

# THE RADIUS OF $\beta$ -CONVEXITY FOR THE CLASSES OF $\lambda$ -SPIRALLIKE ORDER $\alpha$ FUNCTIONS

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ABSTRACT. We get sharp bounds for the radius of  $\beta$ -convexity for the classes of  $\lambda$ -spirallike of order  $\alpha$  and  $p$ -fold  $\lambda$ -spirallike of order  $\alpha$  functions.

## 1. Introduction

Let  $A$  denote the class of functions of the form

$$(1.1) \quad s(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in unit disk  $D = \{z : |z| < 1\}$ . And let  $S$  denote the subclass of  $A$  consisting of analytic and univalent function  $s(z)$  in unit disk  $D$ .

A function  $s(z)$  in  $S$  is said to be starlike if

$$(1.2) \quad \operatorname{Re} \left\{ \frac{zs'(z)}{s(z)} \right\} > 0 \quad (z \in D).$$

We denote by  $S^*$  the class of all starlike functions. A function  $s(z)$  in  $S$  is said to be convex if

$$(1.3) \quad \operatorname{Re} \left\{ 1 + \frac{zs''(z)}{s'(z)} \right\} > 0 \quad (z \in D).$$

And we denote by  $K$  the class of all convex functions. These classes  $S^*$

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**Definition 1.1.** A function  $s(z)$  in  $S$  is said to be  $\lambda$ -spirallike if

$$(1.4) \quad \operatorname{Re} \left\{ e^{i\lambda} z \frac{s'(z)}{s(z)} \right\} > 0 \quad (z \in D)$$

for some real  $\lambda$   $\left(|\lambda| < \frac{\pi}{2}\right)$ . The class of these functions is denoted by  $S_\lambda^*$

**Definition 1.2.** A function  $s(z)$  in  $S$  is said to be  $\lambda$ -spirallike of order  $\alpha$  if

$$(1.5) \quad \operatorname{Re} \left\{ e^{i\lambda} z \frac{s'(z)}{s(z)} \right\} > \alpha \cos \lambda \quad (z \in D)$$

for some real  $\lambda$   $\left(|\lambda| < \frac{\pi}{2}\right)$  and  $\alpha$   $(0 \leq \alpha < 1)$ . The above classes were introduced by Spacek ([12]). For  $\lambda = 0$  in (1.4) the class is a starlike function (1.2).

**Definition 1.3.** Let  $F$  denote a non-empty collection of functions  $s(z)$  each of which is univalent in  $D$ , and let  $\beta$  be given  $0 \leq \beta \leq 1$ . Then the real number

$$(1.6) \quad R_\alpha(F) = \sup \{ R | \operatorname{Re} J(\beta, s(z)) > 0, |z| < R, s(z) \in F \}$$

is called the radius of  $\beta$ -convexity of  $F$ , where  $J(\beta, s(z))$  is defined by the relation,

$$(1.7) \quad J(\beta, s(z)) = (1 - \beta) z \frac{s'(z)}{s(z)} + \beta \left( 1 + z \frac{s''(z)}{s'(z)} \right).$$

The radius of  $\beta$ -convexity was introduced by S. S. Miller, P. T. Mocanu and M. O. Reade ([4]). For  $\beta = 0$  and  $\beta = 1$  in (1.7), we define a starlike function (1.2) and a convex function (1.3), respectively.

**Definition 1.4.** Consider a function  $s(z) = z + a_2z^2 + a_3z^3 + \dots$  which is univalent in  $U$ . Then the function defined by the relation

$$(1.8) \quad f(z) = (s(z^p))^{\frac{1}{p}} = z + \sum_{n=1}^{\infty} a_{np+1} z^{np+1}$$

is also univalent in  $U$ , and  $f(z)$  is called  $p$ -fold univalent function. If the function  $f(z)$  defined by the relation (1.8) satisfies the collection

$$(1.9) \quad \operatorname{Re} \left\{ e^{i\lambda} z \frac{f'(z)}{f(z)} \right\} > 0 \quad (z \in D),$$

then the function  $f(z)$  is called a  $p$ -fold  $\lambda$ -spirallike function in  $U$ , for some real  $\lambda$   $\left(|\lambda| < \frac{\pi}{2}\right)$  ([1]), and the class of these functions is denoted by  $S_{\lambda p}^*$ . And also we can define a  $p$ -fold  $\lambda$ -spirallike function of order  $\alpha$  in  $U$ , denoted by  $S_{\lambda p}^*(\alpha)$ .

The radius of  $\beta$ -convexity was introduced by S. S. Miller, P. T. Mocanu and M. O. Reade ([4]). There are many open problems about the radius of starlikeness, convexity and  $\beta$ -convexity for the classes  $S$  ([1]). So, we get sharp bounds for the radius of  $\beta$ -convexity for the classes of  $\lambda$ -spirallike of order  $\alpha$  and  $p$ -fold  $\lambda$ -spirallike of order  $\alpha$  functions.

## 2. The radius of $\beta$ -convexity

**Lemma 2.1** ([5]). If  $s(z) \in S_{\lambda}^*(\alpha)$ , then

$$(2.1) \quad \left| z \frac{s'(z)}{s(z)} - \frac{1 + \{2(1 - \alpha) \cos \lambda e^{-i\lambda} - 1\} r^2}{1 - r^2} \right| \leq \frac{2(1 - \alpha) r \cos \lambda}{1 - r^2}.$$

**Lemma 2.2 ([10]).** If  $p(z) = 1 + p_1z + p_2z^2 + p_3z^3 + \dots$  is analytic in  $D$ , and satisfies the conditions  $\operatorname{Re} p(z) > 0$ ,  $p(0) = 1$ . Then

$$(2.2) \quad \left| z \frac{p'(z)}{p(z)} \right| \leq \frac{2r}{1-r^2}.$$

**Lemma 2.3 ([7]).** If  $p(z) = 1 + p_1z + p_2z^2 + p_3z^3 + \dots$  is analytic in  $D$ , and satisfies the conditions  $\operatorname{Re} p(z) > 0$ , then

$$(i) \quad |p_n| \leq 2 \quad \text{for } n \geq 1,$$

$$(i) \quad |p(z)| \leq \frac{1+|z|}{1-|z|},$$

$$\operatorname{Re} p(z) \geq \frac{1-|z|}{1+|z|}.$$

**Lemma 2.4.** If  $s(z) \in S_\lambda^*(\alpha)$ , then

(2.3)

$$(i) \quad \text{for } \lambda \neq 0, \quad \left| 1 + z \frac{s''(z)}{s'(z)} - \frac{1 + \{2(1-\alpha) \cos \lambda e^{-i\lambda} - 1\} r^2}{1-r^2} \right| \\ \leq \frac{2(1-\alpha)r\{1+r+(1-r)|\sin \lambda|\} \cos \lambda}{(1-r)^2(1+r)|\sin \lambda|}$$

and

$$(ii) \quad \text{for } \lambda = 0, \quad \left| 1 + z \frac{s''(z)}{s'(z)} - \frac{1 + (1-2\alpha)r^2}{1-r^2} \right| \\ \leq \frac{4r(1-\alpha)\{1+(1-\alpha)r\}}{(1-r^2)\{(1-\alpha)(1+r) + \alpha(1-r)\}}.$$

**Proof.** (i) for  $\lambda \neq 0$ , since  $s(z) \in S_\lambda^*(\alpha)$ , then

$$(2.4) \quad \frac{e^{i\lambda} \frac{zs'(z)}{s(z)} - \alpha \cos \lambda - i \sin \lambda}{(1-\alpha) \cos \lambda} = p(z),$$

where  $p(z)$  is analytic in  $D$ , and satisfies the conditions  $\operatorname{Re} p(z) > 0$ ,  $p(0) = 1$ . Logarithmic differentiation yields

$$(2.5) \quad 1 + z \frac{s''(z)}{s'(z)} - z \frac{s'(z)}{s(z)} = \frac{z(1-\alpha) \cos \lambda p'(z)}{(1-\alpha) \cos \lambda p(z) + \alpha \cos \lambda + i \sin \lambda}.$$

By Lemma 2.2 and putting  $\frac{1}{p(z)} = U + iV$ , we have

$$\begin{aligned} & \left| 1 + z \frac{s''(z)}{s'(z)} - z \frac{s'(z)}{s(z)} \right| \\ &= \left| \frac{\frac{zp'(z)}{p(z)}}{1 + \frac{\alpha}{1-\alpha} \frac{1}{p(z)} + i \frac{1}{1-\alpha} \tan \lambda \frac{1}{p(z)}} \right| \\ (2.6) \quad &= (1-\alpha) \left| \frac{\frac{zp'(z)}{p(z)}}{(1-\alpha) + \alpha \frac{1}{p(z)} + i \tan \lambda \frac{1}{p(z)}} \right| \\ &\leq \frac{(1-\alpha) \frac{2r}{1-r^2}}{(1-\alpha) + \alpha \frac{1}{p(z)} + i \tan \lambda \frac{1}{p(z)}} \\ &\leq \frac{(1-\alpha) \frac{2r}{1-r^2}}{U |\tan \lambda|}. \end{aligned}$$

Using Lemma 2.3 and (2.6), we have the following results.

$$(2.7) \quad \left| 1 + z \frac{s''(z)}{s'(z)} - z \frac{s'(z)}{s(z)} \right| \leq \frac{2(1-\alpha)r}{(1-r)^2 |\tan \lambda|}.$$

And by Lemma 2.3 and (2.7), we get

$$\begin{aligned} (2.8) \quad & \left| 1 + z \frac{s''(z)}{s'(z)} - \frac{1 + \{2(1-\alpha) \cos \lambda e^{i\lambda} - 1\} r^2}{1-r^2} \right| \\ &\leq \frac{2(1-\alpha)r \{1+r+(1-r)|\sin \lambda|\} \cos \lambda}{(1-r)^2(1+r)|\sin \lambda|}. \end{aligned}$$

(ii) for  $\lambda = 0$ , from (2.4) we get

$$(2.9) \quad \frac{zs'(z)}{s(z)} - \alpha = (1 - \alpha)p(z).$$

Using Lemma 2.2 and (2.9), by similar method as  $\lambda \neq 0$ ,

$$(2.10) \quad \left| 1 + z \frac{s''(z)}{s'(z)} - z \frac{s'(z)}{s(z)} \right| \leq \frac{2r(1 - \alpha)}{\{(1 - \alpha)(1 + r) + \alpha(1 - r)\}(1 - r)}.$$

From Lemma 2.1 ( $\lambda = 0$ ), we get

$$(2.11) \quad \left| 1 + z \frac{s''(z)}{s'(z)} - \frac{1 + (1 - 2\alpha)r^2}{1 - r^2} \right| \leq \frac{4r(1 - \alpha)\{1 + (1 - \alpha)r\}}{(1 - r^2)\{(1 - \alpha)(1 + r) + \alpha(1 - r)\}}.$$

**Theorem 2.5.** If  $s(z) \in S_\lambda^*(\alpha)$  ( $\lambda \neq 0$ ), then  $s(z)$  is convex in  $|z| < R(\lambda, \alpha)$ , where  $R(\lambda, \alpha)$  is the smallest positive root of the equation

$$(2.12) \quad \begin{aligned} T(r) = & r^3 |\sin \lambda| \{2(1 - \alpha) \cos^2 \lambda - 1\} - r^2 [2(1 - \alpha) \{\cos^2 \lambda |\sin \lambda| \\ & - (1 - |\sin \lambda|) \cos \lambda\} - |\sin \lambda|] + r \{|\sin \lambda| \\ & + 2(1 - \alpha)(1 + |\sin \lambda|) \cos \lambda\} - |\sin \lambda|, \end{aligned}$$

the result is sharp.

**Proof.** From Lemma 2.4, we obtain

$$(2.13) \quad \begin{aligned} & \operatorname{Re} \left( 1 + z \frac{s''(z)}{s'(z)} \right) \\ & \geq \frac{-r^3 |\sin \lambda| \{2(1 - \alpha) \cos^2 \lambda - 1\}}{(1 - r)^2(1 + r) |\sin \lambda|} \\ & \quad + \frac{r^2 [2(1 - \alpha) \{\cos^2 \lambda |\sin \lambda| - (1 - |\sin \lambda|) \cos \lambda\} - |\sin \lambda|]}{(1 - r)^2(1 + r) |\sin \lambda|} \\ & \quad - \frac{r \{|\sin \lambda| + 2(1 - \alpha)(1 + |\sin \lambda|) \cos \lambda\} + |\sin \lambda|}{(1 - r)^2(1 + r) |\sin \lambda|}. \end{aligned}$$

Since  $T(0) < 0$  and  $T(1) > 1$ , there exists a real root of  $T(r) = 0$  in  $(0, 1)$ . Let  $R(\lambda, \alpha)$  be the smallest positive root of  $T(r) = 0$  in  $(0, 1)$ . Then  $s(z)$  is convex in  $|z| < R(\lambda, \alpha)$ . Sharpness is attained for the function,

$$(2.14) \quad s(z) = \frac{z}{(1-z)^{2(1-\alpha)} \cos \lambda \exp(-i\lambda)}.$$

**Remark 1.** In the case  $\lambda = 0$ , from Lemma 2.4(i) we get

$$(2.15) \quad \begin{aligned} & \operatorname{Re} \left( 1 + z \frac{s''(z)}{s'(z)} \right) \\ & \geq \frac{1 + (1-2\alpha)r^2}{1-r^2} - \frac{4r(1-\alpha)\{1+r-\alpha r\}}{\{(1+r)(1-\alpha) + \alpha(1-r)\}(1-r^2)} \\ & = \frac{(1-2\alpha)^2 r^3 - (4\alpha^2 - 6\alpha + 3)r^2 + (2\alpha - 3)r + 1}{(1-r^2)\{(1-\alpha)(1+r) + \alpha(1-r)\}}. \end{aligned}$$

We have  $s(z)$  is convex in  $|z| < R(\alpha)$ , where  $R(\alpha)$  is the smallest positive root of the equation

$$(2.16) \quad T(r) = (1-2\alpha)^2 r^3 - (4\alpha^2 - 6\alpha + 3)r^2 + (2\alpha - 3)r + 1.$$

**Remark 2.** If  $\alpha = 0$  in (2.16), we get  $r = 2 - \sqrt{3}$ . This result is obtained by R. J. Libera [2].

**Theorem 2.6.** If  $s(z) \in S_\lambda^*(\alpha)$  ( $\lambda \neq 0$ ), then  $s(z)$  is  $\beta$ -convex in  $|z| < R(\lambda, \alpha, \beta)$ , where  $R(\lambda, \alpha, \beta)$  is the smallest positive root of the equation

$$(2.17) \quad \begin{aligned} T(r) = & r^3 |\sin \lambda| \{2(1-\alpha) \cos^2 \lambda - 1\} - r^2 [2(1-\alpha) \{\cos^2 \lambda |\sin \lambda| \\ & - (\beta - |\sin \lambda|) \cos \lambda\} - |\sin \lambda|] \\ & + r \{2(1-\alpha) \cos \lambda (\beta + |\sin \lambda|) + |\sin \lambda|\} - |\sin \lambda|, \end{aligned}$$

the result is sharp.

**Proof.** From inequatlity (2.1) we have

$$(2.18) \quad \operatorname{Re} z \frac{s'(z)}{s(z)} \geq \frac{\{2(1-\alpha) \cos^2 \lambda - 1\} r^2 - 2(1-\alpha) r \cdot \cos \lambda + 1}{1-r^2}.$$

Let  $0 \leq \beta \leq 1$ . If we multiply both sides of (2.18) by  $(1-\beta)$  and of (2.13) by  $\beta$

$$(2.19) \quad \begin{aligned} & \operatorname{Re} J(\beta, s(z)) \\ & \geq \frac{-r^3 |\sin \lambda| \{2(1-\alpha) \cos^2 \lambda - 1\} + r^2 [2(1-\alpha) \{\cos^2 \lambda |\sin \lambda| \\ & - (\beta - |\sin \lambda|) \cos \lambda\} - |\sin \lambda|] - r \{2(1-\alpha) \cos \lambda (\beta + |\sin \lambda|) \\ & + |\sin \lambda|\} + |\sin \lambda|}{(1-r)^2 (1+r) |\sin \lambda|}. \end{aligned}$$

Since  $T(0) < 0$  and  $T(1) > 0$ , there exist a real root of  $T(r) = 0$  in  $(0, 1)$ . Let  $R(\lambda, \alpha, \beta)$  be the smallest positive root  $T(r) = 0$  in  $(0, 1)$ . Then  $s(z)$  is  $\beta$ -convex in  $|z| < R(\lambda, \alpha, \beta)$ . We obtain sharp for the extremal function is given by (2.14).

**Corollary 2.7.** *If  $\beta = 1$ , then we obtain the radius of convexity for the class of  $\lambda$ -spirallike of order  $\alpha$  functions which is given in Theorem 2.5.*

**Corollary 2.8.** *If  $\beta = 0$ , then*

$$r = \frac{(1-\alpha) \cos \lambda - \sqrt{1 - (1-\alpha^2) \cos^2 \lambda}}{2(1-\alpha) \cos^2 \lambda - 1}.$$



**Remark 3.** If  $\alpha = 0$  in Corollary 2.8, then  $r = \frac{1}{|\sin \lambda| + \cos \lambda}$ .

It is the radius of starlikeness for  $\lambda$ -spirallike functions, which was obtained by M. S. Rebertson [10] and R. J. Libera [2].

### 3. The radius of $\beta$ -convexity for $p$ -fold $\lambda$ -spirallike functions

**Theorem 3.1.** If  $f(z) \in S_{\lambda p}^*(\alpha)$  ( $\lambda \neq 0$ ), then  $f(z)$  is  $\beta$ -convex in  $|z| < R(\lambda, \alpha, \beta, p)$ , where  $R(\lambda, \alpha, \beta, p)$  is the smallest positive root of the equation

$$(3.1) \quad \begin{aligned} T(r) = & r^{3p} |\sin \lambda| \{2(1 - \alpha) \cos^2 \lambda - 1\} - r^{2p} [2(1 - \alpha) \{\cos^2 \lambda |\sin \lambda| \\ & - (\beta p - |\sin \lambda|) \cos \lambda\} - |\sin \lambda|] + r^p \{2(1 - \alpha) \cos \lambda (\beta p \\ & + |\sin \lambda|) + |\sin \lambda|\} - |\sin \lambda|. \end{aligned}$$

**Proof.** From the relation (1.8) we obtain

$$1 + z^p \frac{s''(z^p)}{s'(z^p)} = \frac{1}{p} \left( 1 + z \frac{f'(z)}{f(z)} \right) + \left( 1 - \frac{1}{p} \right) z \frac{f'(z)}{f(z)}.$$

From a simple calculation of (1.8), (2.13) and (2.16) we obtain

$$(3.2) \quad \begin{aligned} & \operatorname{Re} J \left( \frac{1}{p}, f(z) \right) \\ & \geq \frac{-r^{3p} |\sin \lambda| \{2(1 - \alpha) \cos^2 \lambda - 1\}}{(1 - r^p)^2 (1 + r^p) |\sin \lambda|} \\ & \quad + \frac{r^{2p} [2(1 - \alpha) \{\cos^2 \lambda |\sin \lambda| - (1 - |\sin \lambda|) \cos \lambda\} - |\sin \lambda|]}{(1 - r^p)^2 (1 + r^p) |\sin \lambda|} \\ & \quad - \frac{r^p \{|\sin \lambda| + 2(1 - \alpha)(1 + |\sin \lambda|) \cos \lambda\} + |\sin \lambda|}{(1 - r^p)^2 (1 + r^p) |\sin \lambda|}, \end{aligned}$$

$$(3.3) \quad \operatorname{Re} z \frac{f'(z)}{f(z)} \geq \frac{\{2(1-\alpha) \cos^2 \lambda - 1\} r^{2p} - 2(1-\alpha) r^p \cos \lambda + 1}{1 - r^{2p}}.$$

If we multiply both sides of (3.2) by  $\gamma$  and (3.3) by  $1 - \gamma$ , then add the corresponding members, we obtain

$$(3.4) \quad \begin{aligned} & \operatorname{Re} J\left(\frac{\gamma}{p}, f(z)\right) \\ & \geq \frac{-r^{3p} |\sin \lambda| \{2(1-\alpha) \cos^2 \lambda - 1\}}{(1-r^p)^2(1+r^p) |\sin \lambda|} \\ & \quad + \frac{r^{2p} [2(1-\alpha) \{\cos^2 \lambda |\sin \lambda| - (\gamma - |\sin \lambda|) \cos \lambda\} - |\sin \lambda|]}{(1-r^p)^2(1+r^p) |\sin \lambda|} \\ & \quad - \frac{r^p \{2(1-\alpha) \cos \lambda (\gamma + |\sin \lambda|) + |\sin \lambda|\} + |\sin \lambda|}{(1-r^p)^2(1+r^p) |\sin \lambda|} \end{aligned}$$

where  $0 \leq \gamma \leq 1$ . If we take  $\frac{\gamma}{p} = \beta$  the inequality (3.2) can be written in the form

$$(3.5) \quad \begin{aligned} & \operatorname{Re} J(\beta, f(z)) \\ & \geq \frac{-r^{3p} |\sin \lambda| \{2(1-\alpha) \cos^2 \lambda - 1\}}{(1-r^p)^2(1+r^p) |\sin \lambda|} \\ & \quad + \frac{r^{2p} [2(1-\alpha) \{\cos^2 \lambda |\sin \lambda| - (\beta p - |\sin \lambda|) \cos \lambda\} - |\sin \lambda|]}{(1-r^p)^2(1+r^p) |\sin \lambda|} \\ & \quad - \frac{r^p \{2(1-\alpha) \cos \lambda (\beta p + |\sin \lambda|) + |\sin \lambda|\} + |\sin \lambda|}{(1-r^p)^2(1+r^p) |\sin \lambda|} \end{aligned}$$

where  $0 \leq \beta \leq 1$ .

Since  $T(0) < 0$  and  $T(1) > 0$ , there exist a real root of  $T(r) = 0$  in  $(0, 1)$ . Let  $R(\lambda, \alpha, \beta, p)$  be the smallest positive root  $T(r) = 0$  in  $(0, 1)$ . Then  $f(z)$  is  $\beta$ -convex in  $|z| < R(\lambda, \alpha, \beta, p)$ . We obtain sharp because the extremal  $f(z) = z/(1 - z^p)^{2(1-\alpha) \cos \lambda \exp(-i\lambda)/p}$ . This shows that the theorem is true.

**Corollary 3.2.** If  $p = 1$ , then we obtain the radius of  $\beta$ -convexity for the class of  $\lambda$ -spirallike of order  $\alpha$  functions which is given in Theorem 2.5.

**Corollary 3.3.** If  $\alpha = 0$ , then we obtain the radius of  $\beta$ -convexity for the class of  $\lambda$ -spirallike functions.

**Corollary 3.4.** For  $\beta = 0$  we obtain  $r = \sqrt[p]{\frac{(1-\alpha)\cos\lambda - \sqrt{1-(1-\alpha)\cos^2\lambda}}{2(1-\alpha)\cos^2\lambda - 1}}$ .

This is the radius of starlikeness for the  $p$ -fold  $\lambda$ -spirallike function. If we take  $p = 1$ ,  $\alpha = 0$  and  $\beta = 0$ , we obtain  $r = (|\sin\lambda| + \cos\lambda)^{-1}$ , which was obtained by M. S. Roberston [8] and R. J. Libera [2].

**Corollary 3.5.** In the case  $\lambda = 0$ , we obtain the radius of  $\beta$ -convexity for the class of  $p$ -fold starlike of order  $\alpha$  functions. If we take  $\alpha = 0$  we obtain the radius of  $\beta$ -convexity for the class of  $p$ -fold starlike functions.

**Corollary 3.6.** For  $p = 1$ ,  $\beta = 0$ ,  $\lambda = 0$  and  $\alpha = 0$ , we obtain  $r = 2 - \sqrt{3}$ , the radius obtained by R. J. Libera [2].

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