

Hata-Yamaguti's result on Takagi function and its applications to digital sum problems

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1 Introduction

Let $n \in \mathbb{N}$ and denote its binary expansion by $n = \sum_{k \geq 0} \alpha_k(n) 2^k$ with $\alpha_k(n) \in \{0, 1\}$. We define

$$\begin{aligned} s(n) &= \sum_{k \geq 0} \alpha_k(n) && \text{(the binary digital sum),} \\ F(\xi, N) &= \sum_{n=0}^{N-1} e^{\xi s(n)} && \text{(the exponential sum),} \\ S_p(N) &= \sum_{n=0}^{N-1} s(n)^p && \text{(the power sum)} \end{aligned}$$

for $N \in \mathbb{N}$ and $p, \xi \in \mathbb{R}$. We first review some fundamental results on these sums. If N is a power of 2, we immediately have $F(\xi, N) = N^{\log_2(1+e^\xi)}$ and $S_1(N) = N \log_2 N / 2$. However it is not so easy to obtain explicit formulas for arbitrary $N \in \mathbb{N}$. In early times the asymptotic behavior of $S_1(N)$ was studied:

$$\begin{aligned} S_1(N) &\sim N \frac{\log_2 N}{2} \quad (N \rightarrow \infty) && \text{(Bush [1]),} \\ S_1(N) &= N \frac{\log_2 N}{2} + O(N) \quad (N \rightarrow \infty) && \text{(Mirsky [8]).} \end{aligned}$$

Finally, Trollope [18] obtained a precise formula for $S_1(N)$ and Delange [4] gave its elegant proof. Let F be a nowhere differentiable continuous periodic function of period 1 given by

$$F(x) = 1 - x - 2^{1-x} T\left(\frac{1}{2^{1-x}}\right), \quad 0 \leq x \leq 1$$

with T the Takagi function. Then

$$S_1(N) = N \frac{\log_2 N}{2} + \frac{N}{2} F(\log_2 N) \quad (\text{Trollope, Delange}).$$

Coquet [3] obtained a precise formula for positive integer powers.

Theorem 1.1 (Coquet [3]) *There are periodic functions $G_{p,\tau} : \mathbb{R} \rightarrow \mathbb{R}$, $0 \leq \tau \leq p$, of period 1, such that*

$$S_p(N) = N \left(\frac{\log_2 N}{2}\right)^p + N \sum_{0 \leq \tau < p} (\log_2 N)^\tau G_{p,\tau}(\log_2 N).$$

for every integer $p \geq 1$. Furthermore $G_{p,\tau}$ verify

$$(1) \quad 2^{-\tau} \binom{d}{\tau} + \sum_{\tau < p < d} \binom{d}{p} G_{p,\tau} = 2^{1-d} \binom{d}{\tau} + \sum_{\tau < q < d} 2 \binom{q}{\tau} G_{d,q}$$

for $d \geq 2$ and $\tau \leq d - 2$.

Theorem 1.2 (Coquet [3])

$$S_2(N) = N \left(\frac{\log_2 N}{2} \right)^2 + N \frac{\log_2 N}{2} \left\{ \frac{1}{2} + F(\log_2 N) \right\} + NG(\log_2 N),$$

where G is a nowhere differentiable continuous periodic function of period 1.

An explicit form of the function G is stated in Osbaldestin [13]. However, for $p \geq 3$, we cannot get such an explicit formula via induction formulae (1) and the continuity of $G_{p,\tau}$ is unknown.

Concerning $F(\xi, N)$, Stolarsky [17] proved that $F(\log_2 N)/N^{\log_2 3}$ is not well-behaved asymptotically. Harborth [6] obtained the following estimates:

$$(2) \quad \begin{aligned} \limsup_{N \rightarrow \infty} \frac{F(\log_2 N)}{N^{\log_2 3}} &= 1, \\ 0.812556 &< \liminf_{N \rightarrow \infty} \frac{F(\log_2 N)}{N^{\log_2 3}} < 0.812557. \end{aligned}$$

We now introduce a function G_ξ by

$$G_\xi(\log_2 N) = \frac{F(\xi, N)}{N^{\log_2(1+\epsilon^\xi)}}.$$

Coquet [3] and Stein [16] investigated the properties of G_ξ . Stein proved that G_ξ is a continuous periodic function of period 1 by giving a formula of F . However, it is unknown if G_ξ is differentiable. In this note, we get a simple explicit formula of $F(\xi, N)$ by the use of the connection between $s(n)$ and the binomial measure μ_r . And using the results obtained in Hata-Yamaguti [7] and Sekiguchi-Shiota [15], we derive explicit formulas of the power sum $S_p(N)$. We notice that the higher order derivatives of the distribution function of μ_r with respect to r play an important role in the explicit formula of $S_p(N)$. The results in this note can be extended to the sum of q -adic digits by the use of multinomial measures (see M-O-S-S [9]).

2 Hata-Yamaguti's result

Let $I = I_{0,0} = [0, 1]$ and

$$I_{n,j} = \left[\frac{j}{2^n}, \frac{j+1}{2^n} \right), \quad j = 0, 1, \dots, 2^n - 2, \quad I_{n,2^n-1} = \left[\frac{2^n-1}{2^n}, 1 \right]$$

for $n = 1, 2, 3, \dots$

Define the binomial measure μ_r ($0 < r < 1$) by a probability measure on I such that

$$\mu_r(I_{n+1,2j}) = r\mu_r(I_{n,j}), \quad \mu_r(I_{n+1,2j+1}) = (1-r)\mu_r(I_{n,j})$$

for $n = 0, 1, 2, \dots, j = 0, 1, \dots, 2^n - 1$.

We denote the distribution function of μ_r by L :

$$L(r, x) = \mu_r([0, x]).$$

It is well-known that $L(r, \cdot)$ is a strictly increasing continuous and singular function except for $r = 1/2$ (see Salem [14]). It immediately follows that $L(r, \cdot)$ satisfies the system of infinitely many difference equations:

$$\begin{cases} L(r, \frac{2j+1}{2^{n+1}}) - (1-r)L(r, \frac{j}{2^n}) - rL(r, \frac{j+1}{2^n}) = 0, \\ L(r, 0) = 0, \quad L(r, 1) = 1, \\ n = 0, 1, 2, \dots, \quad j = 0, 1, \dots, 2^n - 1. \end{cases}$$

This system is equivalent to the following functional equation:

$$L(r, x) = \begin{cases} rL(r, 2x), & 0 \leq x \leq \frac{1}{2}, \\ (1-r)L(r, 2x-1) + r, & \frac{1}{2} \leq x \leq 1. \end{cases}$$

Let

$$\begin{aligned} R(x) &= 1_{I_{1,0}}(x) - 1_{I_{1,1}}(x), \quad 0 \leq x \leq 1, \\ \phi(x) &= \begin{cases} 2x & 0 \leq x < 1/2, \\ 2x-1 & 1/2 \leq x \leq 1, \end{cases} \\ \psi(x) &= \int_0^x 2R(t)dt, \quad 0 \leq x \leq 1. \end{aligned}$$

The Takagi function T is defined by

$$T(x) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \psi(\phi^n(x)) = \sum_{n=1}^{\infty} \frac{1}{2^n} \psi^n(x), \quad 0 \leq x \leq 1.$$

It is well-known that T is a nowhere differentiable continuous function. And T satisfies the system of infinitely many difference equations:

$$\begin{cases} T(\frac{2j+1}{2^{n+1}}) - \frac{1}{2}T(\frac{j}{2^n}) - \frac{1}{2}T(\frac{j+1}{2^n}) = \frac{1}{2^{n+1}}, \\ T(0) = 0, \quad T(1) = 0, \\ n = 0, 1, 2, \dots, \quad j = 0, 1, \dots, 2^n - 1. \end{cases}$$

Hata-Yamaguti [7] have obtained the following formula which connects the Takagi function T with the function L .

Theorem 2.1 (Hata-Yamaguti [7]) *We have*

$$\frac{1}{2} \frac{\partial}{\partial r} L(r, x) \Big|_{r=\frac{1}{2}} = T(x).$$

Remark 2.1 We have

$$\begin{cases} \frac{\partial}{\partial r} L(r, \frac{2j+1}{2^{n+1}}) - (1-r) \frac{\partial}{\partial r} L(r, \frac{j}{2^n}) - r \frac{\partial}{\partial r} L(r, \frac{j+1}{2^n}) = L(r, \frac{j+1}{2^n}) - L(r, \frac{j}{2^n}), \\ \frac{\partial}{\partial r} L(r, 0) = 0, \quad \frac{\partial}{\partial r} L(r, 1) = 0, \\ n = 0, 1, 2, \dots, \quad j = 0, 1, \dots, 2^n - 1. \end{cases}$$

Above system has a unique continuous solution (Hata-Yamaguti [7], S-S [15]).

More generally we have the following:

Theorem 2.2 (S-S [15]) $L(r, x)$ is a continuous function valued analytic function of $r \in I$ and the equality

$$\left. \frac{\partial^k L(r, x)}{\partial r^k} \right|_{r=\frac{1}{2}} = k! T_{\frac{1}{2}, k}(x)$$

holds for $k = 1, 2, 3, \dots$. Here

$$\begin{aligned} T_{\frac{1}{2}, 1}(x) &= 2T(x), \\ T_{\frac{1}{2}, k}(x) &= \sum_{n=0}^{\infty} \frac{1}{2^n} R(2^n x) T_{\frac{1}{2}, k-1}(2^{n+1} x). \end{aligned}$$

Remark 2.2 It also follows that $T_{r, k}$ satisfies the system of infinitely many difference equations:

$$\begin{cases} T_{r, k}(\frac{2j+1}{2^{n+1}}) - (1-r) T_{r, k}(\frac{j}{2^n}) - r T_{r, k}(\frac{j+1}{2^n}) = T_{r, k-1}(\frac{j+1}{2^n}) - T_{r, k-1}(\frac{j}{2^n}), \\ T_{r, k}(0) = 0, \quad T_{r, k}(1) = 0, \\ n = 0, 1, 2, \dots, \quad j = 0, 1, \dots, 2^n - 1. \end{cases}$$

3 An explicit formula of exponential sums

We first give a lemma which suggests a close connection between the distribution function L and digital sums. Set $t = \log_2 N$ for $N \in \mathbb{N}$ and denote by $[t]$ its integer part and by $\{t\}$ its decimal part.

Lemma 3.1 We have

$$(3) \quad L(r, \frac{1}{2^{1-\{t\}}}) = \sum_{n=0}^{N-1} r^{[t]+1-s(n)} (1-r)^{s(n)}.$$

Taking $r = \frac{1}{1+e^\xi}$ in (3), we immediately have next theorem.

Theorem 3.1 We have

$$(4) \quad F(\xi, N) = N^{\log_2(1+e^\xi)} 2^{(1-\{t\}) \log_2(1+e^\xi)} L\left(\frac{1}{1+e^\xi}, \frac{1}{2^{1-\{t\}}}\right)$$

for $\xi \in \mathbb{R}$.

Remark 3.1 By (4), we know that G_ξ is differentiable for almost every $t \in \mathbb{R}_+$.

4 Asymptotic behavior of $F(\xi, N)/N^{\log_2(1+e^\xi)}$

Theorem 4.1 *We have*

$$\limsup_{N \rightarrow \infty} \frac{F(\xi, N)}{N^{\log_2(1+e^\xi)}} = \max_{\frac{1}{2} \leq x \leq 1} x^{\log_2 r} L(r, x),$$

$$\liminf_{N \rightarrow \infty} \frac{F(\xi, N)}{N^{\log_2(1+e^\xi)}} = \min_{\frac{1}{2} \leq x \leq 1} x^{\log_2 r} L(r, x).$$

Hence, to obtain the precise values on the left-hand sides of these equations, it suffices to estimate the function $g(x) = x^{\log_2 r} L(r, x)$. However, it is very hard to get the maximum and the minimum of the function g .

Proposition 4.1 *we have*

$$\limsup_{N \rightarrow \infty} \frac{F(\xi, N)}{N^{\log_2(1+e^\xi)}} = 1 \quad \text{for } \xi > 0,$$

$$\liminf_{N \rightarrow \infty} \frac{F(\xi, N)}{N^{\log_2(1+e^\xi)}} = 1 \quad \text{for } \xi < 0.$$

These estimations are obtained by Stein [16].

Proposition 4.2 *For $k = 1, 2, \dots, 2^{n-1} - 1, n = 1, 2, \dots$, we have*

$$g\left(\frac{2k+1}{2^{n+1}}\right) > \min\left\{g\left(\frac{4k+1}{2^{n+2}}\right), g\left(\frac{4k+3}{2^{n+2}}\right)\right\}, \quad \text{if } 0 < r < \frac{1}{2},$$

$$g\left(\frac{2k+1}{2^{n+1}}\right) < \max\left\{g\left(\frac{4k+1}{2^{n+2}}\right), g\left(\frac{4k+3}{2^{n+2}}\right)\right\}, \quad \text{if } \frac{1}{2} < r < 1.$$

Remark 4.1 The above inequalities are essential in Harborth's algorithm concerned with the lower bound of the function $F(\log 2, N)/N^{\log_2 3}$. Harborth's algorithm is that, by starting with $n_0 = 1$ and $n_{r+1} = 2n_r \pm 1$ where $+$ or $-$ is chosen so that $q_{r+1} = F(\log 2, n_{r+1})/n_{r+1}^{\log_2 3}$ becomes minimal. Then $\{q_r\}$ is strictly decreasing and $q = \lim_{n \rightarrow \infty} q_r < 0.812556 \dots$ (c.f. (2)). The question whether $q = \lim_{n \rightarrow \infty} q_r$ gives a true lower bound is still unknown.

5 From exponential sums to power sums

We set

$$E(r, t) = 2^{1-\{t\}} L\left(r, \frac{1}{2^{1-\{t\}}}\right), \quad 0 < r < 1, t \in \mathbf{R}.$$

Evidently $E(r, 0) = 2r$, $E(r, 1-) = 1$, $E(\frac{1}{2}, t) = 1$, and E is continuous except for $t \in \mathbf{Z}$ and periodic of period 1 as a function of t . Furthermore E is analytic in $r \in (0, 1)$. By use of $E(r, t)$,

$$F(\xi, N) = (1 + e^\xi)^t \left(\frac{1 + e^\xi}{2}\right)^{1-\{t\}} E\left(\frac{1}{1 + e^\xi}, t\right).$$

On the other hand, evidently the equality

$$S_k(N) = \frac{\partial^k}{\partial \xi^k} F(\xi, N) \Big|_{\xi=0}$$

holds for $k = 1, 2, 3, \dots$. Hence we can directly derive explicit formulas of power sums of lower order from these equations. We set

$$E^{(k)}\left(\frac{1}{2}, t\right) = \frac{\partial^k}{\partial r^k} E(r, t) \Big|_{r=\frac{1}{2}}.$$

Then we have

$$S_1(N) = N\left(\frac{t}{2} + \frac{1-\{t\}}{2} - \frac{1}{4}E^{(1)}\left(\frac{1}{2}, t\right)\right) \quad (\text{Trollope [18], Delange [4]}),$$

$$S_2(N) = N\left(\left(\frac{t}{2}\right)^2 + H_{2,1}(t)\frac{t}{2} + H_{2,0}(t)\right) \quad (\text{Coquet [3], Osbaldestin [13]})$$

where

$$\begin{aligned} H_{2,1} &= \frac{1}{2} + 1 - \{t\} - \frac{1}{2}E^{(1)}\left(\frac{1}{2}, t\right), \\ H_{2,0} &= \frac{2 - 3\{t\} + \{t\}^2}{4} - \frac{1 - \{t\}}{4}E^{(1)}\left(\frac{1}{2}, t\right) + \frac{1}{16}E^{(2)}\left(\frac{1}{2}, t\right), \end{aligned}$$

$$S_3(N) = N\left(\left(\frac{t}{2}\right)^3 + H_{3,2}(t)\left(\frac{t}{2}\right)^2 + H_{3,1}(t)\frac{t}{2} + H_{3,0}(t)\right)$$

(Grabner, Kirschenhofer, Prodinger and Tichy [5], O-S-S [10])

where

$$\begin{aligned} H_{3,2}(t) &= -\frac{3}{4}E^{(1)}\left(\frac{1}{2}, t\right) - \frac{3\{t\} - 6}{2}, \\ H_{3,1}(t) &= \frac{3}{16}E^{(2)}\left(\frac{1}{2}, t\right) + \frac{6\{t\} - 9}{8}E^{(1)}\left(\frac{1}{2}, t\right) + \frac{3\{t\}^2 - 12\{t\} + 9}{4}, \\ H_{3,0}(t) &= -\frac{1}{64}E^{(3)}\left(\frac{1}{2}, t\right) - \frac{3\{t\} - 3}{32}E^{(2)}\left(\frac{1}{2}, t\right) \\ &\quad - \frac{3\{t\}^2 - 9\{t\} + 4}{16}E^{(1)}\left(\frac{1}{2}, t\right) - \frac{\{t\}^3 - 6\{t\}^2 + 9\{t\} - 4}{8}, \end{aligned}$$

$$S_4(N) = N\left(\left(\frac{t}{2}\right)^4 + H_{4,3}(t)\left(\frac{t}{2}\right)^3 + H_{4,2}(t)\left(\frac{t}{2}\right)^2 + H_{4,1}(t)\frac{t}{2} + H_{4,0}(t)\right) \quad (\text{O-S-S [11]})$$

where

$$\begin{aligned} H_{4,3}(t) &= -E^{(1)}\left(\frac{1}{2}, t\right) - 2\{t\} + 5, \\ H_{4,2}(t) &= \frac{3}{8}E^{(2)}\left(\frac{1}{2}, t\right) + \frac{3\{t\} - 6}{2}E^{(1)}\left(\frac{1}{2}, t\right) + \frac{6\{t\}^2 - 30\{t\} + 27}{4}, \\ H_{4,1}(t) &= -\frac{1}{16}E^{(3)}\left(\frac{1}{2}, t\right) - \frac{6\{t\} - 9}{16}E^{(2)}\left(\frac{1}{2}, t\right) - \frac{3\{t\}^2 - 12\{t\} + 7}{4}E^{(1)}\left(\frac{1}{2}, t\right) \\ &\quad - \frac{2\{t\}^3 - 15\{t\}^2 + 27\{t\} - 13}{4}, \\ H_{4,0}(t) &= \frac{1}{256}E^{(4)}\left(\frac{1}{2}, t\right) + \frac{\{t\} - 1}{32}E^{(3)}\left(\frac{1}{2}, t\right) + \frac{3\{t\}^2 - 9\{t\} + 2}{32}E^{(2)}\left(\frac{1}{2}, t\right) \\ &\quad + \frac{\{t\}^3 - 6\{t\}^2 + 7\{t\} - 2}{8}E^{(1)}\left(\frac{1}{2}, t\right) + \frac{\{t\}^4 - 10\{t\}^3 + 27\{t\}^2 - 26\{t\} + 8}{16}. \end{aligned}$$

We now extend Theorem 1.1 and 1.2 and get a precise formula of S_p .

Theorem 5.1 (O-S-S [10]) *We have*

$$S_k(N) = N \sum_{p=0}^k H_{k,p}(t) \left(\frac{t}{2}\right)^p, \quad k = 0, 1, 2, \dots$$

Here $H_{k,p}(t)$ is a periodic continuous function of period 1, defined inductively as follows:

$$H_{0,0}(t) = E\left(\frac{1}{2}, t\right) = 1,$$

$$\begin{aligned} \frac{(-2)^k}{k!} H_{k,0}(t) &= \frac{1}{2^k k!} E^{(k)}\left(\frac{1}{2}, t\right) - \sum_{j=0}^{k-1} a(k, j, 1 - \{t\}) H_{j,0}(t), \\ \frac{(-2)^k}{k!} H_{k,p}(t) &= - \sum_{j=0}^{k-1} \sum_{q=0 \vee (p-j)}^{p \wedge (k-j)} \frac{2^q}{q!} a^{(q)}(k, j, 1 - \{t\}) H_{j,p-q}(t) \\ &\quad \text{for } p = 1, 2, \dots, k, \end{aligned}$$

where $a(k, j, t)$ is defined by

$$\frac{2^{t-k}}{k!} \frac{\partial^k}{\partial r^k} (r^{t-s} (1-r)^s) \Big|_{r=\frac{1}{2}} = \sum_j a(k, j, t) s^j,$$

$s \in \mathbf{R}$, $t \in \mathbf{R}$, $k, j \in \mathbf{Z}$, $k \geq 0$, and $a^{(p)}(k, j, t) = \partial^p a(k, j, t) / \partial t^p$.

Furthermore the functions $H_{k,p}(t)$ satisfies the induction formulas:

$$\sum_{j=p}^{k-1} \binom{k}{j} H_{j,p}(t) = 2^{p+1} \sum_{j=p+1}^k 2^{-j} \binom{j}{p} H_{k,j}(t)$$

for $k \geq 1$, $0 \leq p \leq k-1$.

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