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On the transfer map for the Hochschild cohomology of Frobenius algebras

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1 Introduction

We describe a transfer map between the complete Hochschild cohomologies of Frobenius algebras Λ and Γ. For a Frobenius algebra Λ over a commutative ring R which is finitely generated projective R-module, we can define a complete Hochschild cohomology $H^i(\Lambda, M)$ with a coefficient Λ-bimodule M (see [Na]). If, in addition, we assume that Γ is a Frobenius extension of Λ, then Γ is a Frobenius R-algebra. Under this assumption, we can define $\mathrm{Res} : H^r(\Gamma, {}_{\Gamma}M_{\Gamma}) \to H^r(\Lambda, M)$ and $\mathrm{Cor} : H^r(\Lambda, {}_{\Gamma}M_{\Gamma}) \to H^r(\Gamma, M)$. Res for $r \geqslant 0$ and Cor for $r \leqslant -1$ are defined naturally, and, particularly for Frobenius algebras, Res and Cor can also be defined for other integers r, which we may call them the transfer maps (see [S3], [S4], and also [No1], [No2]).

In this summary, we show the explicit description of Res and Cor by means of the standard resolutions of the Frobenius algebras above.

2 Complete Hochschild cohomology of Frobenius algebras

Let R be a commutative ring with identity, Λ an R-algebra which is finitely generated projective as R-module. $\Lambda^e = \Lambda \otimes_R \Lambda^{\mathrm{opp}}$ denotes the enveloping algebra of Λ and $Z\Lambda$ denotes the center of Λ.

If M is a left Λ^e-module (i.e. Λ-bimodule), we define the Hochschild cohomology of Λ with coefficient module M:

$$H^n(\Lambda, M) = \mathrm{Ext}^n_{\Lambda^e}(\Lambda, M) \quad (n \geqslant 0).$$

It is easily verified that this is a $Z\Lambda$-module.

We denote $H^n(\Lambda, \Lambda)$ by $HH^n(\Lambda)$ in the following. By the definition, we see that

$$H^0(\Lambda, M) \cong M^{\Lambda} = \{x \in M \mid ax = xa \text{ for any } a \in \Lambda\}$$

and so we have $HH^0(\Lambda) = Z\Lambda$.

2.1 Standard resolution, cup product

Let $n \geqslant 0$ be an integer, and we put $X_n = \Lambda \otimes \cdots \otimes \Lambda$ ($n+2$-times tensor products over R). Then we have the following Λ^e-projective resolution of Λ which is called the standard resolution of Λ:

$$\cdots \longrightarrow X_{n+1} \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} X_{n-1} \longrightarrow \cdots \longrightarrow X_1 \xrightarrow{d_1} X_0 \xrightarrow{d_0} \Lambda \longrightarrow 0,$$

$$d_n(x_0 \otimes x_1 \otimes \cdots \otimes x_n \otimes x_{n+1}) = \sum_{i=0}^{n} (-1)^i x_0 \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots \otimes x_{n+1},$$

$$d_1(x_0 \otimes x_1 \otimes x_2) = x_0 x_1 \otimes x_2 - x_0 \otimes x_1 x_2,$$

$$d_0(x_0 \otimes x_1) = x_0 x_1$$

We can define the cup product $H^i(\Lambda, M) \otimes H^j(\Lambda, N) \xrightarrow{\smile_{i,j}} H^{i+j}(\Lambda, M \otimes_\Lambda N)$, which satisfies the anti-commutativity:

$$\alpha \smile_{i,j} \beta = (-1)^{ij} \beta \smile_{j,i} \alpha \quad \text{for } \alpha \in HH^i(\Lambda), \beta \in HH^j(\Lambda, M).$$

and the cup product $Z\Lambda \otimes H^i(\Lambda, M) \xrightarrow{\smile_{0,i}} H^i(\Lambda, M)$ gives the $Z\Lambda$-module structure for $H^i(\Lambda, M)$.

Furthermore, we have $HH^i(\Lambda) \otimes HH^j(\Lambda) \xrightarrow{\smile} HH^{i+j}(\Lambda)$, so this makes

$$HH^*(\Lambda) := \bigoplus_{k \geqslant 0} HH^k(\Lambda)$$

a ring containing $HH^0(\Lambda) = Z\Lambda$ as a subring, which is called the Hochschild cohomology ring of Λ.

2.2 Frobenius extensions and Frobenius algberas

Let Γ/Λ be a Frobenius extension. That is,

$$\Gamma = a_1\Lambda \oplus \cdots \oplus a_m\Lambda = \Lambda b_1 \oplus \cdots \oplus \Lambda b_m;$$

$$x a_i = \sum_{j=1}^{m} a_j \beta_{ji}(x), \quad b_j x = \sum_{i-1}^{m} \beta_{ji}(x) b_i \quad (x \in \Gamma, \beta_{ji}(x) \in \Lambda)$$

and there exist the following isomorphisms:

$$\phi_{\Gamma/\Lambda} : {}_\Gamma\Gamma_\Lambda \xrightarrow{\sim} \mathrm{Hom}_{\Lambda,-}(\Gamma_\Gamma, \Lambda_\Lambda), \quad \phi_{\Gamma/\Lambda}(a_i)(b_j) = \delta_{ij},$$

$$\phi'_{\Gamma/\Lambda} : {}_\Lambda\Gamma_\Gamma \xrightarrow{\sim} \mathrm{Hom}_{-,\Lambda}({}_\Gamma\Gamma, {}_\Lambda\Lambda), \quad \phi'_{\Gamma/\Lambda}(b_j)(a_i) = \delta_{ij}.$$

We set

$$\mu_{\Gamma/\Lambda} = \phi_{\Gamma/\Lambda}(1), \quad N_{\Gamma/\Lambda}(x) = \sum_{i=1}^{m} a_i x b_i \quad (x \in \Gamma).$$

Then $\mu_{\Gamma/\Lambda} : \Gamma \to \Lambda$ is a two-sided Λ-module homomorphism and

$$x = \sum_{i-1}^{m} \mu_{\Gamma/\Lambda}(xa_i)b_i = \sum_{j=1}^{m} a_j \mu_{\Gamma/\Lambda}(b_j x) \quad (x \in \Gamma).$$

Furthermore, let R be a commutative ring and Λ a Frobenius R-algebra which is finitely generated free R-module:

$$\Lambda = u_1 R \oplus \cdots \oplus u_n R = Rv_i \oplus \cdots \oplus Rv_n;$$
$$yu_i = \sum_{j=1}^{n} u_j \alpha_{ji}(y), \quad v_j y = \sum_{i=1}^{n} \alpha_{ji}(y) v_i \quad (y \in \Lambda, \alpha_{ji}(y) \in R),$$
$$\phi_\Lambda : {}_\Lambda\Lambda \xrightarrow{\sim} \mathrm{Hom}_R(\Lambda_\Lambda, R), \quad \phi_\Lambda(u_i)(v_j) = \delta_{ij}.$$

We set

$$\mu_\Lambda = \phi_\Lambda(1), \quad N_\Lambda(y) = \sum_{i=1}^{n} u_i y v_i,$$
$$y^{\tau'} = \sum_{i=1}^{n} \mu_\Lambda(u_i y) v_i \text{ (Nakayama automorphism of } \Lambda/R).$$

Then Γ is a Frobenius R-algebra of rank mn with R-bases $(a_i u_j), (v_j b_i)$ $(1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n)$:

$$xa_i u_j = \sum_{k-1}^{m} \sum_{l=1}^{n} a_k u_l \beta_{ij}(\alpha_{ki}(x)),$$
$$v_l b_k x = \sum_{i=1}^{m} \sum_{j=1}^{n} \beta_{ij}(\alpha_{ki}(x)) v_j b_i \quad (x \in \Gamma),$$
$$\phi_\Gamma : {}_\Gamma\Gamma \xrightarrow{\sim} \mathrm{Hom}_R(\Gamma_\Gamma, R), \quad \phi_\Gamma(a_i u_j)(v_l b_k) = \delta_{(i,j),(k,l)}.$$

If we set $\mu_\Gamma = \phi_\Gamma(1)$, then

$$x = \sum_{i=1}^{m} \sum_{j=1}^{n} \mu_\Gamma(xa_i u_j) v_j b_i = \sum_{k=1}^{m} \sum_{l=1}^{n} a_k u_l \mu_\Gamma(v_l b_k x) \quad (x \in \Gamma).$$

We set

$$N_\Gamma(x) := \sum_{i=1}^{m} \sum_{j=1}^{n} a_i u_j x v_j b_i,$$
$$x^\tau := \sum_{i-1}^{m} \sum_{j=1}^{n} \mu_\Gamma(a_i u_j x) v_j b_i \quad (x \in \Gamma).$$

Then we have

$$N_{\Gamma/\Lambda} \circ N_\Lambda = N_\Gamma, \quad \mu_\Lambda \circ \mu_{\Gamma/\Lambda} = \mu_\Gamma, \quad \tau|_\Lambda = \tau'.$$

3 Restriction and corestriction maps

We set

$$(X_\Gamma)_p = \Gamma \otimes_R \cdots \otimes_R \Gamma \quad (p+2 \text{ times tensor products of } \Gamma),$$
$$(X_\Lambda)_p = \Lambda \otimes_R \cdots \otimes_R \Lambda \quad (p+2 \text{ times tensor products of } \Lambda),$$

and we define

$$\begin{aligned} &d_p : (X_\Gamma)_p \longrightarrow (X_\Gamma)_{p-1}, \\ &d_p(x_0 \otimes x_1 \otimes \cdots \otimes x_p \otimes x_{p+1}) \\ &\quad = x_0x_1 \otimes \cdots \otimes x_p \otimes x_{p+1} \\ &\qquad + \sum_{i=1}^{p-1}(-1)^r x_0 \otimes \cdots \otimes x_ix_{i+1} \otimes \cdots \otimes x_{p+1} + (-1)^p x_0 \otimes \cdots \otimes x_{p-1} \otimes x_px_{p+1}. \end{aligned}$$

Then we have the following commutative diagram:

$$\begin{tikzcd} \cdots & \mathrm{Hom}_{\Gamma^e}((X_\Gamma)_1, M) \arrow[l] \arrow[d, "\mathrm{res}^1"] & M \arrow[l, "d_1^*"'] \arrow[d, "\mathrm{res}^0"] & M \arrow[l, "N_\Gamma"'] \arrow[d, "\mathrm{res}_0"] & (X_\Gamma)_1^\tau \otimes_{\Gamma^e} M \arrow[l, "d_1 \otimes \iota"'] \arrow[d, "\mathrm{res}_1"] & \cdots \arrow[l] \\ \cdots & \mathrm{Hom}_{\Lambda^e}((X_\Lambda)_1, M) \arrow[l] & M \arrow[l, "d_1^*"'] & M \arrow[l, "N_\Lambda"'] & (X_\Lambda)_1^{\tau'} \otimes_{\Lambda^e} M \arrow[l, "d_1 \otimes \iota"'] & \cdots . \arrow[l] \end{tikzcd}$$

Here, $(X_\Gamma)_p^\tau$ is defined by $w(x \otimes y^{opp}) = y^{\tau^{-1}}wx$ for $w \in (X_\Gamma)_p$, $x \otimes y^{opp} \in \Gamma^e$, and

$$\mathrm{res}^0 : M \to M, x \mapsto x, \quad \mathrm{res}_0 : M \to M, x \mapsto \sum_{i-1}^{m} b_ixa_i^\tau,$$

$$\begin{aligned} &\mathrm{res}_q : (X_\Gamma)_q^\tau \otimes_{\Gamma^e} M \to (X_\Lambda)_q^{\tau'} \otimes_{\Lambda^e} M, 1 \otimes y_1 \otimes \cdots \otimes y_q \otimes 1 \otimes_{\Gamma^e} x \mapsto \\ &\sum_{i_1,\dots,i_{q+1}=1}^{m} 1 \otimes \mu_{\Gamma/\Lambda}(b_{i_1}y_1a_{i_2}) \otimes \cdots \otimes \mu_{\Gamma/\Lambda}(b_{i_q}y_qa_{i_{q+1}}) \otimes 1 \otimes_{\Lambda^e} b_{i_{q+1}}xa_{i_1}^\tau, \end{aligned}$$

res^p ($p \geqslant 1$) is defined to be a natural homomorphism induced by $(X_\Lambda)_p \to (X_\Gamma)_p$. Then we have

$$\mathrm{Res}^r : H^r(\Gamma, {}_\Gamma M_\Gamma) \to H^r(\Lambda, M) \quad (r \in \mathbb{Z}).$$

Here, $H^r(\Gamma, -)$ and $H^r(\Lambda, -)$ denotes the complete Hochschild cohomology of Γ and Λ, respectively, and these are obtained by the horizontal sequences.

On the other hand, we have the following commutative diagram:

$$\begin{tikzcd} \cdots & \mathrm{Hom}_{\Gamma^e}((X_\Gamma)_1, M) \arrow[l] & M \arrow[l, "d_1^*"'] & M \arrow[l, "N_\Gamma"'] & (X_\Gamma)_1^\tau \otimes_{\Gamma^e} M \arrow[l, "d_1 \otimes \iota"'] & \cdots \arrow[l] \\ \cdots & \mathrm{Hom}_{\Lambda^e}((X_\Lambda)_1, M) \arrow[l] \arrow[u, "\mathrm{cor}^1"] & M \arrow[l, "d_1^*"'] \arrow[u, "\mathrm{cor}^0"] & M \arrow[l, "N_\Lambda"'] \arrow[u, "\mathrm{cor}_0"] & (X_\Lambda)_1^{\tau'} \otimes_{\Lambda^e} M \arrow[l, "d_1 \otimes \iota"'] \arrow[u, "\mathrm{cor}_1"] & \cdots . \arrow[l] \end{tikzcd}$$

Here,

$$\mathrm{cor}_0 : M \to M, x \mapsto x, \quad \mathrm{cor}^0 : M \to M, x \mapsto N_{\Gamma/\Lambda}(x),$$
$$\mathrm{cor}^p : \mathrm{Hom}_{\Lambda^e}((X_\Lambda)_p, M) \to \mathrm{Hom}_{\Gamma^e}((X_\Gamma)_p, M),$$
$$\mathrm{cor}^p(g)(y_0 \otimes y_1 \otimes \cdots \otimes y_p \otimes y_{p+1})$$
$$= \sum_{i_1,\ldots,i_{p+1}=1}^{m} y_0 a_{i_1} g\left(1 \otimes \mu_{\Gamma/\Lambda}(b_{i_1} y_1 a_{i_2}) \otimes \cdots \otimes \mu_{\Gamma/\Lambda}(b_{i_p} y_p a_{i_{p+1}}) \otimes 1\right) b_{i_{p+1}} y_{p+1},$$

and cor_q ($q \geqslant 1$) is defined to be a natural homomorphism induced by $(X_\Lambda)_q \to (X_\Gamma)_q$. Then we have

$$\mathrm{Cor}^r : H^r(\Lambda, {}_\Gamma M_\Gamma) \to H^r(\Gamma, M) \quad (r \in \mathbb{Z}).$$

Proposition We have following fundamental properties for Res and Cor.

(1) Given $f : {}_{\Gamma^e}M \to {}_{\Gamma^e}N$, we have

$$f^*\mathrm{Res}^r = \mathrm{Res}^r f^* : H^r(\Gamma, M) \to H^r(\Lambda, N),$$
$$f^*\mathrm{Cos}^r = \mathrm{Cor}^r f^* : H^r(\Lambda, M) \to H^r(\Gamma, N).$$

(2) Given a short exact sequence $0 \to L \to M \to N \to 0$ of Γ^e-modules, we have

$$\partial\mathrm{Res}^r = \mathrm{Res}^{r+1}\partial : H^r(\Gamma, N) \to H^{r+1}(\Lambda, L),$$
$$\partial\mathrm{Cor}^r = \mathrm{Cor}^{r+1}\partial : H^r(\Lambda, N) \to H^{r+1}(\Gamma, L).$$

(3) Given a Γ^e-module M, we have

$$\mathrm{Cor}^r\mathrm{Res}^r(w) = N_{\Gamma/\Lambda}(1)w \quad (w \in H^r(\Gamma, M)).$$

Since Res preserves the cup product, it follows that we can define a ring homomorphism $HH^*(\Gamma) \longrightarrow H^*(\Lambda, \Gamma)$. Moreover, using the embedding of Λ-bimodules $\Lambda \to \Gamma$, we can define

$$HH^*(\Lambda) \longrightarrow H^*(\Lambda, \Gamma) \xrightarrow{\mathrm{Cor}} HH^*(\Gamma).$$

In particular, we have $Z\Lambda/N_\Lambda(\Lambda) \longrightarrow Z\Gamma/N_\Gamma(\Gamma) : \overline{z} \mapsto \overline{N_{\Gamma/\Lambda}(z)}$ in the zero dimension. Note that the zero dimensional complete Hochschild cohomology is different from the ordinary one (cf. [Br]).

On the other hand, using the Λ-bimodule homomorphism $\mu_{\Gamma/\Lambda} : \Gamma \to \Lambda$, we have

$$HH^*(\Gamma) \xrightarrow{\mathrm{Res}} H^*(\Lambda, \Gamma) \xrightarrow{\mu_{\Gamma/\Lambda}} HH^*(\Lambda).$$

In particular, we have $Z\Gamma/N_\Gamma(\Gamma) \longrightarrow Z\Lambda/N_\Lambda(\Lambda) : \overline{z} \mapsto \overline{\mu_{\Gamma/\Lambda}(z)}$ in the zero dimension.

We have some explicit calculations of Res and Cor for twisted group algebras and crossed products (see [S1] and [S2] for twisted group algebras, and [S4] for crossed products).

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