

A universal framework of the homogenization problem of infinite dimensional diffusions

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Abstract

By generalizing the concrete formulations in [ABRY1,2], A universal framework of the homogenization problem of infinite dimensional diffusions is proposed. The corresponding general structure is considered.

1 Probability space $(\Theta, \bar{\mathcal{B}}, \bar{\mu})$ on which the random coefficients are defined

Suppose that we are given the following:

$\{(\Theta_{\mathbf{k}}, \mathcal{B}_{\mathbf{k}}, \lambda_{\mathbf{k}})\}_{\mathbf{k} \in \mathbb{Z}^d}$: a system of complete probability spaces, where d is a given natural number.

$(\Theta, \bar{\mathcal{B}}, \bar{\lambda})$: the probability space that is the completion of $(\prod_{\mathbf{k}} \Theta_{\mathbf{k}}, \otimes_{\mathbf{k}} \mathcal{B}_{\mathbf{k}}, \prod_{\mathbf{k}} \lambda_{\mathbf{k}})$, i.e., the completion of the direct product probability space.

$(\Theta, \bar{\mathcal{B}}, \mu)$: a complete probability space (corresponding to a Gibbs state) defined as follows: for $\forall D \subset \subset \mathbb{Z}^d$ and for any bounded measurable function φ defined on $\prod_{\mathbf{k} \in D'} \Theta_{\mathbf{k}}$ with some $\forall D' \subset \subset \mathbb{Z}^d$, μ satisfies

$$(\mathbb{E}^D \varphi, \mu) = (\varphi, \mu),$$

where

$$\begin{aligned} (\mathbb{E}^D \varphi)(\theta) &\equiv \int_{\Theta} \mathbb{E}^D(d\theta' | \theta) \\ &\equiv \int_{\Theta} \varphi(\theta'_D \cdot \theta_{D^c}) m_D(\theta'_D \cdot \theta_{D^c}) \bar{\lambda}(d\theta'), \end{aligned}$$

with

$$m_D(\theta'_D \cdot \theta_{D^c}) \equiv \frac{1}{Z_D(\theta_{D^c})} e^{-U_D(\theta'_D \cdot \theta_{D^c})}, \quad U_D \equiv \sum_{\mathbf{k} \in D} U_{\mathbf{k}},$$

$$\Theta \ni \theta \longmapsto \theta_D \in \prod_{\mathbf{k} \in D} \Theta_{\mathbf{k}}$$

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is the natural projection, $\theta'_D \cdot \theta_{D^c}$ is the element $\theta'' \in \Theta$ such that

$$\theta''_D = \theta'_D, \quad \theta''_{D^c} = \theta_{D^c},$$

also, for each $\mathbf{k} \in \mathbb{Z}^d$, $U_{\mathbf{k}}$ is a given bounded measurable function of which support is in $\prod_{|\mathbf{k}' - \mathbf{k}| \leq L} \Theta_{\mathbf{k}'}$, where the number L (the range of interactions) does not depend on $\mathbf{k}$, and $Z_D(\theta_{D^c})$ is the normalizing constant.

2 The ergodic flow

On $(\Theta, \bar{\mathcal{B}}, \bar{\lambda})$ we are given an ergodic flow $T_{\mathbf{x}}$ (which is also a map on $(\Theta, \bar{\mathcal{B}}, \mu)$, but is not a measure preserving map on it) as follows:

Suppose that

$$\exists M_1 < \infty \quad \text{and} \quad \forall \mathbf{k} \in \mathbb{Z}^d \quad \text{there exists a } d_{\mathbf{k}} \text{ such that } d_{\mathbf{k}} < M_1. \quad (2.1)$$

For each $\mathbf{x} \in \prod_{\mathbf{k}} \mathbb{R}^{d_{\mathbf{k}}}$ such that $\mathbf{x} = (\mathbf{x}^{\mathbf{k}})_{\mathbf{k} \in \mathbb{Z}^d}$ with $\mathbf{x}^{\mathbf{k}} = (x_1^{\mathbf{k}}, \dots, x_{d_{\mathbf{k}}}^{\mathbf{k}})$ the map $T_{\mathbf{x}}$ on $(\Theta, \bar{\mathcal{B}}, \bar{\lambda})$ is defined by

i)

$$T_{\mathbf{x}} : \Theta \longrightarrow \Theta$$

that is a measure preserving transformation with respect to the measure $\bar{\lambda}$;

ii)

$$T_{\mathbf{0}} = \text{the identity,}$$

$$\text{for } \mathbf{x}, \mathbf{x}' \in \mathbf{x} \in \prod_{\mathbf{k} \in \mathbb{Z}^d} \mathbb{R}^{d_{\mathbf{k}}} \quad T_{\mathbf{x} + \mathbf{x}'} = T_{\mathbf{x}} \circ T_{\mathbf{x}'},$$

where

$$\mathbf{x} + \mathbf{x}' \equiv (\mathbf{x}^{\mathbf{k}} + \mathbf{x}'^{\mathbf{k}})_{\mathbf{k} \in \mathbb{Z}^d},$$

with

$$\mathbf{x}^{\mathbf{k}} + \mathbf{x}'^{\mathbf{k}} = (x_1^{\mathbf{k}} + x_1'^{\mathbf{k}}, \dots, x_{d_{\mathbf{k}}}^{\mathbf{k}} + x_{d_{\mathbf{k}}}'^{\mathbf{k}}),$$

for

$$\begin{aligned} \mathbf{x} &= (\mathbf{x}^{\mathbf{k}})_{\mathbf{k} \in \mathbb{Z}^d}, & \mathbf{x}^{\mathbf{k}} &= (x_1^{\mathbf{k}}, \dots, x_{d_{\mathbf{k}}}^{\mathbf{k}}), \\ \mathbf{x}' &= (\mathbf{x}'^{\mathbf{k}})_{\mathbf{k} \in \mathbb{Z}^d}, & \mathbf{x}'^{\mathbf{k}} &= (x_1'^{\mathbf{k}}, \dots, x_{d_{\mathbf{k}}}'^{\mathbf{k}}), \end{aligned}$$

and

$$\mathbf{0} \equiv (\mathbf{0}^{\mathbf{k}})_{\mathbf{k} \in \mathbb{Z}^d}, \quad \mathbf{0}^{\mathbf{k}} = (0, \dots, 0) \in \mathbb{R}^{d_{\mathbf{k}}};$$

iii)

$$(\mathbf{x}, \theta) \in \left(\prod_{\mathbf{k} \in \mathbb{Z}^d} \mathbb{R}^{d_{\mathbf{k}}} \right) \times \theta \longrightarrow T_{\mathbf{x}}(\theta) \in \Theta$$

is $\mathcal{B}(\prod_{\mathbf{k} \in \mathbb{Z}^d} \mathbb{R}^{d_{\mathbf{k}}}) \times \bar{\mathcal{B}}/\bar{\mathcal{B}}$ -measurable, where, $\prod_{\mathbf{k} \in \mathbb{Z}^d} \mathbb{R}^{d_{\mathbf{k}}}$ is assumed to be the topological space with the direct product topology;

iv) A function which is $T_{\mathbf{x}}$ invariant for all $\mathbf{x} \in \prod_{\mathbf{k} \in \mathbb{Z}^d} \mathbb{R}^{d_{\mathbf{k}}}$ is a constant function;

v) For $D \subset \mathbb{Z}^d$, let

$$\prod_{\mathbf{k} \in \mathbb{Z}^d} \mathbb{R}^{d_{\mathbf{k}}} \ni \mathbf{x} \longmapsto \mathbf{x}_D \in \prod_{\mathbf{k} \in D} \mathbb{R}^{d_{\mathbf{k}}}$$

be the natural *projection*. If $\mathbf{x}_{D^c} = \mathbf{0}_{D^c}$, then

$$(T_{\mathbf{x}}(\theta))_{D^c} = \theta_{D^c}, \quad \forall \theta \in \Theta, \quad \forall D \subset \subset \mathbb{Z}^d.$$

□

3 The core

We assume that an existence of a *core* $\mathcal{D}$. Namely, there exists $\mathcal{D}$ which is a dense subset of both $L^2(P)$ and $L^1(P)$, and $\forall \varphi \in \mathcal{D}$ satisfies

($\mathcal{D}$ -1) φ is a bounded measurable function having only a finite number of variables θ_D for some $D \subset \subset \mathbb{Z}^d$,

($\mathcal{D}$ -2)

$$\varphi(T_{\mathbf{x}_D}(\theta)) \in C^\infty(\prod_{\mathbf{k} \in D} \mathbb{R}^{d_{\mathbf{k}}} \rightarrow \mathbb{R}), \quad \forall \theta \in \Theta,$$

(cf. v) in the previous section) where we identify $\mathbf{x}_D \in \prod_{\mathbf{k} \in D} \mathbb{R}^{d_{\mathbf{k}}}$ with an $\mathbf{x} \in (\prod_{\mathbf{k} \in \mathbb{Z}^d} \mathbb{R}^{d_{\mathbf{k}}})$ of which projection to $\prod_{\mathbf{k} \in D} \mathbb{R}^{d_{\mathbf{k}}}$ is $\mathbf{x}_D$,

($\mathcal{D}$ -3) in ($\mathcal{D}$ -2) for each $\theta \in \Theta$, all the partial derivatives of all orders of the function $\varphi(T_{\mathbf{x}_D}(\theta))$ (with the variables $\mathbf{x}_D$) are bounded and

$$\forall \varphi \in \mathcal{D}, \exists M < \infty; \quad |\nabla_{\mathbf{k}} \varphi(T_{\mathbf{x}}(\theta))| < M, \quad \forall \theta \in \Theta, \forall \mathbf{x}, \forall \mathbf{k} \in \mathbb{Z}^d, \quad (3.1)$$

where

$$\nabla_{\mathbf{k}} = \left(\frac{\partial}{\partial x_1^{\mathbf{k}}}, \dots, \frac{\partial}{\partial x_{d_{\mathbf{k}}}^{\mathbf{k}}} \right).$$

□

4 Probability space $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$ on which the infinite dimensional diffusions are defined

Suppose that we are given a system of family of functions $a_{ij}^{\mathbf{k}}$, $\mathbf{k} \in \mathbb{Z}^d$, $1 \leq i, j \leq d_{\mathbf{k}}$ on $(\Theta, \overline{\mathcal{B}}, \overline{\mu})$ such that for each $\mathbf{k} \in \mathbb{Z}^d$ and each $1 \leq i, j \leq d_{\mathbf{k}}$, $a_{ij}^{\mathbf{k}}$ is a measurable function on $\Theta_{\mathbf{k}}$ and there exists $M_2 \in (0, \infty)$ and

$$M_2^{-1} \leq \sum_{1 \leq i, j \leq d_{\mathbf{k}}} a_{ij}^{\mathbf{k}}(\theta_{\mathbf{k}}) x_i x_j \leq M_2, \quad \forall \mathbf{k} \in \mathbb{Z}^d, \forall \theta_{\mathbf{k}} \in \Theta_{\mathbf{k}}, \forall (x_1, \dots, x_{d_{\mathbf{k}}}) \in \mathbb{R}^{d_{\mathbf{k}}}. \quad (4.1)$$

We assume that

$$U_{\mathbf{k}}, a_{ij}^{\mathbf{k}} \in \mathcal{D}, \quad \mathbf{k} \in \mathbb{Z}^d, \quad 1 \leq i, j \leq d_{\mathbf{k}}.$$

Also, we assume that there exists a common $M < \infty$ by which the evaluation (3.1) holds for all $a_{i,j}^{\mathbf{k}}$ and $U_{\mathbf{k}}$.

Finally, suppose that we are given a complete probability space $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$, ($t \in \mathbb{R}_+$) with a filtration $\mathcal{F}_t$. On $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$ suppose that there exists a system of independent 1-dimensional $\mathcal{F}_t$ -adapted Brownian motion processes

$$\{(B^{\mathbf{k},i}(t))_{t \geq 0}\}_{\mathbf{k} \in \mathbb{Z}^d, 1 \leq i \leq d_{\mathbf{k}}}.$$

Now, for each $\theta \in \Theta$, let

$$X^\theta \equiv \{(X^{\theta,\mathbf{k},i}(t))_{t \geq 0}\}_{\mathbf{k} \in \mathbb{Z}^d, 1 \leq i \leq d_{\mathbf{k}}}.$$

be the unique solution of

$$\begin{aligned} X^{\theta,\mathbf{k},i}(t) = & X^{\theta,\mathbf{k},i}(0) + \int_0^t \sum_{1 \leq j \leq d_{\mathbf{k}}} \left\{ \frac{\partial}{\partial x_j^{\mathbf{k}}} a_{ij}^{\mathbf{k}}(T_{X^{\theta,\mathbf{k}}(s)}(\theta)) \right. \\ & \left. - a_{ij}^{\mathbf{k}}(T_{X^{\theta,\mathbf{k}}(s)}(\theta)) \left(\frac{\partial}{\partial x_j^{\mathbf{k}}} \left(\sum_{|\mathbf{k}-\mathbf{k}'| \leq L} U_{\mathbf{k}'}(T_{X^{\theta}(s)}(\theta)) \right) \right) \right\} ds \\ & + \int_0^t \sum_{1 \leq j \leq d_{\mathbf{k}}} \sigma_{ij}^{\mathbf{k}}(T_{X^{\theta,\mathbf{k}}(s)}(\theta)) dB^{\mathbf{k},j}(s), \quad t \geq 0, \end{aligned} \quad (4.2)$$

where

$$(\sigma_{ij}^{\mathbf{k}}) = (2a_{ij}^{\mathbf{k}})^{\frac{1}{2}},$$

and

$$X^{\theta,\mathbf{k}}(t) = (X^{\theta,\mathbf{k},1}(t), \dots, X^{\theta,\mathbf{k},d_{\mathbf{k}}}(t)),$$

also, by $X^\theta(t)$ we denote the vector

$$(X^{\theta,\mathbf{k}}(t))_{\mathbf{k} \in \mathbb{Z}^d} \in \prod_{\mathbf{k} \in \mathbb{Z}^d} \mathbb{R}^{d_{\mathbf{k}}}.$$

Then, the random variable on $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$ is the one taking values in

$$C([0, \infty) \rightarrow \prod_{\mathbf{k} \in \mathbb{Z}^d} \mathbb{R}^{d_{\mathbf{k}}}).$$

Through $(X^\theta(t))_{t \geq 0}$ we define a Θ valued process such that

$$\{T_{X^\theta(t)}(\theta)\}_{t \geq 0}.$$

Proposition 4.1 *The following hold:*

i) If $T_{X^\theta(0)}(\theta) = T_{X^{\theta'}(0)}(\theta')$, $P - a.s.$ $\omega \in \Omega$,

then

$$(X^\theta(t) - X^\theta(0))_{t \geq 0} = (X^{\theta'}(t) - X^{\theta'}(0))_{t \geq 0}, \quad P - a.s. \quad \omega \in \Omega,$$

ii) For

$$\theta' = T_{X^\theta(0)}(\theta), \quad (4.3)$$

$$(T_{X^\theta(t)}(\theta))_{t \geq 0} = (T_{X_0^{\theta'}(t)}(\theta'))_{t \geq 0}, \quad P - a.s. \quad \omega \in \Omega, \quad (4.4)$$

where $X_0^{\theta'}(t)(\theta')$ is the diffusion defined by (4.2) with $X_0^{\theta'}(0) = \mathbf{0}$ and replacing θ by θ' in it.

By (4.4) such $(T_{X^\theta(t)}(\theta))_{t \geq 0}$ are represented by

$$(Y_{\theta'}(t))_{t \geq 0} \equiv (T_{X_0^{\theta'}(t)}(\theta'))_{t \geq 0}$$

iii) The process $(Y_{\theta'}(t))_{t \geq 0}$ satisfies $Y_{\theta'}(0) = \theta'$ and is a Markov process.

□

Definition 4.1 By Proposition 4.1, we define Markovian semi-groups corresponding to $(X^\theta(t))_{t \geq 0}$ and $(Y(t))_{t \geq 0}$:

For bounded measurable $f \in \mathcal{C}(\prod_{\mathbf{k} \in D} \mathbb{R}^{d_{\mathbf{k}}} \rightarrow \mathbb{R})$ with some dounded $D \subset \mathbb{Z}^d$,

$$(p_t^{X,\theta} f)(\mathbf{x}) \equiv E[f(X^\theta(t)) | X^\theta(0) = \mathbf{x}], \quad \mathbf{x} \in \prod_{\mathbf{k}} \mathbb{R}^{d_{\mathbf{k}}};$$

For $\varphi \in \mathcal{D}$

$$(p_t^Y \varphi)(\mathbf{y}) \equiv E[\varphi(T_{X^\theta(t)}(\theta)) | T_{X^\theta(0)}(\theta) = \mathbf{y}], \quad \mathbf{y} \in \Theta.$$

□

5 Key assumption and the result

As was done in [ABRY1,2], we assume the following:

There exist $K < \infty$ and $\gamma > 0$ such that

$$\sup_{\mathbf{y} \in \Theta} |(p_t^Y \varphi)(\mathbf{y}) - \langle \varphi, \mu \rangle| \leq K e^{-\gamma t} (\|\nabla \varphi\|_{L^\infty} + \|\varphi\|_{L^\infty}), \quad \forall \theta \in \Theta, \quad \forall \varphi \in \mathcal{D}, \quad (5.1)$$

where

$$\|\nabla \varphi\|_{L^\infty} = \sup_{\mathbf{x}, \theta} |\nabla \varphi(T_{\mathbf{x}}(\theta))|, \quad \|\varphi\|_{L^\infty} = \sup_{\theta} |\varphi(\theta)|.$$

□

Definition 5.1 For each $\mathbf{k} \in \mathbb{Z}^d$ and $i = 1, \dots, d_{\mathbf{k}}$, define an operator $D^{\mathbf{k},i} : \mathcal{D} \rightarrow \mathcal{D}$ such that

$$(D^{\mathbf{k},i} \varphi)(\mathbf{y}) \equiv \frac{\partial}{\partial x_i^{\mathbf{k}}} \varphi(T_{\mathbf{x}}(\mathbf{y}))|_{\mathbf{x}=\mathbf{0}}, \quad \varphi \in \mathcal{D}, \quad \mathbf{y} \in \Theta.$$

□

Proposition 5.1 For each $\mathbf{k} \in \mathbb{Z}^d$, $i = 1, \dots, d_{\mathbf{k}}$ let

$$b_i^{\mathbf{k}}(\mathbf{y}) \equiv \sum_{1 \leq j \leq d_{\mathbf{k}}} \{(D^{\mathbf{k},j} a_{ij}^{\mathbf{k}})(\mathbf{y}) - a_{ij}^{\mathbf{k}}(\mathbf{y})(D^{\mathbf{k},j}(\sum_{|\mathbf{k}-\mathbf{k}'| \leq L} U_{\mathbf{k}'})(\mathbf{y}))\}, \quad \mathbf{y} \in \Theta,$$

then

$$\chi_i^{\theta, \mathbf{k}}(\mathbf{y}) \equiv \int_0^\infty (p_t^Y b_i^{\mathbf{k}})(\mathbf{y}) dt \in L^2(\mu).$$

□

Through the same discussions performed in [ABRY1,2], for the present general framework we are also able to show the following (cf. [ABRY1,2] for the exact statement and the terminologies)

Theorem 5.1 *By taking a subsequence of the scaling process with the scaling parameter $\epsilon > 0$ such that $\{\epsilon X^\theta(\frac{t}{\epsilon^2})\}_{t \geq 0}$, for $\mu - a.e. \theta \in \Theta$ it converges weakly to a Gaussian process with a constant covariance matrix, characterized by $\sigma_{ij}^{\mathbf{k}}, \chi_i^{\theta, \mathbf{k}}, \mathbf{k} \in \mathbb{Z}^d, 1 \leq i, j \leq d_{\mathbf{k}}$, and $\theta \in \Theta$, as $\epsilon_n \downarrow 0$ where $\{\epsilon_n\}_{n \in \mathbb{N}}$ is the sequence of the parameter corresponding to the subsequence of $\{\epsilon X^\theta(\frac{t}{\epsilon^2})\}_{t \geq 0}$.*

□

References

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