

Remarks on propagation of singularities

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1. Introduction

The methods used in [5] and [6] are also applicable to the general studies of propagation of singularities both in Gevrey classes and in C^∞ . In this note we shall show that it is sufficient to obtain microlocal basic *a priori* estimates in order to prove theorems on propagation of singularities.

Let $1 < \kappa \leq \infty$. When $1 < \kappa < \infty$, we consider the problem in Gevrey classes $\mathcal{E}^{(\kappa)}$ (or the spaces $\mathcal{D}^{(\kappa)'} of ultradistributions). When $\kappa = \infty$, we consider the problem in $C^{(\infty)}$ (or $\mathcal{D}'$). We write $\mathcal{E}^{(\infty)} = C^\infty$ and $\mathcal{D}^{(\infty)'} = \mathcal{D}'$ formally. We denote by $WF_{(\kappa)}(u)$ the wave front set of u in $\mathcal{E}^{(\kappa)}$. Let $z^0 = (x^0, \xi^0) \in T^*\mathbb{R}^n \setminus 0$ ($|\xi^0| = 1$), and let W be a closed conic subset of $T^*\mathbb{R}^n \setminus 0$ such that $z^0 \in W$. Moreover, let $\{\varphi_j^W(x, \xi)\}_{j=1,2,\dots}$ be a family of real-valued symbols satisfying the following conditions: (i) $\varphi_j^W(x, \xi)$ is positively homogeneous of degree 0 and satisfies$

$$|\varphi_{j(\beta)}^{W(\alpha)}(x, \xi)| \leq C_j A_j^{|\alpha|+|\beta|} (|\alpha|+|\beta|)! \quad \text{for } x \in \mathbb{R}^n, \xi \in \mathbb{R}^n, |\xi|=1$$

if $1 < \kappa < \infty$,

$$|\varphi_{j(\beta)}^{W(\alpha)}(x, \xi)| \leq C_{j,\alpha,\beta} \quad \text{for } x \in \mathbb{R}^n, \xi \in \mathbb{R}^n, |\xi|=1 \quad \text{if } \kappa = \infty.$$

(ii) $\varphi_j^W(z^0) < 0$. (iii) There is a conic neighborhood $\mathcal{U}_0$ of z^0 such that for any conic nbd $\tilde{W}$ of W there is $j_0 \in \mathbb{N}$ satisfying

$\{z \in \mathcal{E}_0; \varphi_j^W(z) \leq 0\} \subset \tilde{W} \cap \mathcal{E}_0$ for $j \geq j_0$. When $1 < \kappa < \infty$, we need another family $\{\psi_j^W(x, \xi)\}_{j=1,2,\dots}$ of real-valued symbols which satisfies the following conditions: (iv) $\psi_j^W(x, \xi)$ is pos. homo. of deg. 0 and satisfies

$$|\psi_j^{W(\alpha)}(x, \xi)| \leq C_j A_j^{|\alpha|+|\beta|} (|\alpha|+|\beta|)! \quad \text{for } x \in \mathbb{R}^n, \xi \in \mathbb{R}^n, |\xi|=1.$$

(i) $\psi_j^W(z) \geq \varphi_j^W(z)$ and $\psi_j^W(z) > 0$ for $z \in \partial W \cap \mathcal{E}_0$. (vi) $\{z \in \mathcal{E}_0; \psi_j^W(z) \leq 0\} \subset \mathcal{E}_0 \cap W$. Define

$$\begin{aligned} \Lambda_{a,b,j}^\delta(x, \xi) &\equiv \Lambda_{a,b}^\delta(x, \xi; \kappa, \varphi_j^W) \\ &= a\psi(\xi)\varphi_j^W(x, \xi) \langle \xi \rangle^{1/\kappa - \delta} + b\psi(\xi)\psi_j^W(x, \xi) \langle \xi \rangle^{1/\kappa} \quad \text{when } 1 < \kappa < \infty, \end{aligned}$$

$$\begin{aligned} \Lambda_{a,b,j}^\delta(x, \xi) &\equiv \Lambda_{a,b}^\delta(x, \xi; \infty, \varphi_j^W) \\ &= a\psi(\xi)\varphi_j^W(x, \xi) \log \langle \xi \rangle + b \log(1 + \delta \langle \xi \rangle) \quad \text{when } \kappa = \infty, \end{aligned}$$

where $a, b, \varepsilon > 0$, $0 \leq \delta \leq 1$ and $\psi(\xi) \in \mathcal{S}^{(\kappa)}(\mathbb{R}^n)$, $\psi(\xi) = 1$ if $|\xi| \geq 1$ and $\psi(\xi) = 0$ if $|\xi| \leq 1/2$.

Assumptions

(A-1) $_{\kappa}$ $L(x, D) \equiv (L_{ij}(x, D))$: $N \times N$ matrix whose entries are properly supported Ψ .D.Ops in $\mathcal{S}^{(\kappa)}$.

(A-2) $_{\kappa}$ ($1 < \kappa < \infty$) $\exists j_0 \in \mathbb{N}$, $\exists \chi_k(x, \xi) \in C^\infty(T^*\mathbb{R}^n)$ ($k=1,2$) s.t.

' $\chi_k(x, \xi)$: pos. homo. of deg. 0 for $|\xi| \geq 1$, $\chi_k(z) = 1$ near z^0 ($k=1,2$) and $\forall j \geq j_0$, $\exists a_0 > 0$, $\exists b_0 > 0$ s.t. $\forall a \geq a_0$, $\forall b \geq b_0$,

$\exists \delta_0 > 0$ ($\delta_0 \leq 1$), $\exists \ell_k \in \mathbb{R}$ ($1 \leq k \leq 3$), $\exists C > 0$ satisfying

$$\|\chi_1(x, D)v\|_{\ell_1} \leq C\{\|L_\Lambda(x, D)v\|_{\ell_2} + \|v\|_{\ell_1-1} + \|(1-\chi_2(x, D))v\|_{\ell_3}\}$$

if $v \in C_0^\infty$ and $0 < \delta \leq \delta_0$, where $L_\Lambda(x, D) = {}^R(e^{-\Lambda})(x, D)L(x, D) \times$

$(e^\Lambda)(x, D)$, $\Lambda(x, \xi) = \Lambda_{a,b,j}^\delta(x, \xi)$ and

$${}^R(e^{-\Lambda})(x, D)u(x) = \int \left(\int e^{i(x-y) \cdot \xi} e^{-\Lambda(y, \xi)} u(y) dy \right) d\xi.$$

(A-2) $_{\infty}$ ($\kappa = \infty$) $\exists j_0 \in \mathbb{N}$, $\exists \chi_k(x, \xi) \in C^\infty(T^*\mathbb{R}^n)$ ($k=1,2$), $\exists \ell_k \in \mathbb{R}$ ($1 \leq k \leq 3$)

s.t. " $\chi_k(x, \xi)$: pos. homo. of deg. 0 for $|\xi| \geq 1$, $\chi_k(z) = 1$ near z^0 ($k=1, 2$) and $\forall j \geq j_0$, $\exists a_0 > 0$, $\exists \Lambda_1(x, \xi) \in C^\infty(T^*\mathbb{R}^n)$

s.t. $|\Lambda_{1(\beta)}^{(\alpha)}(x, \xi)| \leq \exists C_{\alpha, \beta} \langle \xi \rangle^{\delta' |\beta| - \rho' |\alpha|} \log(1 + \langle \xi \rangle)$ with $0 \leq \delta' < \rho' \leq 1$ and $\forall a \geq a_0$, $\exists b_0 > 0$, $\exists \Lambda_2(x, \xi) \in S_{1,0}^0$ s.t.

' $|\Lambda_{2(\beta)}^{(\alpha)}(x, \xi)| \leq \exists C'_{\alpha, \beta} \langle \xi \rangle^{-|\alpha|}$ and $\forall b \geq b_0$, $\exists \delta_0 > 0$ ($\delta_0 \leq 1$),

$\exists C > 0$ satisfying

$$\|\chi_1(x, D)v\|_{\ell_1} \leq C\{\|L_\Lambda(x, D)v\|_{\ell_2} + \|v\|_{\ell_1-1} + \|(1-\chi_2(x, D))v\|_{\ell_3}\}$$

if $v \in C_0^\infty$ and $0 < \delta \leq \delta_0$, where $\Lambda(x, \xi) = \Lambda_{a,b,j}^\delta(x, \xi) + \Lambda_1(x, \xi) +$

$\Lambda_2(x, \xi)$.' "

Theorem 1. Assume that $(A-1)_\kappa$ and $(A-2)_\kappa$ are satisfied. If $u \in \mathcal{D}^{(\kappa)}$, $z^0 \notin WF_{(\kappa)}(Lu)$ and $\exists \mathcal{E}$: conic nbd of z^0 s.t. $WF_{(\kappa)}(u) \cap \mathcal{E} \cap (W \setminus \{(x^0, \lambda \xi^0); \lambda > 0\}) = \emptyset$, then $z^0 \notin WF_{(\kappa)}(u)$. Moreover, if $\kappa = \infty$, there is a conic nbd $\mathcal{E}_1$ of z^0 s.t. 'if $u \in \mathcal{D}$ and $WF(Lu) \cap \overline{\mathcal{E}} = \emptyset$ and $WF(u) \cap \partial \mathcal{E} \cap W = \emptyset$ for some conic nbd $\mathcal{E}$ of z^0 with $\mathcal{E} \subset \subset \mathcal{E}_0$, then $z^0 \notin WF(u)$.'

Remark. If $z^0 \in \hat{W}$, then Theorem 1 is obvious. Giving $\{\varphi_j^W(x, \xi)\}$ one may determine W so that $\{\varphi_j^W\}$ satisfies the condition (iii). If $W = \{z \in T^*\mathbb{R}^n; \varphi(z) \leq 0\}$ and $\varphi(x, \xi)$ is analytic ($1 < \kappa < \infty$) and smooth ($\kappa = \infty$), then we can choose $\varphi_j^W(z) = \varphi(z) - 1/j$ and $\psi_j^W(z) = \varphi(z) + 1$.

Corollary 1 ([6]). Assume that $W = \{(x^0, \lambda \xi^0); \lambda > 0\}$ and that $(A-1)_\kappa$ and $(A-2)_\kappa$ are satisfied. If $u \in \mathcal{D}^{(\kappa)}$ and $z^0 \notin WF_{(\kappa)}(Lu)$, then $z^0 \notin WF_{(\kappa)}(u)$.

Corollary 2. Assume that $(A-1)_\kappa$ is satisfied. Let $\psi(x, \xi) \in C(T^*\mathbb{R}^n \setminus 0)$ be a real-valued function s.t. $\psi(x, \xi)$: pos. homo. of

deg. 0, $\psi(z^0)=0$, and assume that $\exists \varphi \in \mathcal{E}^{(\kappa)}(T^*\mathbb{R}^n \setminus 0)$ s.t. $\varphi(x, \xi)$: pos. homo. of deg. 0, $\varphi(z^0)=0$, $\varphi(z) > \psi(z)$ for $z \in \mathcal{E}_0 \setminus \{(x^0, \lambda \xi^0); \lambda > 0\}$, $\varphi(x, \xi)$: analytic if $1 < \kappa < \infty$, and $(A-2)_\kappa$ is satisfied, where $\mathcal{E}_0$: conic nbd of z^0 , $\varphi_j^W(z) = \varphi(z) - 1/j$ and $\psi_j^W(z) = \varphi(z) + 1$. Then $z^0 \notin WF_{(\kappa)}(u)$ if $z^0 \notin WF_{(\kappa)}(Lu)$ and $\exists \mathcal{E}$: conic nbd of z^0 s.t. $\{z \in \mathcal{E}; \psi(z) < 0\} \cap WF_{(\kappa)}(u) = \emptyset$.

Remark. This corollary is a microlocal version of Holmgren's uniqueness theorem. When $\kappa = \infty$, we can obtain a little stronger results, corresponding to Theorem 1.

Theorem 2. Let Ω be a conic set in $T^*\mathbb{R}^n \setminus 0$, and assume that $(A-1)_\kappa$ are satisfied and $\exists g: \Omega \ni z \longmapsto g(z) \in T_z \Omega$: cont. and $\exists \{\mathcal{E}_z\}_{z \in \Omega}$: a family of closed convex cones s.t. $\mathcal{E}_z \subset \{\delta z; \sigma(g(z), \delta z) > 0\} \cup \{0\}$ ($\subset T_z \Omega$), $\{\mathcal{E}_z\}_{z \in \Omega}$ is outer semi-cont. and $(A-2)_\kappa$ is valid for $\forall z^0 = (x^0, \xi^0) \in \Omega$ ($|\xi^0| = 1$) if $g^0 \in -\text{int}(\mathcal{E}_{z^0}^\sigma)$, $\sigma(r(z^0), g^0) = 0$, $k > 0$, $\varphi_j^W(z) = \varphi_k(z) - 1/j$, $\psi_j^W(z) = \varphi_k(z) + 1$, $\varphi_k(z) = \tilde{\varphi}_k(z)(1 + \tilde{\varphi}_k(z)^2)^{-1/2}$ and $\tilde{\varphi}_k(x, \xi) = \sigma(g^0, (x - x^0, \xi/|\xi| - \xi^0)) + k(|x - x^0|^2 + |\xi/|\xi| - \xi^0|^2)$, where $\sigma((\delta x, \delta \xi), (\delta y, \delta \eta)) = \delta y \cdot \delta \xi - \delta x \cdot \delta \eta$, $r(x, \xi) = \sum_{j=1}^n \xi_j \partial / \partial \xi_j$, $\mathcal{E}^\sigma = \{\delta z; \sigma(\delta w, \delta z) \geq 0 \text{ for } \forall \delta w \in \mathcal{E}\}$ and $\text{int}(\mathcal{E})$: the interior of $\mathcal{E}$. If $u \in \mathcal{D}^{(\kappa)}$, $z^0 \in WF_{(\kappa)}(u) \cap \Omega$ and $WF_{(\kappa)}(Lu) \cap \Omega = \emptyset$, then $\exists a \in (-\infty, 0] \cup \{-\infty\}$, $\exists \{z(t)\}_{t \in (a, 0]}$: Lip. cont. curve in Ω s.t. $z(t) \in WF_{(\kappa)}(u)$ for $t \in (a, 0]$, $(d/dt)z(t) \in \mathcal{E}_{z(t)} \cap \{|\delta z| = 1\}$ for a.e. $t \in (a, 0]$, $z(0) = z^0$, and $\lim_{t \rightarrow a+0} z(t) \in \partial \Omega$ if $a > -\infty$, where $\partial \Omega$: the boundary of Ω in $T^*\mathbb{R}^n$.

Corollaries 1 and 2 are immediate consequences of Theorem 1. Theorem 2 follows from Corollary 2 of Theorem 1, applying the same arguments as in [5], [10] and [11]. We shall prove Theorem 1 in the case where $\kappa = \infty$ in §2. Theorem 1 for $1 < \kappa < \infty$ can

be proved in the same manner. In §3 we shall consider several examples, applying the results in §1.

2. Proof of Theorem 1 ($\kappa=\infty$)

We may assume that $u \in \delta' \cap H^{s'}$ for some $s' \in \mathbb{R}$. Let $\mathcal{C}_1$ be a conic nbd of z^0 s.t. $\mathcal{C}_1 \cap \{|\xi| \geq 1\} \subset \{\chi_1(x, \xi) = \chi_2(x, \xi) = 1\}$. Assume that $WF(f) \cap \bar{\mathcal{C}} = \emptyset$ and $WF(u) \cap \partial \mathcal{C} \cap W = \emptyset$ for $\exists \mathcal{C}$: conic nbd of z^0 with $\mathcal{C} \subset \mathcal{C}_1$, where $f = Lu$. Choose $\chi(x, \xi) \in C^\infty(T^*\mathbb{R}^n)$ so that $\chi(x, \xi)$: pos. homo. of deg. 0 for $|\xi| \geq 1$, $\chi(z) = 1$ near z^0 , $\text{supp } \chi \cap \{|\xi| \geq 1\} \subset \mathcal{C}_1$, $WF(f) \cap \text{supp } \chi \cap \{|\xi| \geq 1\} = \emptyset$ and $WF(u) \cap W \cap \text{supp } d\chi \cap \{|\xi| \geq 1\} = \emptyset$. Then $\exists j \geq j_0$, $\exists \varepsilon > 0$ s.t. $(x, \xi) \notin WF(u)$ if $(x, \xi) \in \text{supp } d\chi$, $|\xi| \geq 1$ and $\varphi_j^W(x, \xi) \leq 2\varepsilon$. For a fixed $s > s'$ we can choose $a \geq a_0$ so that

$$a\varepsilon + \liminf_{\lambda \rightarrow \infty} \int_{|\xi| \geq \lambda} \Lambda_1(x, \xi) (\log |\xi|)^{-1} > \ell_2 + m - 1 - s',$$

$$-a\varepsilon'/2 + \limsup_{\lambda \rightarrow \infty} \int_{|\xi| \geq \lambda} \Lambda_1(x, \xi) (\log |\xi|)^{-1} < \ell_1 - s,$$

where $\varepsilon' = -\varphi_j^W(z^0)$ (> 0). By the assumptions in (A-2) $_\infty$ $\exists b_0 > 0$ and $\exists \Lambda_2(x, \xi) \in S^0$ for each a . Choose $b \geq b_0$ so that

$$b > b(u) \equiv -a \inf \varphi_j^W(x, \xi) - \liminf_{\lambda \rightarrow \infty} \int_{|\xi| \geq \lambda} \Lambda_1(x, \xi) (\log |\xi|)^{-1} + \max\{\ell_2 + m, \ell_1 - 1, \ell_3\}.$$

It follows from calculus of Ψ .D.Op. that $\exists q(x, \xi) \in S_{\rho', \delta'}^{\delta' - \rho' + d}$ ($\forall d > 0$) s.t.

$$(e^\Lambda)(x, D)(1 + q(x, D))^R (e^{-\Lambda})(x, D) = 1 \quad (\text{mod } L^{-\infty}).$$

Put $v_\delta(x) \equiv v(x) = (1 + q(x, D))^R (e^{-\Lambda})(x, D) \chi(x, D) u(x)$. Then $\chi u \equiv (e^\Lambda)(x, D) v(x)$ ($\text{mod } L^{-\infty}$) and

$$L_\Lambda(x, D) v \equiv {}^R(e^{-\Lambda}) \chi f + {}^R(e^{-\Lambda}) [L, \chi] u \quad (\text{mod } L^{-\infty}).$$

Since $u \in C^\infty$ near $\{(x, \xi); \varphi_j^W(x, \xi) \leq 2\varepsilon, (x, \xi) \in \text{supp } d\chi \text{ and } |\xi| \geq 1\}$ and $a\varphi_j^W(x, \xi) + \liminf_{\lambda \rightarrow \infty} \int_{|\xi| \geq \lambda} \Lambda_1(x, \xi) (\log |\xi|)^{-1} > \ell_2 + m - 1 - s'$ if

$\varphi_j^W(x, \xi) \geq \varepsilon$, we have $\|L_\Lambda v\|_{\ell_2} \leq C$, where C is indep. of δ ($0 < \delta \leq \delta_0$).

Noting that $v \in H^{\max\{\ell_2+m, \ell_1-1, \ell_3\}}$ for $\delta > 0$, we have $\|x_1(x, D)v\|_{\ell_1}$

$\leq C$, where C is indep. of δ ($0 < \delta \leq \delta_0$). Therefore, $v_\delta \rightarrow v_0$ weakly

in H^{ℓ_1} as $\delta \downarrow 0$. This implies that $v_0 \in H^{\ell_1}$. Since $\Lambda^0(x, \xi) \equiv$

$a\varphi_j^W(x, \xi) \log \langle \xi \rangle + \Lambda_1(x, \xi) + \Lambda_2(x, \xi) < (\ell_1 - s) \log \langle \xi \rangle$ if $\varphi_j^W(x, \xi) \leq -\varepsilon'/2$

and $|\xi| > 1$, we have $u \in H^s$ at z^0 . This proves Theorem 1 for $\kappa = \infty$.

Roles of e^Λ and $R(e^{-\Lambda})$

(i) To reduce the problem in Gevrey classes (or C^∞) to the problem in the Sobolev spaces.

(ii) To deal with $[L, x]$, i.e., to neglect the term $R(e^{-\Lambda})[L, x]u$ in $R(e^{-\Lambda})Lxu = R(e^{-\Lambda})\chi f + R(e^{-\Lambda})[L, x]u$.

(iii) To change norms of u and Lu .

3. Applications of Theorems 1 and 2

3.1. Microhyperbolic operators

Let $1 < \kappa \leq 2$, and let Ω be an open conic subset of $T^*\mathbb{R}^n \setminus 0$.

(L-1) $\exists m_1, \dots, m_N, n_1, \dots, n_N \in \mathbb{R}$, $\exists L^0(x, \xi) = (L_{ij}^0(x, \xi))$ and $\rho > 0$ s.t.
 $L_{ij}^0(x, \xi)$: pos. homo. of deg. $m_i + n_j$, $L_{ij}(x, D) - L_{ij}^0(x, D)\psi(D)$
 is a Ψ .D.Op. of order $m_i + n_j - \rho$ and $0 < \rho \leq 1$ if $N > 1$.

(L-2) $\exists g: \Omega \ni z \mapsto g(z) \in T_z \Omega$: cont. s.t. $p(x, \xi) \equiv \det L^0(x, \xi)$ is
 microhyperbolic w.r.t. $g(z)$ at $z \in \Omega$, i.e., $\forall z^0 \in \Omega$, $\exists \mathcal{U}$: nbd
 of z^0 in $T^*\mathbb{R}^n \setminus 0$, $\exists \ell \in \mathbb{N} \cup \{0\}$, $\exists c > 0$ and $\exists t_0 > 0$ s.t.
 $|\sum_{j=0}^{\ell} (-itg(z^0))^j p(z)/j!| \geq ct^{\ell}$ for $z \in \mathcal{U}$ and $0 \leq t \leq t_0$.

(L-3) $\mu_0 = \sup_{z \in \Omega} \mu(z) < \infty$, where $\mu(z^0)$ is the multiplicity of
 $p(z)$ at z^0 , i.e.,

$$p(z^0 + s\delta z) = s^{\mu(z^0)} (p_{z^0}(\delta z) + o(1)) \quad \text{as } s \rightarrow 0,$$

where $p_{z^0}(\delta z) \neq 0$ in δz .

(L-4) $1 < \kappa \leq \min\{2, \mu_0/(\mu_0 - \rho)\}$, where $\kappa \leq 2$ if $\mu_0 \leq \rho$.

Theorem 3 ([5]). Assume that $(A-1)_\kappa$ and (L-1)-(L-4) are satisfied. Then the conclusion of Theorem 2 is valid, taking $\mathcal{G}_{z^0} = \Gamma(p_{z^0}, g(z^0))^\sigma$.

3.2. Symmetric hyperbolic systems

Let $1 < \kappa \leq \infty$, and let Ω be an open conic subset of $T^*\mathbb{R}^n \setminus 0$.

(D-1) $L_{ij}(x, D)$: properly supported $\Psi.D.Op.$ of order 1 in $\mathcal{S}^{(\kappa)}$ and $\exists L^0(x, \xi) = (L_{ij}^0(x, \xi))$ s.t. $L_{ij}^0(x, \xi)$: pos. homo. of deg. 1 and $L_{ij}(x, D) - L_{ij}^0(x, D)\psi(D)$ is a $\Psi.D.Op.$ of order $1/\kappa$, where $1/\kappa = 0$ if $\kappa = \infty$.

(D-2) L is dissipative in Ω , i.e.,
 $-i(L^0(x, \xi) - L^0(x, \xi)^*) \leq 0$ for $(x, \xi) \in \Omega$.

(D-3) $\exists g: \Omega \ni z \mapsto g(z) \in T_z \Omega$: cont. s.t.
 $\operatorname{Re}((g L^0)(z^0)v, v) > 0$ for $\forall z^0 \in \Omega$ and $\forall v \in \operatorname{Ker} L^0(z^0) \setminus \{0\}$.

Lemma 4. Under the assumptions (D-1)-(D-3), $p(x, \xi) \equiv \det L^0(x, \xi)$ is microhyp. w.r.t. $g(z^0)$ at z^0 ($\in \Omega$). Moreover, for $z^0 \in \Omega$

$\operatorname{Re}((g_x \cdot (\partial L^0 / \partial x)(z^0) + g_\xi \cdot (\partial L^0 / \partial \xi)(z^0))v, v) > 0$
 if $v \in \operatorname{Ker} L^0(z^0) \setminus \{0\}$ and $g = (g_x, g_\xi) \in \Gamma(p_{z^0}, g(z^0))$.

Theorem 5. Assume that (D-1)-(D-3) are satisfied. Then the conclusion of Theorem 2 is valid, taking $\mathcal{G}_{z^0} = \Gamma(p_{z^0}, g(z^0))^\sigma$.

Remark. When $\kappa = \infty$, Ivrii [3] proved results corresponding to Corollary 2 of Theorem 1 and Wakabayashi proved Theorem 5 in

[10]. Similarly we can prove results on wave front sets in the Sobolev spaces.

3.3. Effectively hyperbolic operators

From now on we consider the problems in C^∞ ($\kappa=\infty$). Let Ω be an open conic subset of $T^*\mathbb{R}^n \setminus 0$.

(E-1) $P(x,D)$: properly supported Ψ .D.Op. whose symbol is $P(x,\xi) \in S^m$, and $\exists p(x,\xi)$: real-valued and pos. homo. of deg. m s.t. $P(x,\xi) - \psi(\xi)p(x,\xi) \in S^{m-1}$.

(E-2) $p(x,\xi)$: effectively hyperbolic in Ω , i.e., $\exists g: \Omega \ni z \mapsto g(z) \in T_z\Omega$: cont. s.t. $p(x,\xi)$: microhyp. w.r.t. $g(z)$ at $\forall z \in \Omega$ and for $\forall z^0 \in \{z \in \Omega; dp(z)=0\}$ the fundamental matrix (Hamilton map) $F_p(z^0)$ has non-zero real eigenvalues, where

$$F_p(x,\xi) = \begin{pmatrix} p_{\xi x}(x,\xi) & p_{\xi\xi}(x,\xi) \\ -p_{xx}(x,\xi) & -p_{x\xi}(x,\xi) \end{pmatrix}.$$

Theorem 6 ([7],[8]). Assume that (E-1) and (E-2) are satisfied. Then the conclusion of Theorem 2 is valid, taking

$$\mathcal{G}_{z^0} = \Gamma(p_{z^0}, g(z^0))^\sigma.$$

Remark. In the theorem the curve $\{z(t)\}_{t \in (a,0]}$ near z^0 is one of two possible curves, which are limiting bicharacteristics of p , if $dp(z^0)=0$ (see [8] and [4]).

Following Melrose [7], we can assume that $z^0 = (0; 0, \dots, 0, 1)$ ($\in \Omega$) and $p(x,\xi) = \xi_1^2 - x_1^2 \alpha(x,\xi') - \beta(x,\xi')$, where $\xi' = (\xi_2, \dots, \xi_n)$, $\alpha(x,\xi') \geq c|\xi'|^2$ ($\exists c > 0$), $\beta(x,\xi') \geq 0$ and $\beta(0; 0, \dots, 1) = 0$. To prove Theorem 6 we choose

$$\Lambda_1(x,\xi) = \gamma \log(\sqrt{1+x_1^2 \langle \xi_n \rangle} + x_1 \langle \xi_n \rangle^{1/2}) \text{ near } z^0, \quad \Lambda_2(x,\xi) = 0,$$

where $\gamma \gg 1$. Finally, we have

$$\| \chi_1 v \|_{3/4} \leq C \{ \| P_\Lambda v \|_0 + \| v \|_{-1/4} + \| (1 - \chi_1) v \|_2 \} \quad \text{if } v \in C_0^\infty, \quad 0 < \delta \leq \delta_0.$$

3.4. Branching of singularities

Let us consider a special class of effectively hyperbolic operators. Let $z^0 = (x^0, \xi^0) \in T^*\mathbb{R}^n \setminus 0$ ($|\xi^0| = 1$).

(B-1) $P(x, D) = p_1(x, D)\psi(D)p_2(x, D)\psi(D) + Q(x, D)$: properly supported, classical Ψ .D.Op., $p_j(x, \xi)$: pos. homo. of deg. m_j ($j=1, 2$) and $Q(x, \xi) \in S^{m_1+m_2-1}$.

(B-2) $p_j(z^0) = 0$ ($j=1, 2$) and $\{p_1, p_2\}(z^0) \neq 0$.

Let $\gamma_j = \{\gamma_j(t)\}_{-\delta \leq t \leq \delta}$ ($j=1, 2$) be bicharacteristics of p_j s.t. $\gamma_j(0) = z^0$, where $\delta > 0$, and put $\gamma_j^\pm = \{\gamma_j(t)\}_{0 < \pm t \leq \delta}$.

Theorem 7 ([2], [1]). Let $j=1$ or 2 . Assume that

$$(-1)^{j+1} i \text{ sub } \sigma(P)(z^0) / \{p_1, p_2\}(z^0) - 1/2 \notin \{0, 1, 2, \dots\},$$

where $\text{sub } \sigma(P)(x, \xi) = Q_1(x, \xi) + (i/2) \sum_{k=1}^n (\partial^2 / \partial x_k \partial \xi_k)(p_1 p_2)(x, \xi)$ and $Q_1(x, \xi)$ is the principal symbol of $Q(x, D)$. Then $z^0 \notin \text{WF}(u)$ if $u \in \mathcal{D}'$, $z^0 \notin \text{WF}(Pu)$ and $\exists \mathcal{C}$: conic nbd of z^0 s.t. $\text{WF}(u) \cap \mathcal{C} \cap (\gamma_j \setminus \{z^0\}) = \emptyset$.

Corollary. Assume that

$$i \text{ sub } \sigma(P)(z^0) / \{p_1, p_2\}(z^0) - 1/2 \notin \{0, \pm 1, \pm 2, \dots\}.$$

Then $z^0 \notin \text{WF}(u)$ if $z^0 \notin \text{WF}(Pu)$ and $\exists \mathcal{U}$: nbd of z^0 and $\exists \gamma^1, \gamma^2 \in \{\gamma_1^+, \gamma_1^-, \gamma_2^+, \gamma_2^-\}$ s.t. $\gamma^1 \neq \gamma^2$ and $\text{WF}(u) \cap \mathcal{U} \cap (\gamma^1 \cup \gamma^2) = \emptyset$.

We may assume that $z^0 = (0; 0, \dots, 0, 1)$ and $P(x, \xi) = x_1 \xi_1 + q(x', \xi')$, where $q \in S_{1,0}^0$ (see [1], [9]). Choosing $W = \{x' = 0, \xi'' = 0\}$, and $\varphi_j^W(x, \xi) = |\xi''|^2 \xi_n^{-2} + |x'|^2 - 1/j$ near z^0 , we can apply Theorem 1 if $\text{Im } q_0(z^0) > -1/2$, where $\xi'' = (\xi_1, \dots, \xi_{n-1})$ and $q_0(x, \xi)$ is the principal symbol of $q(x, D)$. Therefore, we can prove

Theorem 7 for $j=2$ if $\text{Im } q_0(z^0) > -1/2$. Following Hanges [1], we can prove Theorem 7 (for $j=2$) in the case where $\text{Im } q_0(z^0) \leq -1/2$. In fact, $x_1^\ell P(x, D)u = (P(x, D) + i\ell)x_1^\ell u$. So, if $\ell \in \mathbb{N}$, $\text{Im } q_0(z^0) + \ell > -1/2$, $z^0 \notin \text{WF}(Pu)$ and $\text{WF}(u) \cap \mathcal{B} \cap (\gamma_2 \setminus \{z^0\}) = \emptyset$, then $z^0 \notin \text{WF}(x_1^\ell u)$. Assume that $z^0 \notin \text{WF}(x_1^k u)$ for some k ($1 \leq k \leq \ell$). Then we can write

$$x_1^{k-1}u = f_1 + (x_1 + i0)^{-1}f_2 + \delta(x_1) \otimes g(x'),$$

where $z^0 \notin \text{WF}(f_j)$ ($j=1, 2$). We have also

$$\begin{aligned} x_1^{k-1}P(x, D)u &= (P(x, D) + i(k-1))f_1 + D_1 f_2 \\ &\quad + (q + ik)\{(x_1 + i0)^{-1}f_2 + \delta(x_1) \otimes g(x')\}. \end{aligned}$$

So we have $z^0 \notin \text{WF}((q + ik)((x_1 + i0)^{-1}f_2 + \delta(x_1) \otimes g(x')))$. The assumptions in Theorem 7 imply that $iq_0(z^0) \notin \{1, 2, 3, \dots\}$, i.e., $q + ik$ is elliptic at z^0 . Thus we have $z^0 \notin \text{WF}(x_1^{k-1}u)$. Theorem 7 for $j=1$ can be reduced to the case where $j=2$.

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