

Block-transitive 3-designs from $\text{PSL}(2, q)$

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1 Introduction and preliminaries

The projective special linear group $\text{PSL}(2, q)$ acts as linear fractional transformations:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} : z \mapsto \frac{az + b}{cz + d} \quad (z \in \mathbb{F}_q \cup \{\infty\}),$$

where $ad - bc = 1$.

The purpose of this talk is to give a family of nice orbits consisting of $(q-1)/e$ -element subset forming 3-designs.

More precisely,

- e is a positive integer with $e \geq 2$,
- q is a prime power with $q \equiv 1 \pmod{e}$,
- a representative for the orbit is the set of e -th powers in $\mathbb{F}_q^\times$.
- ... (additional conditions, to be stated later).

Let Ω be a finite set, and denote by $\binom{\Omega}{k}$ the family of k -element subsets of Ω .

Definition 1. A pair $(\Omega, \mathcal{B})$ is called a t -**design** if $\mathcal{B} \subseteq \binom{\Omega}{k}$ and, any t points of Ω is contained in a constant number of members of $\mathcal{B}$.

To avoid triviality, we assume $|\Omega| > k > t > 0$ in Definition 1. Members of $\mathcal{B}$ are often called **blocks**. The constant number in Definition 1 is usually denoted by λ , and we say $\mathcal{B}$ is a t -(v, k, λ) **design**, where $v = |\Omega|$.

Definition 2. A subgroup of the symmetric group on a finite set Ω is called a **permutation group** of degree $|\Omega|$. A permutation group G is said to be **t -transitive** if G acts transitively on the set of ordered t -tuples of distinct elements of Ω :

$$\{(x_1, \dots, x_t) \in \Omega^t \mid x_1, \dots, x_t : \text{distinct}\}.$$

Examples follow:

- The symmetric group on Ω with $|\Omega| = n$ is n -transitive.
- The alternating group on Ω with $|\Omega| = n$ is $(n - 2)$ -transitive.
- The sporadic simple group M_{24} is a 5-transitive permutation group of degree 24.
- The projective general linear group $\text{PGL}(2, q)$ is 3-transitive on the projective line $\mathbb{F}_q \cup \{\infty\} = \text{PG}(1, q) = \mathbb{P}^1(\mathbb{F}_q)$, the 1-dimensional projective space.

Definition 3. A permutation group G is said to be **t -homogeneous** if G acts transitively on $\binom{\Omega}{t}$.

Recall that $\binom{\Omega}{t}$ is the set of unordered t -tuples, i.e., t -element subsets. Clearly, t -transitivity implies t -homogeneity.

If G is a t -homogeneous permutation group on Ω , and $B \in \binom{\Omega}{k}$ with $|\Omega| > k > t$, then $(\Omega, G \cdot B)$ is a t -design, where $G \cdot B$ is the orbit of B under G . If the set of blocks $\mathcal{B}$ is of the form $G \cdot B$, then the design $(\Omega, \mathcal{B})$ is called **block-transitive**, and a representative B is called a **starter** of the design $(\Omega, \mathcal{B})$ under G .

The group $\text{PGL}(2, q)$ acts on $\mathbb{F}_q \cup \{\infty\}$ in terms of linear fractional transformations

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} : z \mapsto \frac{az + b}{cz + d} \quad (z \in \mathbb{F}_q \cup \{\infty\}).$$

This action is 3-transitive: $(\infty, 0, 1) \mapsto$ any triple of distinct elements of $\mathbb{F}_q \cup \{\infty\}$. Thus, any $B \in \binom{\mathbb{F}_q \cup \{\infty\}}{k}$, $k > 3$, is a starter of a block-transitive design under $\text{PGL}(2, q)$.

To what extent is this true for $\text{PSL}(2, q)$? Recall

$$\begin{aligned} \text{PGL}(n, q) &= \text{GL}(n, q) / Z(\text{GL}(n, q)), \\ \text{PSL}(n, q) &= \text{SL}(n, q) / (\text{SL}(n, q) \cap Z(\text{GL}(n, q))) \\ &\cong (\text{SL}(n, q) Z(\text{GL}(n, q))) / Z(\text{GL}(n, q)), \\ \text{PSL}(2, q) &= \text{SL}(2, q) / \{\pm I\}. \end{aligned}$$

We already mentioned that $\text{PGL}(2, q)$ is 3-transitive, hence 3-homogeneous on $\mathbb{F}_q \cup \{\infty\}$. If $q = 2^m$, then $\text{PSL}(2, q) = \text{PGL}(2, q)$ is 3-transitive and hence 3-homogeneous. If q is odd, then $|\text{PGL}(2, q) : \text{PSL}(2, q)| = 2$. The following fact is well known.

- If $q \equiv -1 \pmod{4}$, then $\text{PSL}(2, q)$ is 3-homogeneous.
- If $q \equiv 1 \pmod{4}$, then $\text{PSL}(2, q)$ is not 3-homogeneous.

If $G = \text{PSL}(2, q)$ is 3-homogeneous on $\mathbb{F}_q \cup \{\infty\}$. then for $B \in \binom{\mathbb{F}_q \cup \{\infty\}}{k}$, $(\mathbb{F}_q \cup \{\infty\}, G \cdot B)$ is a 3- $(q+1, k, \lambda)$ design for some λ , where

$$\lambda = \frac{|G \cdot B|k(k-1)(k-2)}{(q+1)q(q-1)}.$$

can be computed from

$$|G \cdot B| = |G : \text{Stab}_G(B)|.$$

Keränen and Kreher [9] investigated such designs for the case $q = 2^m$. For the case $q \equiv -1 \pmod{4}$, see [15, 16, 18].

Since $\text{PGL}(2, q) \supsetneq \text{PSL}(2, q)$, in general,

$$\text{PGL}(2, q) \cdot B \supsetneq \text{PSL}(2, q) \cdot B \quad \text{for } B \in \binom{\mathbb{F}_q \cup \{\infty\}}{k}.$$

However, it can happen that $\text{PGL}(2, q) \cdot B = \text{PSL}(2, q) \cdot B$ for some B .

Since $|\text{PGL}(2, q) : \text{PSL}(2, q)| = 2$, we have the following.

Lemma 4. For $B \subseteq \mathbb{F}_q \cup \{\infty\}$, the following are equivalent:

- (i) $\text{PGL}(2, q) \cdot B = \text{PSL}(2, q) \cdot B$
- (ii) $\exists \sigma \in \text{PGL}(2, q)$ such that $\sigma(B) = B$ and $\sigma \notin \text{PSL}(2, q)$.

Theorem 5. Suppose $q \equiv 1 \pmod{4e}$, e is odd. Let $B = \langle \alpha^e \rangle \subseteq \mathbb{F}_q^\times = \langle \alpha \rangle$. Then B is a starter of a block-transitive 3-design under $\text{PSL}(2, q)$.

Proof. Let $\sigma =$ multiplication by α^e , and use Lemma 4 □

2 3-Designs not coming from Lemma 4 or Theorem 5

Lemma 4 and Theorem 5 give a sufficient condition for B to be a starter of a block-transitive 3-design under $\text{PSL}(2, q)$. This condition is, however, not necessary.

Bonnecaze and Solé [2] found a block-transitive 3-design under $\text{PSL}(2, 41)$ which is not invariant under $\text{PGL}(2, 41)$. We give a description of this design. Let q be an odd prime power, and define $\chi: \mathbb{F}_q \rightarrow \{0, \pm 1\}$ by

$$\chi(a) = \begin{cases} 0 & \text{if } a = 0, \\ 1 & \text{if } a \in (\mathbb{F}_q^\times)^2, \\ -1 & \text{otherwise.} \end{cases}$$

This is known as the Legendre symbol, or quadratic residue character.

Let $q = 41$. The linear span over $\mathbb{F}_2$ of the rows of the $q \times q$ matrix

$$\frac{1}{2}((\chi(a-b) + 1)_{a,b \in \mathbb{F}_q} - I)$$

is the binary quadratic residue code of length 41, denoted QR_{41} . Then $QR_{41} \subseteq \mathbb{F}_2^{41}$, $\dim QR_{41} = 21$. The extended binary quadratic residue code XQR_{42} of length 42 is obtained from QR_{41} by adding the “parity check coordinate.” Then $XQR_{42} \subseteq \mathbb{F}_2^{42}$, $\dim XQR_{42} = 21$.

For $x \in \mathbb{F}_2^n$,

$$\begin{aligned}\text{supp}(x) &= \{i \mid 1 \leq i \leq n, x_i = 1\}, \\ \text{wt}(x) &= |\text{supp}(x)|.\end{aligned}$$

Let

$$\begin{aligned}\Omega &= \{1, 2, \dots, 42\}, \\ \mathcal{B} &= \{\text{supp}(x) \mid x \in XQR_{42}, \text{wt}(x) = 10\}.\end{aligned}$$

Then $(\Omega, \mathcal{B})$ is a 3-(42, 10, 18) design (verified by computer). WHY?

Let

$$\begin{aligned}\Omega &= \{1, 2, \dots, 42\}, \\ \mathcal{B} &= \{\text{supp}(x) \mid x \in XQR_{42}, \text{wt}(x) = k\}.\end{aligned}$$

Then $(\Omega, \mathcal{B})$ is a 3-design only if $k = 10, 32$ (verified by computer, according to [2]). It is known that $\text{Aut } XQR_{42} = \text{PSL}(2, 41)$, and it acts transitively on $\mathcal{B}$ if $k = 10, 32$.

The design invariant under $\text{PGL}(2, 41)$ is formed by taking the union of XQR_{42} and $XQR_{42}^\perp$ as follows:

$$\tilde{\mathcal{B}} = \{\text{supp}(x) \mid x \in XQR_{42} \cup XQR_{42}^\perp, \text{wt}(x) = 10\}.$$

Then $(\Omega, \tilde{\mathcal{B}})$ is a 3-design (this fact can be theoretically generalized, but $|\tilde{\mathcal{B}}| = 2|\mathcal{B}|$. In fact, $\tilde{\mathcal{B}}$ is a $\text{PGL}(2, 41)$ -orbit.)

In fact, we may identify Ω with $\mathbb{F}_{41} \cup \{\infty\}$. Let β be a primitive 10-th root of 1 in $\mathbb{F}_{41}$, and let

$$B = \{1, \beta, \beta^2, \dots, \beta^9\},$$

Equivalently, B is the set of quartic (4th power) residues in $\mathbb{F}_{41}$, i.e.,

$$B = \langle \alpha^4 \rangle, \quad \mathbb{F}_{41}^\times = \langle \alpha \rangle.$$

Then $\mathcal{B} = \text{PSL}(2, 41) \cdot B$.

3 Main results

In this section, we let q be a prime power with $q \equiv 1 \pmod{4}$, and let $G = \text{PSL}(2, q)$. For some particular choice of B , $(\mathbb{F}_q \cup \{\infty\}, G \cdot B)$ can happen to be a 3-design.

Theorem 6 (Keränen–Kreher–Shiue [10]). Suppose $q \equiv 5$ or $13 \pmod{24}$. Let $B = \{\infty, 0, 1, -1\} \subseteq \mathbb{F}_q \cup \{\infty\}$. Then B is a starter of a block-transitive 3- $(q+1, 4, 3)$ design under G .

Theorem 7 (Li–Deng–Zhang [12]). Suppose $q \equiv 1 \pmod{20}$. Let $B = \langle \alpha^{(q-1)/5} \rangle \subseteq \mathbb{F}_q^\times = \langle \alpha \rangle$. Then B is a starter of a block-transitive 3- $(q+1, 5, 3)$ design under G , if and only if there exists $\theta \in \mathbb{F}_q^\times$ such that $\chi(\theta) = -1$ and $\theta^2 - 4\theta - 1 = 0$.

For the remainder of this section, by a starter, we mean a starter $B \subseteq \mathbb{F}_q \cup \{\infty\}$ of a 3-design under $G = \text{PSL}(2, q)$. Note that there has been no systematic work on finding a starter with $|B| > 5$. Earlier work include Keranen, Kreher and Shiue [10] for $|B| = 4$, Chen and Liu [5] for $|B| = 5$, Balachandran and Ray-Chaudhuri [1] for $|B| = 7$, and Li [11] for $|B| = 12$. Let q be a prime power with $q \equiv 1 \pmod{4}$, and let $e|q-1$. Let

$$\mathbb{F}_q^\times = \langle \alpha \rangle, \quad (1)$$

$$B = \langle \alpha^e \rangle, \quad (2)$$

$$G = \text{PSL}(2, q). \quad (3)$$

Regarding $B \subseteq \mathbb{F}_q \cup \{\infty\}$, we are interested in the question when $(\mathbb{F}_q \cup \{\infty\}, G \cdot B)$ is a 3-design.

- Bonnetcaze–Solé [2]: $q = 41, e = 4$.
- Li–Deng–Zhang [12]: $q \equiv 1 \pmod{20}, e = (q-1)/5$, under some condition.

Observe that $q = 41$ satisfies the condition $q \equiv 1 \pmod{20}$, but $e = 4$ does not satisfy $e = (q-1)/5$. Thus, the above two results look unrelated at the first glance. To connect these two, we need some preparation. There are only two orbits on $\binom{\mathbb{F}_q \cup \{\infty\}}{3}$ under G , namely,

$$\binom{\mathbb{F}_q \cup \{\infty\}}{3} = \mathcal{O}_+ \cup \mathcal{O}_- \quad (\text{disjoint}),$$

where

$$\mathcal{O}_+ = G \cdot \{\infty, 0, 1\}, \quad \mathcal{O}_- = G \cdot \{\infty, 0, \alpha\}.$$

In fact,

$$\binom{\mathbb{F}_q^\times}{3} \cap \mathcal{O}_\pm = \{\{a, b, c\} \mid \chi((a-b)(b-c)(c-a)) = \pm 1\}.$$

Thus, a G -orbit $\mathcal{B} \subseteq \binom{\mathbb{F}_q \cup \{\infty\}}{k}$ is the set of blocks of a 3-design if and only if

$$|\{B \in \mathcal{B} \mid \{\infty, 0, 1\} \subseteq B\}| = |\{B \in \mathcal{B} \mid \{\infty, 0, \alpha\} \subseteq B\}|.$$

Further simplification is as follows.

Lemma 8 (Tonchev [17, Theorem 1.6.1]). Let $B \subseteq \mathbb{F}_q \cup \{\infty\}$ with $|B| > 3$. Then B is a starter of a block-transitive 3-design under G if and only if

$$\left| \binom{B}{3} \cap \mathcal{O}_+ \right| = \left| \binom{B}{3} \cap \mathcal{O}_- \right|.$$

Theorem 9 (Bonnecaze–Solé [2], reformulated). Let $q = 41$, $G = \text{PSL}(2, q)$. Let $e = 4$, $B = \langle \alpha^e \rangle \subseteq \mathbb{F}_q^\times = \langle \alpha \rangle$. Then B is a starter of a block-transitive 3-design under G .

The proof of Theorem 9 using Lemma 8 amounts to showing

$$\left| \binom{B}{3} \cap \mathcal{O}_+ \right| = \left| \binom{B}{3} \cap \mathcal{O}_- \right|,$$

which can be verified directly:

$$B = \langle 6^4 \rangle = \{1, 25, 10, 4, 18, 40, 16, 31, 37, 23\} \subseteq \mathbb{F}_{41}^\times = \langle 6 \rangle.$$

$\{1, 25, 10\} \in \mathcal{O}_+$ since $\chi((1-25)(25-10)(10-1)) = 1, \dots$, and so on.

For $q = 41$, Theorem 9 says $B = \langle \alpha^4 \rangle$ is a starter of size $|B| = 10$, while Theorem 7 says $B = \langle \alpha^8 \rangle$ is a starter of size $|B| = 5$. Since $\langle \alpha^4 \rangle$ is a union of two cosets of $\langle \alpha^8 \rangle$, it may not be too surprising that there is a connection.

Let us go back to the general setting (1)–(3). Let

$$\begin{aligned} B &= \langle \alpha^{(q-1)/10} \rangle, \\ B' &= \langle \alpha^{(q-1)/5} \rangle. \end{aligned}$$

Then $|B| = 10$, $|B'| = 5$, and (by computer)

$$\begin{aligned} &B \text{ is a starter of a 3-design under } \text{PSL}(2, q) \\ &\quad \text{if } q = 41, 61, 241, 281, 421, 601, 641, \dots, \\ &B' \text{ is a starter of a 3-design under } \text{PSL}(2, q) \\ &\quad \text{if } q = 41, 61, 241, 281, 421, 601, 641, \dots, \end{aligned}$$

The latter condition is, by [12]:

$$\exists \theta \in \mathbb{F}_q^\times, \chi(\theta) = -1, \theta^2 - 4\theta - 1 = 0. \quad (4)$$

The sequence of primes

$$41, 61, 241, 281, 421, 601, 641, \dots$$

satisfying (4) was found to be in coincidence with the sequence OEIS A325072 [14]: prime numbers $p \equiv 1 \pmod{20}$ with

$$p \neq x^2 + 20y^2, x^2 + 100y^2. \quad (5)$$

It turns out that various conditions mentioned above are all equivalent to each other.

Theorem 10. Let q be a prime power with $q \equiv 1 \pmod{20}$, let $\mathbb{F}_q^\times = \langle \alpha \rangle$ and $\beta = \alpha^{(q-1)/10}$. Let χ denote the quadratic residue character of $\mathbb{F}_q^\times$. Then the following are equivalent:

- (LDZ1) There exists $\theta \in \mathbb{F}_q^\times$ such that $\chi(\theta) = -1$ and $\theta^2 - 4\theta - 1 = 0$.
- (LDZ2) $B = \langle \beta^2 \rangle$ is a starter of a 3-design with block size 5 under $\text{PSL}(2, q)$.
- (BS) $B = \langle \beta \rangle$ is a starter of a 3-design with block size 10 under $\text{PSL}(2, q)$.
- (M) $\chi(\beta - 1) = -1$.
- (OEIS) q is an odd power of a prime p with $p \equiv 1 \pmod{20}$ satisfying (5).

It is shown in [12] that (LDZ1) is equivalent to (LDZ2). So the new part is

$$(\text{LDZ1}) \iff (\text{BS}) \iff (\text{M}) \iff (\text{OEIS}).$$

It is shown in [4] that, for a prime p with $p \equiv 1 \pmod{20}$, (5) is equivalent to

$$p \neq x^2 + 100y^2, \tag{6}$$

which is then equivalent to

$$5 \notin \langle \alpha^4 \rangle \tag{7}$$

by [8, p. 69]. The proof of Theorem 10 consists of establishing the equivalence of arithmetic conditions (LDZ1), (M) and (7), and of showing the equivalence of (BS) and (M) by using Lemma 8.

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