

An Assmus-Mattson Theorem for Matroids

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Abstract

This note is a summary of the results in the preprints [2], [3] and [4] which are the joint works with Thomas Britz.

1 Introduction

The most celebrated result to connect coding theory and design theory is undoubtedly the Assmus-Mattson Theorem [1]. It offers a sufficient condition for the codewords of a given weight in a linear code over a finite field to form a simple t -design. Consequently, it has been used to construct t -designs from linear codes; for instance, 5-designs are in [1] obtained from the extended Golay code, the extended ternary Golay code, and other codes.

The MacWilliams identity [8] for the weight enumerator of a linear code over a finite field plays an important role in the proof of the Assmus-Mattson Theorem. Recently [2], we proved a matroid theoretical analogue of this identity. In [3], we apply this MacWilliams identity for matroids in order to establish the matroid theoretical analogue of the Assmus-Mattson theorem. We prove the Assmus-Mattson theorem for subcode supports of linear codes in [4].

Our matroid theoretic terminology essentially follows that of Whitney [13], Tutte [11], Oxley [10] and Welsh [12].

2 Notation and Terminology

We begin by introducing matroids, as in [10]. A *matroid* is an ordered pair $M = (E, \mathcal{I})$ consisting of a finite set E and a collection $\mathcal{I}$ of subsets of E satisfying the following three conditions:

(I1) $\emptyset \in \mathcal{I}$.

(I2) If $I \in \mathcal{I}$ and $I' \subseteq I$, then $I' \in \mathcal{I}$.

(I3) If I_1 and I_2 are in $\mathcal{I}$ and $|I_1| < |I_2|$, then there is an element e of $I_2 - I_1$ such that $I_1 \cup e \in \mathcal{I}$.

The members of $\mathcal{I}$ are the *independent sets* of M , and a subset of E that is not in $\mathcal{I}$ is called *dependent*. A minimal dependent set in M is called a *circuit* of M , and a maximal independent set in M is called a *base* of M . For a subset X of E , we define the *rank* of X as follows:

$$\rho(X) := \max\{|Y| : Y \subseteq X, Y \in \mathcal{I}\}.$$

The *dual matroid* M^* of M is defined as the matroid, the set of bases of which is

$$\{E - B : B \text{ is a base of } M\}.$$

When we denote the rank of M^* by ρ^* , the following is well-known:

$$\rho^*(X) = |X| - \rho(M) + \rho(E - X).$$

For a matroid $M = (E, \mathcal{I})$ and a subset T of E , it is easy to check that

$$M \setminus T = (E - T, \{I \subseteq E - T : I \in \mathcal{I}\})$$

is a matroid which is called the *deletion of T from M* . The *contraction of T from M* is given by

$$M/T = (M^* \setminus T)^*.$$

For an $m \times n$ matrix A over $\mathbb{F}_q$, if E is the set of column labels of A and $\mathcal{I}$ is the set of subsets X of E for which the multiset of columns labeled by X is linearly independent in the vector space $\mathbb{F}_q^m$, then $M[A] := (E, \mathcal{I})$ is a matroid and is called a *matroid of A over $\mathbb{F}_q$* .

For a vector $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{F}_q^n$ and a subset $D \subseteq \mathbb{F}_q^n$, we define the *supports* of $\mathbf{x}$ and D respectively as follows:

$$\begin{aligned} \text{supp}(\mathbf{x}) &:= \{i \mid x_i \neq 0\}, \\ \text{Supp}(D) &:= \bigcup_{\mathbf{x} \in D} \text{supp}(\mathbf{x}). \end{aligned}$$

A t -(v, k, μ) *design* is a collection $\mathcal{B}$ of k -subsets (called *blocks*) of a set V with v points, such that any t -subset of V is contained in exactly μ blocks. In [1], E. F. Assmus, Jr. and H. F. Mattson, Jr. proved the following result, which is thus widely known as the *Assmus-Mattson Theorem* (cf. [7]).

Theorem 2.1 *Let C be a linear code on E over $\mathbb{F}_q$ with minimum nonzero weight d , and let $d^\perp$ denote the minimum nonzero weight of $C^\perp$. Let $w = n$ when $q = 2$ and otherwise let w be the largest integer satisfying*

$$w - \left\lfloor \frac{w + q - 2}{q - 1} \right\rfloor < d,$$

defining $w^\perp$ similarly. Suppose there is an integer t with $0 < t < d$ that satisfies the following condition: the number of indices i ($1 \leq i \leq n - t$) such that $A_{C^\perp}(i) \neq 0$ is at most $d - t$. Then for each i with $d \leq i \leq w$, the supports of codewords in C of weight i , provided there are any, yield a t -design. Similarly, for each j with $d^\perp \leq j \leq \min\{w^\perp, n - t\}$, the supports of codewords in $C^\perp$ of weight j , provided there are any, form a t -design.

3 Main Results

For any subset $T \subseteq E$ and a matroid M , let $M.T$ denote the contraction $M/(E - T)$ and let $M|T$ denote the deletion $M \setminus (E - T)$. The *characteristic polynomial* $p(M; \lambda)$ of a matroid M on the set E is given by the sum

$$p(M; \lambda) = \sum_{T \subseteq E} (-1)^{|T|} \lambda^{\rho(E) - \rho(T)},$$

where ρ is the rank function of M .

The *characteristic enumerator* of a matroid M on a set E is given by

$$\begin{aligned} W_M(\lambda, x, y) &= \sum_{T \subseteq E} p(M.T; \lambda) x^{|E-T|} y^{|T|} \\ &= \sum_{i=0}^n A_M(i, \lambda) x^{n-i} y^i, \end{aligned}$$

where $A_M(i, \lambda) = \sum_{T \in \binom{E}{i}} p(M.T; \lambda)$. Then we proved the following MacWilliams type identity in [2].

Theorem 3.1 *If M is a matroid on the set E , then*

$$\lambda^{\rho(M)} W_{M^*}(\lambda, x, y) = W_M(\lambda, x + (\lambda - 1)y, x - y), \quad (1)$$

and for $i = 0, 1, \dots, n$,

$$\lambda^{\rho(M)} A_{M^*}(i, \lambda) = \sum_{j=0}^n A_M(j, \lambda) \sum_{\nu=0}^j (-1)^\nu (\lambda - 1)^{i-\nu} \binom{j}{\nu} \binom{n-j}{i-\nu}. \quad (2)$$

Let $\mathbb{F}$ be a (not necessarily finite) field, and let $\mathbb{F}[z]$ denote the ring of polynomials in an indeterminate z with coefficients in $\mathbb{F}$. Furthermore, define $\mathbb{G} := \mathbb{F}[z] - \{0, 1\}$. For a matroid M on E with at least one cocircuit, we define for positive integers i and t ,

$$\begin{aligned} \mathcal{R}_{M,t}^\lambda &= \{i \in \{1, \dots, n-t\} : A_{M^*}(i, \lambda) \neq 0\}; \\ d_M &= \min\{|X| : X \text{ is a cocircuit in } M\}; \\ \mathcal{C}_{M,i} &= \{X : X \text{ is a cocircuit of } M \text{ with } |X| = i\} \\ \mathcal{H}_{M,i} &= \{X : X \text{ is a hyperplane of } M \text{ with } |X| = i\} \\ e_M &= \max\{i : \text{no subset } X \in \binom{E}{i} \text{ contains two distinct cocircuits of } M\}. \end{aligned}$$

Using the above theorem, we have a generalization of the Assmus-Mattson theorem for matroids.

Theorem 3.2 *Let M be a matroid on E with at least one circuit and one cocircuit, and suppose that t ($0 < t < d_M$) is an integer with $|\mathcal{R}_{M,t}^\lambda| \leq d_M - t$ for some $\lambda \in \mathbb{G}$ such that*

1. *for all $T \in \binom{E}{t}$ and $l = 1, \dots, n-t$, $A_{M^*/T}(l, \lambda) = 0$ whenever $A_{M^*}(l, \lambda) = 0$.*

Then for $m = \min\{e_{M^*}, n - t\}$,

$$\mathcal{C}_{M,d_M}, \dots, \mathcal{C}_{M,e_M}, \mathcal{C}_{M^*,d_{M^*}}, \dots, \mathcal{C}_{M^*,m}, \mathcal{H}_{M,n-e_M}, \dots, \mathcal{H}_{M,n-d_M}, \mathcal{H}_{M^*,n-m}, \dots, \mathcal{H}_{M^*,n-d_{M^*}}$$

each forms a t -design.

Example 3.3 The binary affine matroid $M = AG(3, 2)$, represented by the binary matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix},$$

has minimal cocircuit size $d_M = 4$, and the characteristic enumerator of M^* (and of M) is

$$\begin{aligned} W_{M^*}(\lambda, x, y) &= (\lambda - 1)(\lambda^3 - 7\lambda^2 + 21\lambda - 21)y^8 + 8(\lambda - 1)(\lambda - 2)(\lambda - 4)xy^7 \\ &\quad + 28(\lambda - 1)(\lambda - 2)x^2y^6 + 14(\lambda - 1)x^4y^4 + x^8. \end{aligned}$$

so $|\mathcal{R}_{M,3}^\lambda| = |\{4\}| = 1 \leq d_M - 3$. By letting λ be an indeterminate (resp., by setting $\lambda := 2$), Condition 1 in Theorem 3.2 is satisfied for λ and $t = 3$. Hence, $\mathcal{C}_{M,4}$ and $\mathcal{H}_{M,4}$ each form a 3-design.

Let C be an $[n, k]$ code over $\mathbb{F}_q$. Let r, i be integers with $1 \leq r \leq k$ and $1 \leq i \leq n$, and define

$$\begin{aligned} \mathcal{D}_r(C) &= \{D : D \text{ is an } r\text{-dimensional subcode of } C\}; \\ \mathcal{S}_r(C) &= \{\text{Supp}(D) : D \in \mathcal{D}_r(C)\}; \\ \mathcal{S}_{r,i}(C) &= \{X \in \mathcal{S}_r(C) : |X| = i\}; \\ d_r(C) &= d_r = \min\{|X| : X \in \mathcal{S}_r(C)\}. \end{aligned}$$

For an r with $1 \leq r \leq k$, the r -th support weight enumerator $A_C^{(r)}(x, y)$ of C is defined as follows:

$$A_C^{(r)}(x, y) = \sum_{i=0}^n A_i^{(r)} x^{n-i} y^i,$$

where

$$A_i^{(r)} = A_i^{(r)}(C) = |\{D : \text{Supp}(D) \in \mathcal{S}_{r,i}(C)\}|.$$

Using Theorem 3.1 and Theorem 3.2, we have the Assmus-Mattson type theorem for subcode supports of linear codes.

Theorem 3.4 Let C be an $[n, k, d]$ code over $\mathbb{F}_q$ and let m be an integer with $1 \leq m \leq \min\{k, n - k\}$. Suppose that t ($0 < t < d$) is an integer with

$$|\{i \in \{d_m^\perp, \dots, n - t\} : A_{M^*}(i, q^m) \neq 0\}| \leq d_m - t.$$

If each $\mathcal{S}_{r,i}(C)$ form t -designs and $|\mathcal{S}_{r,i}(C)| = A_i^{(r)}(C)$ whenever $A_i^{(r)}(C) \neq 0$, for all r ($1 \leq r \leq m-1$) and all i ($d_r \leq i < d_{m+1}$), then each $\mathcal{S}_{m,i}(C)$ ($d_m \leq i < d_{m+1}$) forms a t -design. Moreover, if each $\mathcal{S}_{r,j}(C^\perp)$ form t -designs and $|\mathcal{S}_{r,j}(C^\perp)| = A_j^{(r)}(C^\perp)$ whenever $A_j^{(r)}(C^\perp) \neq 0$, for all r ($1 \leq r \leq m-1$) and all j ($d_r^\perp \leq j < d_{m+1}^\perp$), then each $\mathcal{S}_{m,j}(C^\perp)$ ($d_m^\perp \leq j < d_{m+1}^\perp$) form t -design.

From this theorem, we have the following result for doubly-even self-dual codes of length 24, 32 and 48.

Corollary 3.5 For $n = 24, 32$ or 48 , let C be a binary doubly-even self-dual $[n, n/2, 4\lfloor n/24 \rfloor + 4]$ code. then each $\mathcal{S}_{m,i}(C)$ forms a t -design as follows:

length n	m	support weights i	t -designs
24	2	12	5-(24, 12, 660)
24	2	14	5-(24, 14, 8008)
24	2	16	5-(24, 16, 65598)*
24	3	14	5-(24, 14, 4290)
24	3	15	5-(24, 15, 40040)*
32	2	12	3-(32, 12, 385)
32	2	14	3-(32, 14, 10192)
48	2	18	5-(48, 18, 13328)
48	2	20	5-(48, 20, 581400)
48	2	22	5-(48, 22, 15853068)

*by computer search

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