

Notes on Sakaguchi functions

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Abstract

By using the definition for certain univalent functions $f(z)$ in the open unit disk $\mathbb{U}$ given by K.Sakaguchi(J.Math.Soc.Japan, 11(1959)), two classes $\mathcal{S}(\alpha)$ and $\mathcal{T}(\alpha)$ of analytic functions in $\mathbb{U}$ are introduced. The object of the present paper is to discuss some properties of functions $f(z)$ belonging to the classes $\mathcal{S}(\alpha)$ and $\mathcal{T}(\alpha)$.

1 Introduction

Let $\mathcal{A}$ be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (1.1)$$

that are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} \mid |z| < 1\}$. A function $f(z) \in \mathcal{A}$ is said to be in the class $\mathcal{S}(\alpha)$ if it satisfies

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z) - f(-z)} \right\} > \alpha \quad (1.2)$$

for some $\alpha (0 \leq \alpha < \frac{1}{2})$ and for all $z \in \mathbb{U}$. The class $\mathcal{S}(0)$ when $\alpha = 0$ was introduced by Sakaguchi [2]. Therefore, a function $f(z) \in \mathcal{S}(\alpha)$ is called Sakaguchi function of order α . We also denote by $\mathcal{T}(\alpha)$ the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ such that $zf'(z) \in \mathcal{S}(\alpha)$.

For $f(z)$ belonging to $\mathcal{S}(\alpha)$ and $\mathcal{T}(\alpha)$, Cho, Kwon and Owa [1] have given

Lemma 1 *If $f(z) \in \mathcal{A}$ satisfies*

$$\sum_{n=2}^{\infty} \{2(n-1)|a_{2n-2}| + (2n-1-2\alpha)|a_{2n-1}|\} \leq 1-2\alpha \quad (1.3)$$

for some $\alpha (0 \leq \alpha < \frac{1}{2})$, then $f(z) \in \mathcal{S}(\alpha)$.

Lemma 2 *If $f(z) \in \mathcal{A}$ satisfies*

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$$\sum_{n=2}^{\infty} \{4(n-1)^2 |a_{2n-2}| + (2n-1)(2n-1-2\alpha) |a_{2n-1}|\} \leq 1-2\alpha \quad (1.4)$$

for some $\alpha (0 \leq \alpha < \frac{1}{2})$, then $f(z) \in \mathcal{T}(\alpha)$.

In view of the above lemmas, we see

Example 1.1 Let us consider a function $f(z)$ given by

$$f(z) = z + \frac{1}{3}\delta_2 z^2 + \left(1 - \frac{8}{3(3-2\alpha)}\right) \delta_3 z^3 \quad (1.5)$$

with $|\delta_2| = |\delta_3| = 1$. Then, since

$$\sum_{n=2}^{\infty} \{2(n-1) |a_{2n-2}| + (2n-1-2\alpha) |a_{2n-1}|\} < 1-2\alpha$$

we see that $f(z) \in \mathcal{S}(\alpha)$.

Example 1.2 Let us consider a function $f(z)$ given by

$$f(z) = z + \frac{1}{6}\delta_2 z^2 + \frac{1}{3} \left(1 - \frac{8}{3(3-2\alpha)}\right) \delta_3 z^3 \quad (1.6)$$

with $|\delta_2| = |\delta_3| = 1$. Then, since

$$zf'(z) = z + \frac{1}{3}\delta_2 z^2 + \left(1 - \frac{8}{3(3-2\alpha)}\right) \delta_3 z^3 \in \mathcal{S}(\alpha),$$

we have that $f(z) \in \mathcal{T}(\alpha)$.

2 Coefficient inequalities

Applying Carathéodory function

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \quad (2.1)$$

in $\mathbb{U}$, we first discuss the coefficient inequalities for functions $f(z)$ in $\mathcal{S}(\alpha)$ and $\mathcal{T}(\alpha)$.

Theorem 2.1 If $f(z) \in \mathcal{S}(\alpha)$, then

$$|a_{2n}| \leq \frac{\prod_{j=1}^{n+1} (j-2\alpha)}{n(n!)} \quad (n \geq 1) \quad (2.2)$$

and

$$|a_{2n+1}| \leq \frac{\prod_{j=1}^n (j-2\alpha)}{n!} \quad (n \geq 1). \quad (2.3)$$

Proof We define the function $p(z)$ by

$$p(z) = \frac{1}{1-2\alpha} \left(\frac{2zf'(z)}{f(z) - f(-z)} - 2\alpha \right) = 1 + \sum_{n=1}^{\infty} p_n z^n \quad (2.4)$$

for $f(z) \in \mathcal{S}(\alpha)$. Then $p(z)$ is a Carathéodory function and satisfies $|p_n| \leq 2$ ($n \geq 1$). Since

$$2zf'(z) = (f(z) - f(-z))((1-2\alpha)p(z) + 2\alpha),$$

we obtain that

$$a_{2n} = \frac{1-2\alpha}{2n} (p_{2n+1} + a_3 p_{2n-1} + \cdots + a_{2n+1} p_1) \quad (2.5)$$

and

$$a_{2n+1} = \frac{1-2\alpha}{2n} (p_{2n} + a_3 p_{2n-2} + \cdots + a_{2n-1} p_2). \quad (2.6)$$

Taking $n = 1$, we see that

$$|a_3| \leq 1 - 2\alpha \quad (2.7)$$

and

$$|a_2| = \frac{1-2\alpha}{1+|a_3|} \leq (1-2\alpha)(2-2\alpha). \quad (2.8)$$

Thus, using the mathematical induction, we complete the proof of the theorem.

Remark 2.1 Equalities in Theorem 2.1 are attended for $f(z)$ given by

$$\frac{zf'(z)}{f(z) - f(-z)} = \frac{1 + (1-4\alpha)z}{2(1-z)}.$$

Theorem 2.2 If $f(z) \in \mathcal{T}(\alpha)$, then

$$|a_{2n}| \leq \frac{\prod_{j=1}^{n+1} (j-2\alpha)}{2n^2(n!)} \quad (n \geq 1) \quad (2.9)$$

and

$$|a_{2n+1}| \leq \frac{\prod_{j=1}^n (j-2\alpha)}{(2n+1)(n!)} \quad (n \geq 1). \quad (2.10)$$

3 Distortion inequalities

In view of Lemma 1 and Lemma 2, we introduce the subclasses $\mathcal{S}_0(\alpha)$ and $\mathcal{T}_0(\alpha)$. If $f(z) \in \mathcal{S}(\alpha)$ satisfies the coefficient inequalities (1.3), then we say that $f(z) \in \mathcal{S}_0(\alpha)$. Also, if $f(z) \in \mathcal{T}(\alpha)$ satisfies the coefficient inequalities (1.4), then we say that $f(z) \in \mathcal{T}_0(\alpha)$. For $f(z)$ belonging to $\mathcal{S}_0(\alpha)$ and $\mathcal{T}_0(\alpha)$, Cho, Kwon and Owa [1] have shown that

Theorem A *If $f(z) \in \mathcal{S}_0(\alpha)$, then*

$$|z| - \frac{1-2\alpha}{2}|z|^2 - \frac{1-2\alpha}{3-2\alpha}|z|^3 \leq |f(z)| \leq |z| + \frac{1-2\alpha}{2}|z|^2 + \frac{1-2\alpha}{3-2\alpha}|z|^3 \quad (3.1)$$

and

$$1 - (1-2\alpha)|z| - \frac{3(1-2\alpha)}{3-2\alpha}|z|^2 \leq |f'(z)| \leq 1 + (1-2\alpha)|z| + \frac{3(1-2\alpha)}{3-2\alpha}|z|^2 \quad (3.2)$$

for $z \in \mathbb{U}$.

Theorem B *If $f(z) \in \mathcal{T}_0(\alpha)$, then*

$$|z| - \frac{1-2\alpha}{4}|z|^2 - \frac{1-2\alpha}{3(3-2\alpha)}|z|^3 \leq |f(z)| \leq |z| + \frac{1-2\alpha}{4}|z|^2 + \frac{1-2\alpha}{3(3-2\alpha)}|z|^3 \quad (3.3)$$

and

$$1 - \frac{1-2\alpha}{2}|z| - \frac{1-2\alpha}{3-2\alpha}|z|^2 \leq |f'(z)| \leq 1 + \frac{1-2\alpha}{2}|z| + \frac{1-2\alpha}{3-2\alpha}|z|^2 \quad (3.4)$$

for $z \in \mathbb{U}$.

Now, we show

Theorem 3.1 *If $f(z) \in \mathcal{S}_0(\alpha)$, then*

$$|z| - \sum_{n=2}^j |a_n||z|^n - A_j|z|^{j+1} \leq |f(z)| \leq |z| + \sum_{n=2}^j |a_n||z|^n + A_j|z|^{j+1} \quad (3.5)$$

and

$$1 - \sum_{n=2}^{2j-2} n|a_n||z|^{n-1} - B_j|z|^{2j-2} \leq |f'(z)| \leq 1 + \sum_{n=2}^{2j-2} n|a_n||z|^{n-1} + B_j|z|^{2j-2} \quad (3.6)$$

where

$$A_j = \frac{1-2\alpha - \sum_{n=2}^j \{n - (1 + (-1)^{n+1})\alpha\}|a_n|}{j+1 - (1 + (-1)^j)\alpha} \quad (j \geq 2) \quad (3.7)$$

and

$$B_j = (2j-1) \frac{1-2\alpha - \sum_{n=2}^{2j-2} \{n - (1 + (-1)^{n+1})\alpha\}|a_n|}{2j-1-2\alpha} \quad (j \geq 2). \quad (3.8)$$

Proof Note that the coefficient inequalities (1.3) can be written as

$$\sum_{n=2}^{\infty} \{n - (1 + (-1)^{n+1})\alpha\}|a_n| \leq 1-2\alpha. \quad (3.9)$$

This gives us that

$$\sum_{n=2}^j \{n - (1 + (-1)^{n+1}) \alpha\} |a_n| + \{j+1 - (1 + (-1)^j) \alpha\} \sum_{n=j+1}^{\infty} |a_n| \leq 1 - 2\alpha \quad (3.10)$$

and

$$\sum_{n=2}^{2j-2} \{n - (1 + (-1)^{n+1}) \alpha\} |a_n| + \left(1 - \frac{2\alpha}{2j-1}\right) \sum_{n=2j-1}^{\infty} n |a_n| \leq 1 - 2\alpha. \quad (3.11)$$

Therefore, $f(z) \in \mathcal{S}_0(\alpha)$ satisfies

$$\sum_{n=j+1}^{\infty} |a_n| \leq A_j \quad (3.12)$$

and

$$\sum_{n=2j-1}^{\infty} n |a_n| \leq B_j. \quad (3.13)$$

Thus, the distortion inequality (3.5) follows from (3.12) and the distortion inequality (3.6) follows from (3.13).

Remark 3.1 If we take $j = 2$ in Theorem 3.1, then we have Theorem A due to Cho, Kwon and Owa [1].

Furthermore, we also have

Theorem 3.2 If $f(z) \in \mathcal{T}_0(\alpha)$, then

$$|z| - \sum_{n=2}^j |a_n| |z|^n - C_j |z|^{j+1} \leq |f(z)| \leq |z| + \sum_{n=2}^j |a_n| |z|^n + C_j |z|^{j+1} \quad (3.14)$$

and

$$1 - \sum_{n=2}^j n |a_n| |z|^{n-1} - D_j |z|^j \leq |f'(z)| \leq 1 + \sum_{n=2}^j n |a_n| |z|^{n-1} + D_j |z|^j \quad (3.15)$$

for $z \in \mathbb{U}$ where

$$C_j = \frac{1 - 2\alpha - \sum_{n=2}^j n \{n - (1 + (-1)^{n+1}) \alpha\} |a_n|}{(j+1) \{j+1 - (1 + (-1)^j) \alpha\}} \quad (j \geq 2) \quad (3.16)$$

and

$$D_j = \frac{1 - 2\alpha - \sum_{n=2}^j n \{n - (1 + (-1)^{n+1}) \alpha\} |a_n|}{j+1 - (1 + (-1)^j) \alpha} \quad (j \geq 2). \quad (3.17)$$

Proof Noting that the coefficient inequalities (1.4) satisfy

$$\sum_{n=2}^{\infty} n \{n - (1 + (-1)^{n+1}) \alpha\} |a_n| \leq 1 - 2\alpha, \quad (3.18)$$

we have that

$$\begin{aligned} & \sum_{n=2}^j n\{n - (1 + (-1)^{n+1})\alpha\}|a_n| \\ & + (j+1)\{j+1 - (1 + (-1)^{j+2})\alpha\} \sum_{n=j+1}^{\infty} |a_n| \leq 1 - 2\alpha, \end{aligned} \quad (3.19)$$

which implies that

$$\sum_{n=j+1}^{\infty} |a_n| \leq C_j. \quad (3.20)$$

Further, by virtue of (3.18), we see that

$$\sum_{n=2}^j n\{n - (1 + (-1)^{n+1})\alpha\}|a_n| + \{j+1 - (1 + (-1)^{j+2})\alpha\} \sum_{n=j+1}^{\infty} |a_n| \leq 1 - 2\alpha, \quad (3.21)$$

which derives

$$\sum_{n=j+1}^{\infty} |a_n| \leq D_j. \quad (3.22)$$

Therefore, the proof of the theorem follows from (3.21) and (3.22).

Remark 3.2 If we let $j = 2$ in Theorem 3.2, then we have Theorem B by Cho, Kwon and Owa [1].

4 Relation between the classes

By the definitions for the classes $\mathcal{S}_0(\alpha)$, and $\mathcal{T}_0(\alpha)$, we know that

$$\mathcal{S}_0(\alpha) \subset \mathcal{S}_0(\beta) \subset \mathcal{S}_0(0) \quad \left(0 \leq \beta \leq \alpha < \frac{1}{2}\right)$$

and

$$\mathcal{T}_0(\alpha) \subset \mathcal{T}_0(\beta) \subset \mathcal{T}_0(0) \quad \left(0 \leq \beta \leq \alpha < \frac{1}{2}\right).$$

Let us discuss a relation between $\mathcal{S}_0(\beta)$ and $\mathcal{T}_0(\alpha)$.

Theorem 4.1 If $f(z) \in \mathcal{T}_0(\alpha)$, then $f(z) \in \mathcal{S}_0\left(\frac{1+2\alpha}{4}\right)$.

Proof Let $f(z) \in \mathcal{T}_0(\alpha)$. Then, if $f(z)$ satisfies

$$\frac{n - (1 + (-1)^{n+1})\beta}{1 - 2\beta} \leq n \frac{n - (1 + (-1)^{n+1})\alpha}{1 - 2\alpha} \quad (4.1)$$

for all $n \geq 2$, we have that $f(z) \in \mathcal{S}_0(\beta)$. which satisfies the inequality (4.1). After calculation (4.1), we have that

$$\beta \leq n \frac{n-1+(3+(-1)^{n+1})\alpha}{2n^2-(1+(-1)^{n+1})(2n\alpha-2\alpha+1)}. \quad (4.2)$$

If n is even, then (4.2) becomes

$$\beta \leq \frac{n-1+2\alpha}{2n}. \quad (4.3)$$

This implies that

$$\beta \leq \frac{1+2\alpha}{4} \quad (\text{for even } n). \quad (4.4)$$

On the other hand, if n is odd, then (4.3) becomes

$$\beta \leq \frac{n^2-(1-4\alpha)n}{2n^2-4n\alpha+4\alpha-2}. \quad (4.5)$$

Since, for odd n and $0 \leq \alpha < \frac{1}{2}$,

$$\frac{n^2-(1-4\alpha)n}{2n^2-4n\alpha+4\alpha-2} - \frac{1+2\alpha}{4} = \frac{(1-2\alpha)(n-1)(n-1-2\alpha)}{4(n^2-2n\alpha+2\alpha-1)} > 0, \quad (4.6)$$

we conclude that $\beta \leq \frac{1+2\alpha}{4}$ for all n . Thus we conclude that $\mathcal{T}_0(\alpha) \subset \mathcal{S}_0\left(\frac{1+2\alpha}{4}\right)$.

References

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