l \in \mathbb{N}_0$ we have

$$\widehat{\mathcal{Z}}_{a,\lambda}^{(n)}\left(-l, \beta_{n-1}; z_0, \mathbf{z}\right) = -\frac{1}{l+1} \sum_{k=-1}^l \binom{l+1}{k+1} \widehat{F}_D^{(n)}\left(k-l, \beta_{n-1}; -\frac{\mathbf{z}}{z_0}\right) B_{k+1}(a, e(\lambda)) z_0^{l-k}.$$

Asymptotics for derivatives

We define the generalized Euler-Stieltjes constants $\gamma_m(a, e(\lambda))$ ($m \in \mathbb{N}_0$) and the modified Stirling polynomials $\sigma_{m,n}(x)$ ($m, n \in \mathbb{N}_0$) respectively by the power series

$$\zeta(s, a, \lambda) = \frac{B_0(a, e(\lambda))}{s-1} + \sum_{m=0}^{\infty} \gamma_m(a, e(\lambda))(s-1)^m$$

centered at $s = 1$, and

$$\frac{1}{m!} (1-z)^{-x} \{-\log(1-z)\}^m = \sum_{n=0}^{\infty} \frac{\sigma_{m,n}(x)}{n!} z^n$$

centered at $z = 0$. Note that $\sigma_{m,n}(x) = 0$ for $0 \leq n < m$. We further set

$$C_{k,l,m}(a, e(\lambda)) = \sum_{j=0}^m \frac{m!}{(m-j)!} \sigma_{j,k}(l) \left(\frac{\partial}{\partial s}\right)^{m-j} \zeta(s, a, \lambda) \Big|_{s=l+k}$$

for any $k, l, m \in \mathbb{N}_0$. Then Theorem 1 yields:

Corollary 6. Let $(\beta, \gamma, \mathbf{z})$ be as in Theorem 1. For any integer $K \geq 1$ the following asymptotic expansions hold as $z_0 \rightarrow 0$ through $|\arg z_0 - \theta_0| \leq \pi/2 - \delta$ with any $\delta > 0$, while other z_j 's move through the same sector satisfying the conditions $|z_j| \leq c|z_0|$ ($j = 1, \dots, n$) with some constant $c > 0$:

i) when $s \rightarrow 1$,

$$\begin{aligned} & \lim_{s \rightarrow 1} \left(\frac{\partial}{\partial s}\right)^m \left\{ \mathcal{Z}_{a,\lambda}^{(n)}\left(s, \beta; z_0, \mathbf{z}\right) - \frac{B_0(a, e(\lambda))}{s-1} \right\} \\ &= m! \gamma_m(a, e(\lambda)) + \sum_{k=1}^{K-1} \frac{(-1)^k}{k!} C_{k,1,m}(a, e(\lambda)) F_D^{(n)}\left(-k, \beta; -\frac{\mathbf{z}}{z_0}\right) z_0^k + O(|z_0|^K); \end{aligned}$$

ii) when $s = l$ ($l = 2, 3, \dots$),

$$\left(\frac{\partial}{\partial s}\right)^m \mathcal{Z}_{a,\lambda}^{(n)}\left(s, \beta; z_0, z\right)\Big|_{s=l} = \sum_{k=0}^{K-1} \frac{(-1)^k}{k!} C_{k,l,m}(a, e(\lambda)) F_D^{(n)}\left(-k, \beta; -\frac{z}{z_0}\right) z_0^k + O(|z_0|^K).$$

It is known that $\lim_{s \rightarrow 1} \{\zeta(s, z) - 1/(s-1)\} = \gamma_0(z) = -\psi(z) = -(\Gamma'/\Gamma)(z)$. The case $(\lambda, \beta) = (0, 0)$ above reduces to the classical Taylor series expansion of $\psi(a+z)$ (cf. [4]).

Corollary 7. For $|z| < a$ we have

$$\psi(a+z) = \psi(a) - \sum_{k=1}^{\infty} \left\{ \left(\sum_{h=1}^k \frac{1}{h} \right) \zeta(1+k, a) + \zeta'(1+k, a) \right\} z^k.$$

We next define the polynomials $\mathcal{P}_{l,m}, \mathcal{Q}_{k,l,m} \in \mathbb{C}[[x]][y]$ ($k, l, m \in \mathbb{N}_0$) by

$$\begin{aligned} \mathcal{P}_{l,m}(\beta; x, y) &= \sum_{j=0}^m \frac{m!}{(m-j)!} \left\{ \sum_{i=0}^j \frac{(l+1)^{i-j-1}}{(j-i)!} \right. \\ &\quad \times \left. \left(\frac{\partial}{\partial \alpha} \right)^{j-i} F_D^{(n)}(\alpha, \beta; -x) \Big|_{\alpha=-l-1} \right\} (-y)^{m-j}, \\ \mathcal{Q}_{k,l,m}(\beta; x, y) &= \sum_{j=0}^m \frac{m!}{(m-j)!} \left\{ \sum_{i=0}^j \frac{\sigma_{i,j}(-l)}{(j-i)!} \right. \\ &\quad \times \left. \left(\frac{\partial}{\partial \alpha} \right)^{j-i} F_D^{(n)}(\alpha, \beta; -x) \Big|_{\alpha=k-l} \right\} (-y)^{m-j}. \end{aligned}$$

Corollary 8. Let (β, γ, z) be as in Theorem 2, and $l, m \in \mathbb{N}_0$ arbitrary. Then for any integer $K \geq l+1$ the asymptotic expansion

$$\begin{aligned} \left(\frac{\partial}{\partial s}\right)^m \mathcal{Z}_{a,\lambda}^{(n)}\left(s, \beta; z_0, z\right)\Big|_{s=-l} &= -B_0(a, e(\lambda)) z_0^{l+1} \mathcal{P}_{l,m}\left(\beta; \frac{z}{z_0}, \log z_0\right) \\ &\quad + \sum_{k=0}^{K-1} \frac{(-1)^{k+1}}{(k+1)!} B_{k+1}(a, e(\lambda)) z_0^{l-k} \mathcal{Q}_{k,l,m}\left(\beta; \frac{z}{z_0}, \log z_0\right) \\ &\quad + O(|z_0|^{l-K} \log^m |z_0|) \end{aligned}$$

holds as $z_0 \rightarrow \infty$ through $|\arg z_0 - \theta_0| \leq \pi/2 - \delta$ with any $\delta > 0$, while other z_j 's move through the same sector satisfying the conditions $|z_j| \leq c|z_0|$ ($j = 1, \dots, n$) with some constant $c > 0$.

It is known that $(\partial/\partial s)\zeta(s, z)|_{s=0} = \log\{\Gamma(z)/\sqrt{2\pi}\}$ (cf. [4]). The case $(n, \beta) = (2, 0)$ and $\lambda \in \mathbb{Z}$ above reduces to the following variant of Stirling's formula (cf. [4]).

Corollary 9. For any integer $K \geq 0$ the asymptotic expansion

$$\begin{aligned} \log \Gamma(a+z) &= \left(z + a - \frac{1}{2}\right) \log z + \frac{1}{2} \log(2\pi) + \sum_{k=1}^{K-1} \frac{(-1)^{k+1} B_{k+1}(a)}{k(k+1)} z^{-k} \\ &\quad + O(|z|^{-K} \log |z|) \end{aligned}$$

holds as $z \rightarrow \infty$ through $|\arg z| \leq \pi - \delta$ with any small $\delta > 0$.

Applications of our main formulae with $n = 2$

One can observe that the case $(n, \gamma) = (2, s)$ of (2.2) and (2.4) reduce respectively to the expressions

$$(5.1) \quad \mathcal{Z}_{a,\lambda}^{(2)}\left(s, \beta_1, \beta_2; z_0, z_1, z_2\right) = \sum_{l=0}^{\infty} e(\lambda l) (a+l+z_0)^{-s} \left(1 + \frac{z_1}{a+l+z_0}\right)^{-\beta_1} \\ \times \left(1 + \frac{z_2}{a+l+z_0}\right)^{-\beta_2},$$

and

$$(5.2) \quad \widehat{\mathcal{Z}}_{a,\lambda}^{(2)}\left(s, \beta_1; z_0, z_1, z_2\right) = \sum_{l=0}^{\infty} e(\lambda l) (a+l+z_0)^{-s} \left(1 + \frac{z_1}{a+l+z_0}\right)^{-\beta_1} \\ \times \exp\left(-\frac{z_2}{a+l+z_0}\right).$$

Theorems 1 and 2 in particular assert on (5.1) and (5.2) the following corollaries.

Corollary 10. *Let θ_0 be as in Theorem 1. Then for any integer $K \geq 0$, in the region $\sigma > 1 - K$ except at $s = 1$ Function $\mathcal{Z}_{a,\lambda}^{(2)}(s, \beta_1, \beta_2; z_0, z_1, z_2)$ is represented as (3.1) in the poly-sector $|\arg z_j - \theta_0| < \pi/2$ ($j = 0, 1, \dots, n$) and for all $(\beta_1, \beta_2) \in \mathbb{C}^2$, where*

$$S_{a,\lambda,K}^+\left(s, \beta_1, \beta_2; z_0, z_1, z_2\right) = \sum_{k=0}^{K-1} \frac{(-1)^k (s)_k}{k!} F_1\left(-k, \beta_1, \beta_2; -\frac{z_1}{z_0}, -\frac{z_2}{z_0}\right) \zeta(s+k, a, \lambda) z_0^k,$$

and

$$R_{a,\lambda,K}^+\left(s, \beta_1, \beta_2; z_0, z_1, z_2\right) = \frac{1}{2\pi\sqrt{-1}} \int_{(u_K^+)} \Gamma\left(s+w, -w\right) F_1\left(-w, \beta_1, \beta_2; -\frac{z_1}{z_0}, -\frac{z_2}{z_0}\right) \\ \times \zeta(s+w, a, \lambda) z_0^w dw.$$

These formulae give a complete asymptotic expansion of $\mathcal{Z}_{a,\lambda}^{(n)}(s, \beta_1, \beta_2; z_0, z_1, z_2)$ as $z_0 \rightarrow 0$ through $|\arg z_0 - \theta_0| < \pi/2$ in the sense of Theorem 1.

Corollary 11. *Let θ_0 be as in Theorem 1. Then for any integer $K \geq 0$, in the region $\sigma > -K$ except at $s = 1$ Function $\mathcal{Z}_{a,\lambda}^{(2)}(s, \beta_1, \beta_2; z_0, z_1, z_2)$ is represented as (3.4) in the poly-sector $|\arg z_j - \theta_0| < \pi/2$ ($j = 0, 1, \dots, n$) and for all $(\beta_1, \beta_2) \in \mathbb{C}^2$, where*

$$S_{a,\lambda,K}^-\left(s, \beta_1, \beta_2; z_0, z_1, z_2\right) = \sum_{k=-1}^{K-1} \frac{(-1)^{k+1} (s)_k}{(k+1)!} F_1\left(s+k, \beta_1, \beta_2; -\frac{z_1}{z_0}, -\frac{z_2}{z_0}\right) \\ \times B_{k+1}(a, e(\lambda)) z_0^{-s-k},$$

and

$$R_{a,\lambda,K}^-\left(s, \beta_1, \beta_2; z_0, z_1, z_2\right) = \frac{1}{2\pi\sqrt{-1}} \int_{(u_K^-)} \Gamma\left(s+w, -w\right) F_1\left(-w, \beta_1, \beta_2; -\frac{z_1}{z_0}, -\frac{z_2}{z_0}\right) \\ \times \zeta(s+w, a, \lambda) z_0^w dw.$$

These formulae give a complete asymptotic expansion of $\mathcal{Z}_{a,\lambda}^{(n)}(s, \beta_1, \beta_2; z_0, z_1, z_2)$ as $z_0 \rightarrow \infty$ through $|\arg z_0 - \theta_0| < \pi/2$ in the sense of Theorem 2.

Corollary 12. Let θ_0 be as in Theorem 1. Then for any integer $K \geq 0$, in the region $\sigma > 1 - K$ except at $s = 1$ Function $\widehat{\mathcal{Z}}_{a,\lambda}^{(2)}(s, \beta_1; z_0, z_1, z_2)$ is represented as (3.7) in the poly-sector $|\arg z_j - \theta_0| < \pi/2$ ($j = 0, 1, \dots, n$) and for all $\beta_1 \in \mathbb{C}$. Here

$$\widehat{S}_{a,\lambda,K}^+(s, \beta_1; z_0, z_1, z_2) = \sum_{k=0}^{K-1} \frac{(-1)^k (s)_k}{k!} \Phi_1\left(-k, \beta_1; -\frac{z_1}{z_0}, -\frac{z_2}{z_0}\right) \zeta(s+k, a, \lambda) z_0^k,$$

and

$$\begin{aligned} \widehat{R}_{a,\lambda,K}^+(s, \beta_1; z_0, z_1, z_2) &= \frac{1}{2\pi\sqrt{-1}} \int_{(u_K^+)} \Gamma\left(s+w, -w\right) \Phi_1\left(-w, \beta_1; -\frac{z_1}{z_0}, -\frac{z_2}{z_0}\right) \\ &\times \zeta(s+w, a, \lambda) z_0^w dw. \end{aligned}$$

These formulae give a complete asymptotic expansion of $\widehat{\mathcal{Z}}_{a,\lambda}^{(n)}(s, \beta_1; z_0, z_1, z_2)$ as $z_0 \rightarrow 0$ through $|\arg z_0 - \theta_0| < \pi/2$ in the sense of Theorem 3.

Corollary 13. Let θ_0 be as in Theorem 1. Then for any integer $K \geq 0$, in the region $\sigma > -K$ except at $s = 1$ Function $\widehat{\mathcal{Z}}_{a,\lambda}^{(2)}(s, \beta_1; z_0, z_1, z_2)$ is represented as (3.10) in the poly-sector $|\arg z_j - \theta_0| < \pi/2$ ($j = 0, 1, \dots, n$) and for all $\beta_1 \in \mathbb{C}$. Here

$$\widehat{S}_{a,\lambda,K}^-(s, \beta_1; z_0, z_1, z_2) = \sum_{k=-1}^{K-1} \frac{(-1)^{k+1} (s)_k}{(k+1)!} \Phi_1\left(s+k, \beta_1; -\frac{z_1}{z_0}, -\frac{z_2}{z_0}\right) B_{k+1}(a, e(\lambda)) z_0^{-s-k},$$

and

$$\begin{aligned} \widehat{R}_{a,\lambda,K}^-(s, \beta_1; z_0, z_1, z_2) &= \frac{1}{2\pi\sqrt{-1}} \int_{(u_K^-)} \Gamma\left(s+w, -w\right) \Phi_1\left(-w, \beta_1; -\frac{z_1}{z_0}, -\frac{z_2}{z_0}\right) \\ &\times \zeta(s+w, a, \lambda) z_0^w dw. \end{aligned}$$

These formulae give a complete asymptotic expansion of $\widehat{\mathcal{Z}}_{a,\lambda}^{(2)}(s, \beta_1; z_0, z_1, z_2)$ as $z_0 \rightarrow \infty$ through $|\arg z_0 - \theta_0| < \pi/2$ in the sense of Theorem 4.

Further applications

We define for $x, y \in \mathbb{R}_+$ and for $\sigma > 1$ the functions

$$\begin{aligned} \mathcal{C}_{a,\lambda}(s, \beta; x, y) &= \sum_{l=0}^{\infty} e(\lambda l) (a+l+x)^{-s} \frac{\cos\left\{\beta \operatorname{Arctan}\left(\frac{y}{a+l+x}\right)\right\}}{\left\{1 + \left(\frac{y}{a+l+x}\right)^2\right\}^{\beta/2}}, \\ \mathcal{S}_{a,\lambda}(s, \beta; x, y) &= \sum_{l=0}^{\infty} e(\lambda l) (a+l+x)^{-s} \frac{\sin\left\{\beta \operatorname{Arctan}\left(\frac{y}{a+l+x}\right)\right\}}{\left\{1 + \left(\frac{y}{a+l+x}\right)^2\right\}^{\beta/2}}, \end{aligned}$$

and their confluent forms

$$\widehat{\mathcal{C}}_{a,\lambda}(s; x, y) = \sum_{l=0}^{\infty} e(\lambda l) (a+l+x)^{-s} \cos\left(\frac{y}{a+l+x}\right),$$

$$\widehat{S}_{a,\lambda}(s; x, y) = \sum_{l=0}^{\infty} e(\lambda l) (a + l + x)^{-s} \sin\left(\frac{y}{a + l + x}\right).$$

It is in fact possible to show that Theorems 1 and 2 are valid when $n = 1$ in a wider sector

$$\max(-\pi, \arg z_0 - \pi) < \arg z_1 < \min(\pi, \arg z_0 + \pi),$$

and this allows us to take $z_0 = x$ and $z_1 = e^{\pm\pi i/2}y$ with $\arg x = 0$ and $\arg y = 0$; the following Corollaries 14 and 15 are derived.

Corollary 14. *Let (s, β) be as in Theorem 1. Then for any $s \in \mathbb{C}$ except at $s = 1 - k$ ($k \in \mathbb{N}_0$), and any $x, y \in \mathbb{R}$ with $|x|, |y| < a$ the following formulae hold:*

$$\mathcal{C}_{a,\lambda}(s, \beta; x, y) = \frac{1}{2} \sum_{k=0}^{\infty} \frac{(-1)^k (s)_k}{k!} \left\{ {}_2F_1\left(-k, \beta; \frac{iy}{x}\right) + {}_2F_1\left(-k, \beta; \frac{-iy}{x}\right) \right\} \zeta(s + k, a, \lambda) x^k,$$

$$\widehat{\mathcal{C}}_{a,\lambda}(s; x, y) = \frac{1}{2} \sum_{k=0}^{\infty} \frac{(-1)^k (s)_k}{k!} \left\{ {}_1F_1\left(-k; \frac{iy}{x}\right) + {}_1F_1\left(-k; \frac{-iy}{x}\right) \right\} \zeta(s + k, a, \lambda) x^k,$$

and similarly,

$$\mathcal{S}_{a,\lambda}(s, \beta; x, y) = \frac{1}{2i} \sum_{k=0}^{\infty} \frac{(-1)^k (s)_k}{k!} \left\{ {}_2F_1\left(-k, \beta; \frac{iy}{x}\right) - {}_2F_1\left(-k, \beta; \frac{-iy}{x}\right) \right\} \zeta(s + k, a, \lambda) x^k,$$

$$\widehat{\mathcal{S}}_{a,\lambda}(s; x, y) = \frac{1}{2i} \sum_{k=0}^{\infty} \frac{(-1)^k (s)_k}{k!} \left\{ {}_1F_1\left(-k; \frac{iy}{x}\right) - {}_1F_1\left(-k; \frac{-iy}{x}\right) \right\} \zeta(s + k, a, \lambda) x^k.$$

Corollary 15. *Let (s, β) be as in Theorem 2. Then for any integer $K \geq 0$ in the region $\sigma > -K$ except at $s = 1 - k$ ($k \in \mathbb{N}_0$) the following asymptotic expansions hold as $x \rightarrow +\infty$, while y satisfies $y \ll x$:*

$$\begin{aligned} \mathcal{C}_{a,\lambda}(s, \beta; x, y) &= \frac{1}{2} \sum_{k=-1}^{K-1} \frac{(-1)^{k+1} (s)_k}{(k+1)!} \left\{ {}_2F_1\left(s+k, \beta; \frac{iy}{x}\right) + {}_2F_1\left(s+k, \beta; \frac{-iy}{x}\right) \right\} \\ &\quad \times B_{k+1}(a, e(\lambda)) x^{-s-k} + O(x^{-\sigma-K}), \end{aligned}$$

$$\begin{aligned} \widehat{\mathcal{C}}_{a,\lambda}(s; x, y) &= \frac{1}{2} \sum_{k=-1}^{K-1} \frac{(-1)^{k+1} (s)_k}{(k+1)!} \left\{ {}_1F_1\left(s+k; \frac{iy}{x}\right) + {}_1F_1\left(s+k; \frac{-iy}{x}\right) \right\} \\ &\quad \times B_{k+1}(a, e(\lambda)) x^{-s-k} + O(x^{-\sigma-K}), \end{aligned}$$

and similarly,

$$\begin{aligned} \mathcal{S}_{a,\lambda}(s, \beta; x, y) &= \frac{1}{2i} \sum_{k=-1}^{K-1} \frac{(-1)^{k+1} (s)_k}{(k+1)!} \left\{ {}_2F_1\left(s+k, \beta; \frac{iy}{x}\right) - {}_2F_1\left(s+k, \beta; \frac{-iy}{x}\right) \right\} \\ &\quad \times B_{k+1}(a, e(\lambda)) x^{-s-k} + O(x^{-\sigma-K}), \end{aligned}$$

$$\begin{aligned} \widehat{\mathcal{S}}_{a,\lambda}(s; x, y) &= \frac{1}{2i} \sum_{k=-1}^{K-1} \frac{(-1)^{k+1} (s)_k}{(k+1)!} \left\{ {}_1F_1\left(s+k; \frac{iy}{x}\right) - {}_1F_1\left(s+k; \frac{-iy}{x}\right) \right\} \\ &\quad \times B_{k+1}(a, e(\lambda)) x^{-s-k} + O(x^{-\sigma-K}). \end{aligned}$$

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