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Isolated singularities and positive solutions of elliptic equations in R^n

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I would like to talk about isolated singularies of positive solutions of a second order linear elliptic equation

$$Pu \equiv -\sum_{j,k=1}^{n} \partial_j(a_{jk}\partial_k u) + \sum_{j=1}^{n} b_j\partial_j u + cu = 0 \qquad (*)$$

in a domain of R^n. Here $\partial_j = \partial/\partial x_j$. The purpose of this talk is two-fold: (i) To give a relationship between positive solutions in a punctured ball and those in R^n; (ii) to apply it to the study of isolated positive singularites of solutions.

1. Consider the equation (*) in a domain D of R^n, $n \geq 2$. We assume that the coefficients are real-valued and satisfy the condition

$a_{jk} \in L_{\infty,loc}(D)$, $b_j \in L_{2p,loc}(D)$ and $c \in L_{p,loc}(D)$ for some $p > n/2$, and $[a_{jk}(x)]_{j,k}$ is locally uniformly positive definite in D.

A positive solution of (*) means a positive continuous function in the Sobolev space $H^1_{loc}(D)$ of order 1 satisfying (*) in the weak sense. Let $\{D_j\}_{j=1}^{\infty}$ be a sequence of regular bounded domains which exhausts D. Let P_j be the Dirichlet realization of P on $L_2(D_j)$, and $\sigma(P_j)$ the spectrum of P_j. Put

$$\Gamma(P,D) = \inf_{j=1,2,..} \inf \operatorname{Re} \sigma(P_j) .$$

Theorem (Allegretto-Piepenbrink-Agmon). There exists a positive solution of (*) in D if and only if $\Gamma(P,D) \geqq 0$.

Definition. We say that (P,D) is subcritical if there exists a positive Green's function in D, and that (P,D) is critical if $\Gamma(P,D) \geqq 0$ and there exists no positive Green's function in D.

Theorem. (i) (P,D) is subcritical if and only if there exists a function q in $L_{p,loc}(D)$ such that $q \geqq 0$, $q \neq 0$, and $\Gamma(P-q,D) \geqq 0$. (ii) (P,D) is critical if and only if $\Gamma(P,D) \geqq 0$ but there exists no function q as above.

We denote by $H_+(P,D)$ the metric space of all positive solutions of (*) in D, where the metric is the usual one generated by the maximum norms on D_j , j=1,2,...

<u>Theorem</u>. Suppose that (P,D) is critical. Then $H_+(P,D) = \{Cu;\ C > 0\}$, where u is a particular solution satisfying the integral equation

$$u(x) = \int G(x,y)q(y)u(y)dy$$

with q being a nonnegative continuous function with compact support which is not identically zero and G being the Green's function for $P+q$ in D.

2. Now consider the equation (*) in a punctured ball $B^* = \{0 < |x| < R\}$, where the coefficients satisfy the condition in Section 1 with D replaced by $\{0 < |x| < R+1\}$. Suppose that (P,B^*) is subcritical. Choose a positive continuous function g on $\{0 < |x| \leqq R\}$ satisfying

$$-\ \Sigma_{j,k}\ \partial_j(a_{jk}\partial_k g) + \Sigma_j\ b_j\partial_j g = 0 \quad \text{in} \quad B^*.$$

Let $H_+(P,B^*,\{0\}) = \{u \in H_+(P,B^*);\ u\big|_{|x|=R} = 0\}$. For u in $H_+(P,B^*,\{0\})$, define a generalized Kelvin transform κu by

$$(\kappa u)(y) = (u/g)(y/|y|^2).$$

Then κu is a solution of $P^1(\kappa u) = 0$ in $B^{-1} \equiv \{y;\ |y| > R^{-1}\}$, where P^1 is an elliptic differential operator determined by P and g. We can show that

P^1 admits an extension $P^{\sim}$ to R^n such that $(P^{\sim},R^n)$ is subcritical.

<u>Theorem</u>. There exists an isomorphism from $H_+(P, B^*, \{0\})$ onto $H_+(P^\sim, R^n)$ such that for any u in $H_+(P, B^*, \{0\})$

$$(u/g)(y/|y|^2) \leqq Tu(y) \leqq C(u/g)(y/|y|^2) \quad \text{in} \quad \{|y| > 2R^{-1}\},$$

where C is a positive constant independent of u. Furthermore, u is a minimal solution if and only if Tu is so.

<u>Hint of proof</u>. For v in $H_+(P^\sim, R^n)$, define Πv by

$$\Pi v = v - Bv,$$

where Bv is a solution of the boundary value problem: $P^\sim(Bv) = 0$ in B^{-1}, $Bv = v$ on $\{|x| = R^{-1}\}$, and Bv is of minimal growth at infinity. Then Π is an isomorphism from $H_+(P^\sim, R^n)$ onto $H_+(P^\sim, B^{-1}, \{\infty\}) \equiv \{u \in H_+(P^\sim, B^{-1});\ u|_{\partial B^{-1}} = 0\}$. We put $T = \Pi^{-1}\kappa$.

Q.E.D.

We can also obtain analogous results starting with a subcritical operator in R^n.

3. I will give a few results concerning isolated positive singularities which I obtained by using the above theorem, although we could also show them directly.

Let $P = -\Delta + V_0 + \lambda V_1$ in $B^* = \{x \in R^2;\ 0 < |x| < R\}$, where $\lambda \in R^1$, V_1 belongs to $L_p(B^*)$ for some $p \geqq 2$ and satisfy $\max(\pm V_1, 0) \neq 0$, and

$$\begin{aligned} V_0(x) &= |x|^{\alpha_{2j}} \quad \text{for } x \text{ with } \theta_{2j-1} \leqq \arg x \leqq \theta_{2j},\quad j=1,..,k, \\ &= 0 \qquad \text{otherwise,} \end{aligned}$$

where k is a natural number, $0 = \Theta_0 < \Theta_1 < \cdots < \Theta_{2k} = 2\pi$, and $\alpha_{2j} < -2$ for $j=1,..,k$. We put $\theta_i = \Theta_i - \Theta_{i-1}$ and $\Theta_{i-1/2} = \Theta_{i-1} + \theta_i/2$. Regarding λ as a parameter we have

<u>Theorem</u>. There exist $a < 0$ and $b > 0$ such that (i) (P,B^*) is subcritical if and only if $a < \lambda < b$; and (ii) (P,B^*) is critical if and only if $\lambda = a$ or b.

<u>Theorem</u>. If $\lambda = a$ or b, then $H_+(P,B^*,\{0\}) = \{Cp;\ C > 0\}$, where the positive solution $p(r,\phi)$, with (r,ϕ) being polar coordinates of R^2, has the following decay property as $r \to 0$:

$$\begin{aligned} p(r,\phi) &= o(r^{-\pi/\theta_{2j-1}}), && \Theta_{2j-2} < \phi < \Theta_{2j-1}, \\ &= O(\exp[-q_{2j}(\phi) r^{\alpha_{2j}/2 + 1}]), && \Theta_{2j-1} < \phi < \Theta_{2j}, \end{aligned}$$

where q_{2j} is a positive continuous function in $(\Theta_{2j-1}, \Theta_{2j})$.

<u>Theorem</u>. Suppose that $a < \lambda < b$. Then:

(i) The metric space

$$ExH_+(P,B^*,\{0\}) \equiv \{u \in H_+(P,B^*,\{0\});\ u \text{ is extremal and } u(R/2,0) = 1\}$$

is homeomorphic to

$$\sigma = \bigcup_{i=1}^{2k} \sigma_i ,$$

where $\sigma_{2j} = \{\psi \in [0,2\pi);\ |\psi - \Theta_{2j-1/2}| \leqq \theta_{2j}/2 + \pi/(\alpha_{2j} + 2)\}$ and $\sigma_{2j-1} = \{\Theta_{2j-3/2}\}$.

(ii) The minimal solution $P(r,\phi;\psi)$ corresponding to ψ in σ has the following asymptotics as $r \to 0$:

(a) For $\psi = \theta_{2j-1}$,

$$P(r,\phi;\psi) = C(\psi)\chi_{2j-1}(\phi)r^{-\pi/\theta_{2j-1}} \sin[\pi(\phi-\theta_{2j-2})/\theta_{2j-1}] + p(r,\phi;\psi),$$

where $C(\psi)$ is a positive constant, χ_{2j-1} is the characteristic function of the set $[\theta_{2j-2},\theta_{2j-1}]$, and $p(r,\phi;\psi)$ has the same asymptotic property as in the above theorem.

(b) For ψ in σ_{2j},

$$P(r,\phi;\psi) = C(\psi)\chi_{2j}(\phi)\exp[-\frac{r^{\alpha_{2j}/2+1}}{\alpha_{2j}/2+1} \cos(\alpha_{2j}/2+1)(\phi-\psi)][1+o(1)]$$
$$+ p(r,\phi;\psi).$$

Of course, any positive solution is represented uniquely by integrating $P(r,\phi;\psi)$ with respect to ψ by a positive Borel measure on σ.

An interesting open problem is: What is the Martin boundary over $\{0\}$?

I conclude this talk with a theorem concerning the Dirichlet problem at $\{0\}$. Put

$$g(x) = \sum_{j=1}^{k} \{\theta_{2j-1} P(x;\theta_{2j-3/2}) + \theta_{2j} \int_{\sigma_{2j}} P(x;\psi)\frac{d\psi}{|\sigma_{2j}|}\} + 1.$$

By convention, the second integral equals $P(x;\Theta_{2j})$ when $\sigma_{2j} = \{\Theta_{2j}\}$, and is zero when $\sigma_{2j} = \emptyset$.

Theorem. For any continuous function f on σ there exists a unique solution u of the following problem

$$Pu = 0 \quad \text{in} \quad B^*, \quad u\big|_{|x|=R} = 0, \quad u/g \in L_\infty(B^*),$$

$$\lim_{r\to 0} u(r,\phi)/g(r,\phi) = f(\phi) \quad \text{for any} \quad \phi \in \sigma .$$

This theorem means, loosely speaking, the order of singularity is represented by that of g.

Finally, I should mention that there are several related results concerning the cardinal number of the set ExH_+ of all normalized extremal positive solutions obtained by M. Nakai and his collaborators.

References

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