

Abundance of σ -compact non-compactly generated groups witnessed by the Bohr topology of an abelian group *

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Abstract

Answering negatively a question of Fujita and Shakhmatov [2], Tkachenko and Torres Falcón [6] have constructed an example of a countable (and thus, σ -compact) totally bounded group that is not compactly generated. We observe that *any* countable non-finitely generated abelian group G equipped with its Bohr topology fails to be compactly generated, thereby obtaining an abundant supply of totally bounded groups providing a counter-example to the question of Fujita and Shakhmatov [2].

A group G is *finitely generated* if G is algebraically generated by its finite subset.

A group G is called *σ -compact* provided that G can be represented as a union of a countable family of its compact subsets, and G is called *compactly*

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generated if G is algebraically generated by its compact subset. One can easily see that a compactly generated group is σ -compact.

Fujita and Shakhmatov [2] proved that a σ -compact metric group is compactly generated. The same result also holds for a wider class of groups that contains both metric and locally compact groups, see [3].

Answering a question of Fujita and Shakhmatov [2], Tkachenko and Torres Falc3n [6] have constructed an example of a countable (thus, σ -compact) totally bounded group that is not compactly generated. (Recall that a group G is *totally bounded* if it is (topologically and algebraically) isomorphic to a subgroup of some compact group.)

The main purpose of this note is to observe that *any* countable non-finitely generated abelian group G equipped with its Bohr topology fails to be compactly generated, thereby obtaining an abundant supply of totally bounded groups providing a counter-example to the question of Fujita and Shakhmatov [2].

Let G be an abelian group. The *Bohr topology* of G is the weakest group topology on G making all characters $\chi : G \rightarrow \mathbb{T}$ continuous, where $\mathbb{T}$ is the torus group. The Bohr topology of G is totally bounded, and the group G equipped with this topology is usually denoted by $G^\#$.

According to the celebrated result of Glicksberg [4], $G^\#$ has no infinite compact subsets. Thus, $G^\#$ is compactly generated if and only if $G^\#$ is finitely generated. This immediately yields the following

Theorem 1. *Let G be a countable abelian group that is not finitely generated. Then $G^\#$ is a σ -compact totally bounded group that is not compactly generated.*

It should be noted that there are only countably many finitely generated abelian groups, as every such group has the form

$$\mathbb{Z}^n \times \mathbb{Z}(m_1) \times \dots \times \mathbb{Z}(m_k), \quad (1)$$

where $n, m_1, \dots, m_k$ are integer numbers and $\mathbb{Z}(k)$ denotes the cyclic group of order k . On the other hand, there are continuum many pairwise non-isomorphic countable abelian groups. (In fact, even the group $\mathbb{Q}$ of rational numbers contains continuum many pairwise non-isomorphic subgroups.)

Corollary 2. *Let G be a countable abelian group that is not isomorphic to a group of the form (1). Then $G^\#$ is a σ -compact totally bounded group that is not compactly generated.*

Let us finish with some concrete examples.

Corollary 3. *$\mathbb{Q}^\#$ is a (σ -compact) divisible totally bounded group that is not compactly generated.*

Corollary 4. *If G is a countably infinite torsion group, then $G^\#$ is a $(\sigma$ -compact) totally bounded group that is not compactly generated.*

It follows from Corollary 4 that $(\mathbb{Q}/\mathbb{Z})^\#$ is a $(\sigma$ -compact) divisible totally bounded group that is not compactly generated.

Furthermore, from Corollary 4 one can easily obtain an infinite family of $(\sigma$ -compact) non-compactly-generated totally bounded torsion groups that are pairwise non-homeomorphic even as topological spaces. Indeed, if G and H are countably infinite torsion groups of distinct prime exponent, then $G^\#$ and $H^\#$ are not homeomorphic as topological spaces ([5, 1]).

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