

TAIL PROBABILITIES OF SOME CONTINUOUS FUNCTIONALS

OF GAUSSIAN PROCESSES

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1. Let $X = \{X(t), 0 \leq t \leq 1\}$ be a path continuous Gaussian process with mean zero, and let T be a real continuous functional on $C[0,1]$ such that $T(cx) = c^p T(x)$ with $p > 0$ for any positive constant c . In this note the following asymptotic estimate for the tail probabilities of $T(X)$ is obtained:

$$\lim_{\alpha \rightarrow \infty} (1/\alpha^{2/p}) \cdot \log P\{T(X) > \alpha\} = -(1/2)b^2,$$

where b^2 is a constant determined as the solution of certain extremal problem. For example, it is shown that if X is Brownian motion, then

$$\lim_{\alpha \rightarrow \infty} (1/\alpha^{2/p}) \cdot \log P\left\{\int_0^1 |X(t)|^p dt > \alpha\right\} = -(1/2)(c(p))^{-2/p},$$

where $p \geq 1$ and

$$c(p) = 2(p+2)^{(p/2)-1} / \left(\int_0^1 (1-t^p)^{-1/2} dt\right)^p p^{p/2},$$

and also, if X is Brownian bridge, then the same formula holds with $c(p)$ replaced by $2^{-p}c(p)$.

In his thesis [3] and also in [4], N. A. Marlow obtained a similar

asymptotic formula for tail probabilities of uniformly Hölder continuous, asymptotically homogeneous functionals F of path continuous Gaussian processes. His method of proof is to first estimate $\log P\{F(X) > \alpha\}$ in the finite-dimensional case by a Laplace asymptotic formula, and then to pass to the limit to obtain the function space version. Note also that H. P. McKean [5] obtained a similar asymptotic estimate for tail probabilities of multiple Wiener integrals.

Our method is different from Marlow's and is based on the following Fredlin-Wentzell type estimates for Gaussian measures given in [7] and [2]. Let $C = C[0,1]$ be the space of all continuous functions on $[0,1]$ with the supremum norm $\|\cdot\|_\infty$, and let A be the σ -field of Borel subsets of C . Let μ be a Gaussian measure on (C, A) with mean zero and covariance function $R(s,t)$, i.e., $\int_C x(t) \mu(dx) = 0$, for $0 \leq t \leq 1$, and $R(s,t) = \int_C x(s)x(t) \mu(dx)$, for $0 \leq s, t \leq 1$, where $x \in C$. Let $H = H(R)$ be the reproducing kernel Hilbert space (RKHS) with reproducing kernel (r.k.) R , whose norm is denoted by $\|\cdot\|_H$. Note that $H \subset C$, since R is continuous.

Theorem 1. Let $\phi \in H$. Then, for any $\delta, h > 0$, there is a number $\alpha_0 = \alpha_0(\delta, h, \|\phi\|_H)$ such that

$$\begin{aligned} \mu\{x \mid \|(x/\alpha) - \phi\|_\infty < \delta\} &\geq \mu\{x \mid \|x - \alpha\phi\|_\infty < \delta\} \\ &\geq \exp[-(\alpha^2/2)(\|\phi\|_H^2 + h)] \end{aligned}$$

for all $\alpha \geq \alpha_0$.

Theorem 2. Let $K_r = \{\phi \in H \mid \|\phi\|_H \leq r\}$ and let $d(x, K_r)$ be the distance from $x \in C$ to K_r in the sup norm $\|\cdot\|_\infty$. Then, for any $\delta, h > 0$,

there is a number $\alpha_0 = \alpha_0(\delta, h, r)$ such that

$$\mu\{x \mid d(x/\alpha, K_r) > \delta\} \leq \exp[-(\alpha^2/2)(r^2 - h)]$$

for all $\alpha \geq \alpha_0$.

For the proofs see [7] or [2]. From Theorems 1 and 2 we obtain the following

Theorem 3. Let T be a real continuous functional on C such that $T(cx) = c^p T(x)$ with $p > 0$ for any positive constant c and $T(\phi) > 0$ for some $\phi \in H$.

Then

$$\lim_{\alpha \rightarrow \infty} (1/\alpha^{2/p}) \cdot \log \mu\{x \mid T(x) > \alpha\} = -(1/2)b^2,$$

where $b^2 = \inf \{ \|\phi\|_H^2 \mid T(\phi) > 1 \} = \sup \{ r^2 \mid \sup\{T(\phi) \mid \phi \in K_r\} < 1 \}$.

Proof. Let $D = \{x \mid T(x) > 1\}$. D is open and its closure $\bar{D} = \{x \mid T(x) \geq 1\}$. For any $\phi \in H \cap D$, there is a $\delta > 0$ such that $\|x - \phi\|_\infty < \delta$ implies $x \in D$. Hence, using Theorem 1, we obtain

$$\begin{aligned} \mu\{x \mid T(x) > \alpha\} &= \mu\{x \mid T(x/\alpha^{1/p}) > 1\} \\ &\geq \mu\{x \mid \|(x/\alpha^{1/p}) - \phi\|_\infty < \delta\} \\ &\geq \exp[-(\alpha^{2/p}/2)(\|\phi\|_H^2 + h)] \end{aligned}$$

for any $h > 0$, if α is sufficiently large. Thus, for any $\phi \in H \cap D$,

$$\liminf_{\alpha \rightarrow \infty} (1/\alpha^{2/p}) \cdot \log \mu\{x \mid T(x) > \alpha\} \geq -(1/2)\|\phi\|_H^2,$$

and hence,

$$\begin{aligned} \liminf_{\alpha \rightarrow \infty} (1/\alpha^{2/p}) \cdot \log \mu\{x \mid T(x) > \alpha\} \\ \geq -(1/2) \cdot \inf\{ \|\phi\|_H^2 \mid T(\phi) > 1 \}. \end{aligned}$$

Since K_r is compact in C (see, e.g. [6]) and T is continuous, there is a number $r > 0$ such that $\sup\{T(\phi) \mid \phi \in K_r\} < 1$, and for any such a number r , there is a $\delta > 0$ such that $d(K_r, \bar{D}) > \delta$, where $d(K_r, \bar{D})$ is the distance between K_r and $\bar{D}$. If $T(x) > \alpha$, then $x/\alpha^{1/p} \in D$, and by Theorem 2,

$$\begin{aligned} \mu\{x \mid T(x) > \alpha\} &\leq \mu\{x \mid d(x/\alpha^{1/p}, K_r) > \delta\} \\ &\leq \exp[-(\alpha^{2/p}/2)(r^2 - h)] \end{aligned}$$

for any $h > 0$, if α is sufficiently large. Therefore,

$$\limsup_{\alpha \rightarrow \infty} (1/\alpha^{2/p}) \cdot \log \mu\{x \mid T(x) > \alpha\} \leq -(1/2)r^2,$$

and hence

$$\begin{aligned} \limsup_{\alpha \rightarrow \infty} (1/\alpha^{2/p}) \cdot \log \mu\{x \mid T(x) > \alpha\} \\ \leq -(1/2) \cdot \sup\{r^2 \mid \sup\{T(\phi) \mid \phi \in K_r\} > 1\}. \end{aligned}$$

It is easy to see that $\inf\{\|\phi\|_H^2 \mid T(\phi) > 1\} = \inf\{\|\phi\|_H^2 \mid T(\phi) \geq 1\}$ (in fact, $= \inf\{\|\phi\|_H^2 \mid T(\phi) = 1\}$), and if $r^2 < \inf\{\|\phi\|_H^2 \mid T(\phi) \geq 1\}$, then $\sup\{T(\phi) \mid \phi \in K_r\} < 1$, and so $\sup\{r^2 \mid \sup\{T(\phi) \mid \phi \in K_r\} > 1\} = \inf\{\|\phi\|_H^2 \mid T(\phi) > 1\}$. This completes the proof.

Remark. Note that $\sup\{T(\phi) \mid \phi \in K_b\} = 1$. Since $\sup\{T(\phi) \mid \phi \in K_b\} = b^p \cdot \sup\{T(\phi) \mid \phi \in K_1\}$, we have $b^2 = (\sup\{T(\phi) \mid \phi \in K_1\})^{-2/p}$.

2. In what follows we consider several examples for which the values of b^2 can be explicitly given by evaluating $\sup\{T(\phi) \mid \phi \in K_1\}$.

(i) Let X be a path continuous Gaussian process with mean zero and covariance function $R(s,t)$. Then

$$\lim_{\alpha \rightarrow \infty} (1/\alpha) \cdot \log P\left\{\int_0^1 X^2(t) dt > \alpha\right\} = -1/(2\lambda_1),$$

where λ_1 is the largest eigenvalue of the covariance operator R with kernel $R(s,t)$ on $L^2[0,1]$.

This is a known result, and so we just indicate briefly how it can be derived from Theorem 3. In this case $T(x) = \int_0^1 x^2(t) dt = \|x\|_2^2$ and $p = 2$. Let $\{\lambda_i\}$ and $\{\psi_i\}$ be the eigenvalues and the corresponding normalized eigenfunctions of R . Then $\{\phi_i = \lambda_i^{1/2} \psi_i\}$ is a complete orthonormal system in $H(R)$. It can be shown that $\|\phi\|_2^2 \leq \lambda_1 \|\phi\|_H^2$ for any $\phi \in H(R)$. Hence $\sup\{T(\phi) \mid \phi \in K_1\} \leq \lambda_1$. Since $\|\phi_1\|_2^2 = \lambda_1$, we have $\sup\{T(\phi) \mid \phi \in K_1\} = \lambda_1$, and hence the result.

(ii) Let μ be the Wiener measure and let $T(x) = \int_0^1 |x(t)|^p dt$, $p \geq 1$. The RKHS $H(R)$ associated with the Wiener measure is the space of all absolutely continuous functions ϕ on $[0,1]$ such that $\phi(0) = 0$ and $d\phi/dt \in L^2[0,1]$, and $(\phi, \psi)_H = \int_0^1 (d\phi/dt)(d\psi/dt) dt$, where $(\cdot, \cdot)_H$ denotes the inner product of $H(R)$. V. Strassen ([8], p.220) proved that $\sup\{T(\phi) \mid \phi \in K_1\} = c(p)$, where

$$c(p) = 2(p+2)^{(p/2)-1} / \left(\int_0^1 (1-t)^p dt\right)^{p/2}.$$

We thus obtain the result for Brownian motion stated at the beginning of this note. In particular, $c(1) = 3^{-1/2}$ and $c(2) = 4/\pi^2$. The case $p = 1$ has been previously obtained by Marlow [3] by a different method, and the

case $p = 2$ is of course a particular case of (i). If p is an integer, then the same formula holds for $T(x) = \int_0^1 (x(t))^p dt$.

(iii) Let μ be the Wiener measure and let

$$T(x) = \int_0^1 |x(t)|^2 dt / \int_0^1 |x(t)| dt.$$

Then $\sup\{T(\phi) \mid \phi \in K_1\} = 2q$, where $0 < q < 1$ is the largest solution of

$$(1-q)^{1/2} \sin((1-q)^{1/2}/q) + \cos((1-q)^{1/2}/q) = 0$$

(see [8], p.222). Hence, if X is Brownian motion, then

$$\lim_{\alpha \rightarrow \infty} (1/\alpha^2) \cdot \log P\left\{ \int_0^1 |X(t)|^2 dt / \int_0^1 |X(t)| dt > \alpha \right\} = -1/(8q)^2.$$

(iv) Let X be Brownian bridge. We shall show that

$$\lim_{\alpha \rightarrow \infty} (1/\alpha^{2/p}) \cdot \log P\left\{ \int_0^1 |X(t)|^p dt > \alpha \right\} = -2(c(p))^{-2/p}, \quad p \geq 1,$$

where $c(p)$ is the same as in (ii).

The covariance function of Brownian bridge is

$$\begin{aligned} R(s,t) &= \begin{cases} s(1-t), & \text{for } s \leq t, \\ t(1-s), & \text{for } s \geq t, \end{cases} \\ &= \int_0^1 Q(u,s)Q(u,t)du, \end{aligned}$$

where

$$Q(u,t) = \begin{cases} 1-t, & \text{for } u \leq t, \\ -t, & \text{for } u > t. \end{cases}$$

Hence the RKHS $H(R)$ with r.k. R is isometrically isomorphic to the closed subspace M of $L^2[0,1]$, spanned by $\{Q(u,t), 0 \leq t \leq 1\}$, and any function ϕ in $H(R)$ has a representation $\phi(t) = \int_0^1 m(u)Q(u,t)du$ with $m \in M$. Note that

$M \perp 1$, i.e., $\int_0^1 m(u) du = 0$ for all $m \in M$, since $\int_0^1 Q(u, t) du = 0$ for all $t \in [0, 1]$ and if $\int_0^1 n(u) du = 0$ and $\int_0^1 n(u) Q(u, t) du = 0$ for all $t \in [0, 1]$, then $n = 0$. Hence $\phi(t) = \int_0^1 m(u) Q(u, t) du = \int_0^t m(u) du$, which shows that ϕ is absolutely continuous. Therefore, $H(R)$ is the space of all absolutely continuous functions ϕ on $[0, 1]$ such that $\phi(0) = \phi(1) = 0$ and $\phi' = d\phi/dt \in L^2[0, 1]$, and $(\phi, \psi)_H = \int_0^1 \phi' \psi' dt$.

As in Strassen's proof [8] for Brownian motion case, we shall evaluate $\sup\{T(\phi) \mid \phi \in K_1\} = \sup\{\int_0^1 |\phi'(t)|^p dt \mid \phi(0) = \phi(1) = 0 \text{ and } \int_0^1 \phi'^2 dt \leq 1\}$ by classical methods of the calculus of variations. Since K_1 is compact and T is continuous, there is a maximizing point ϕ with $\|\phi\|_H^2 = \int_0^1 \phi'^2 dt = 1$. We may assume $\phi \geq 0$, and ϕ satisfies the equation

$$\int_0^1 p \phi^{p-1} \psi dt = 2\lambda \cdot \int_0^1 \phi' \psi' dt, \quad \text{for any } \psi \in H(R),$$

where $\lambda > 0$ is a Lagrange multiplier. Integrating by parts the left-hand side and noting that $\psi' \perp 1$, we obtain

$$\int_0^1 \left\{ \int_t^1 p \phi^{p-1}(s) ds - \int_0^1 \left[\int_s^1 p \phi^{p-1}(u) du \right] ds \right\} \psi'(t) dt = 2\lambda \cdot \int_0^1 \phi' \psi' dt$$

for all $\psi' \in M$. Therefore,

$$(1) \quad \int_t^1 p \phi^{p-1}(s) ds - \int_0^1 \left[\int_s^1 p \phi^{p-1}(u) du \right] ds = 2\lambda \phi'(t); \quad \text{for } 0 \leq t \leq 1.$$

Since $\phi \geq 0$ and $\lambda > 0$, (1) shows that $\phi'(0) \geq 0$, $\phi'(1) \leq 0$ and ϕ' is differentiable and monotone decreasing. Hence there is a point t_0 such that $\phi'(t_0) = 0$ and $\phi'(t) \geq 0$ or ≤ 0 according as $0 \leq t \leq t_0$ or $t_0 \leq t \leq 1$. Differentiating (1), multiplying with ϕ' and integrating again, we have

$$(2) \quad \phi^p(t) + \lambda \phi'^2(t) = \phi^p(1) + \lambda \phi'^2(1) = \lambda \phi'^2(1).$$

Hence $|\phi'(1)| > 0$ and

$$\phi'(t) = \begin{cases} (\phi'^2(1) - (1/\lambda)\phi^p(t))^{1/2} & \text{for } 0 \leq t \leq t_0, \\ -(\phi'^2(1) - (1/\lambda)\phi^p(t))^{1/2} & \text{for } t_0 \leq t \leq 1. \end{cases}$$

Therefore, noting that $\phi(0) = \phi(1) = 0$, we get, for $0 \leq t \leq t_0$,

$$\begin{aligned} (3) \quad t &= \int_0^{\phi(t)} |\phi'(1)|^{-1} (1 - u^p/(\lambda\phi'^2(1)))^{-1/2} du \\ &= \lambda^{1/p} |\phi'(1)|^{(2/p)-1} \int_0^{\phi(t)/(\lambda\phi'^2(1))} (1 - v^p)^{-1/2} dv, \end{aligned}$$

and, for $t_0 \leq t \leq 1$,

$$(4) \quad t - 1 = -\lambda^{1/p} |\phi'(1)|^{(2/p)-1} \int_0^{\phi(t)/(\lambda\phi'^2(1))} (1 - v^p)^{-1/2} dv.$$

Put $t = t_0$ in (3) and (4). Then $t_0 = 1 - t_0$, and so $t_0 = 1/2$. Put $t = 1/2$ in (2) and (3). Then $\phi^p(1/2) = \lambda\phi'^2(1)$ and

$$(5) \quad 1/2 = \lambda^{1/p} |\phi'(1)|^{(2/p)-1} \int_0^1 (1 - v^p)^{-1/2} dv.$$

Integrating (2) and noting $\int_0^1 \phi'^2 dt = 1$, we have

$$(6) \quad \int_0^1 \phi^p(t) dt = \lambda(\phi'^2(1) - 1)$$

Using (3) and (4), we obtain

$$\begin{aligned} \int_0^{1/2} \phi^p(t) dt &= \lambda^{1+(1/p)} |\phi'(1)|^{1+(2/p)} \cdot \int_0^1 v^p (1 - v^p)^{-1/2} dv \\ &= \int_{1/2}^1 \phi^p(t) dt. \end{aligned}$$

Hence

$$\begin{aligned} (7) \quad \int_0^1 \phi^p(t) dt &= 2\lambda^{1+(1/p)} |\phi'(1)|^{1+(2/p)} \cdot \int_0^1 v^p (1 - v^p)^{-1/2} dv \\ &= 4(p+2)^{-1} \lambda^{1+(1/p)} |\phi'(1)|^{1+(2/p)} \cdot \int_0^1 (1 - v^p)^{-1/2} dv. \end{aligned}$$

Eliminating λ and $|\phi'(1)|$ from (5), (6) and (7), we obtain

$$\int_0^1 \phi^p(t) dt = 2^{1-p} (p+2)^{(p/2)-1} / (\int_0^1 (1-t^p)^{-1/2} dt)^{p/2} = 2^{-p} c(p).$$

Thus $b^2 = 4(c(p))^{-2/p}$.

Remark. If p is an integer, $T(x) = \int_0^1 |x(t)|^p dt$ can be replaced by $T(x) = \int_0^1 x^p(t) dt$. The above result $\sup\{T(\phi) \mid \phi \in K_1\} = 2^{-p} c(p)$ can be used to obtain an iterated logarithm result for the functional T of empirical distributions (cf. H. Finkelstein [1]). Finkelstein discusses only the case $p = 2$, which can be obtained as a particular case of (i).

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