

Characteristic classes of symplectomorphism groups as discrete groups

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1 Problem

(M, ω) : closed symplectic manifold, $\dim = 2n$

$\text{Symp } M$: symplectomorphism group, $\text{Symp}^\delta M$: with discrete topology

Problem 1 (widely open problem)

$$H^*(\text{BSymp } M) = ?$$

$$H^*(\text{BSymp}^\delta M) = H^*(\text{Symp}^\delta M) = ?$$

$M = \Sigma_g$ closed oriented surface $g \geq 2$:

$$\text{Symp } \Sigma_g \overset{\text{Moser}}{\sim} \text{Diff}^+ \Sigma_g \overset{\text{Earle-Eells}}{\sim} \mathcal{M}_g \Rightarrow H^*(\text{BSymp } \Sigma_g) \cong H^*(\text{Diff } \Sigma_g) \cong H^*(\mathcal{M}_g)$$

but $H^*(\text{Symp}^\delta \Sigma_g)$ completely different

joint work with D. Kotschick

2 Gel'fand-Fuks cohomology theory

$\mathfrak{a}_n$: Lie algebra of formal vector fields on $\mathbb{R}^n$

$$\mathfrak{a}_n = \left\{ \sum_i f_i \frac{\partial}{\partial x_i}; f_i \in \mathbb{R}[[x_1, \dots, x_n]] \right\}$$

$\text{B}\Gamma_n$: Haefliger classifying space for codimension n foliations

$$H_{GF}^*(\mathfrak{a}_n, \text{O}(n)) \longrightarrow H^*(\text{B}\Gamma_n; \mathbb{R}) \quad \text{Gel'fand-Fuks cohomology}$$

Theorem 2 (Gel'fand-Fuks)

$$H_{GF}^*(\mathfrak{a}_n), H_{GF}^*(\mathfrak{a}_n, \text{O}(n))$$

finite dimensional

Vey found a basis. In particular, the class with the lowest degree:

$$H_{GF}^{2n+1}(\mathfrak{a}_n, \mathcal{O}(n)) \ni h_1 c_1^n$$

close relation with the **Godbillon-Vey** class:

$$GV \in H^{2n+1}(\mathrm{B}\Gamma_n; \mathbb{R})$$

case of $n = 1$:

$\mathcal{F}$: cod. 1 foliation on X defined by 1-form $\theta \Rightarrow d\theta = \eta \wedge \theta$ for some η

$$\eta \wedge d\eta \text{ is a closed 3-form and } GV(\mathcal{F}) = [\eta \wedge d\eta] \in H^3(X; \mathbb{R})$$

Theorem 3 (Thurston) *There is a one parameter family $\mathcal{F}_t$ of cod. 1 foliations on S^3 such that*

$$GV(\mathcal{F}_t) = t \in H^3(S^3; \mathbb{R}) \cong \mathbb{R}$$

essential use of hyperbolic geometry

3 Gel'fand-Fuks cohomology of formal Hamiltonian vector fields

$\mathfrak{ham}_{2n}$: Lie algebra of formal Hamiltonian vector fields on $(\mathbb{R}^{2n}, \omega)$

$\mathrm{B}\Gamma_{2n}^\omega$: Haefliger classifying space for codimension $2n$ transversely symplectic foliation

$H_{GF}^*(\mathfrak{ham}_{2n}, \mathrm{U}(n)) \longrightarrow H^*(\mathrm{B}\Gamma_{2n}^\omega; \mathbb{R})$ Gel'fand-Fuks cohomology

very difficult open question (**mystery!**):

$H_{GF}^*(\mathfrak{ham}_{2n})$ infinitely generated ?

Gel'fand-Kalinin-Fuks: (cohomology is **bigraded**)

$$H_{GF}^*(\mathfrak{ham}_{2n}, \mathrm{Sp}(2n, \mathbb{R}))_{(\leq 0)} \cong \mathbb{R}[\omega, p_1, p_2, \dots, p_n] / \text{Bott vanishing}$$

$$\omega^k p_1^{k_1} \cdots p_n^{k_n} = 0$$

$$\text{for } k + k_1 + 2k_2 + \cdots + nk_n > n$$

For the case $n = 1$, they also found

$$H_{GF}^*(\mathfrak{ham}_2, \mathrm{Sp}(2, \mathbb{R}))_{(k)} = 0 \text{ for } k = 1, 2, 3$$

But, $\exists$ **exotic class!** in $H_{GF}^7(\mathfrak{ham}_2, \mathrm{Sp}(2, \mathbb{R}))_{(4)}$ **Gel'fand-Kalinin-Fuks class** (1972)

In 1999, Metoki found another exotic class in $H_{GF}^9(\mathfrak{ham}_2, \mathrm{Sp}(2, \mathbb{R}))_{(7)}$ **Metoki class**

On the other hand, **Perchik** (1975) obtained a formula for the Euler characteristics

$$\chi(H_{GF}^*(\mathfrak{ham}_2, \mathrm{Sp}(2, \mathbb{R}))_{(k)})$$

$$\Rightarrow \exists 57 - 4(-1) \text{ more exotic classes! He could not prove } \dim H_{GF}^*(\mathfrak{ham}_2) = \infty$$

Conjecture 4 (Folklore)

$$\dim H_{GF}^*(\mathfrak{ham}_{2n}, \mathrm{Sp}(2n, \mathbb{R})) = \infty, \dim H_{GF}^*(\mathfrak{div}_n, \mathrm{SL}(n, \mathbb{R})) = \infty$$

$\mathfrak{div}_n$: Lie algebra of formal divergence free vector fields on $\mathbb{R}^n$, $\mathfrak{ham}_2 = \mathfrak{div}_2$

4 Foliated cohomology

$(X, \mathcal{F})$: codimension n foliated manifold

Definition 5 (foliated cohomology)

$$\Omega_{\mathcal{F}}^*(X) := \Gamma(X, \Lambda^*(T_{\mathcal{F}}^*)) = \Omega^*(X) / (\text{forms vanishing on leaves})$$

$$H_{\mathcal{F}}^*(X) = H^*(\Omega_{\mathcal{F}}^*(X))$$

wild object, hard to compute

5 Foliated cohomology in the symplectic case

$$\mathfrak{ham}_{2n} \cong \mathbb{R}[[x_1, \dots, x_n; y_1, \dots, y_n]]/\mathbb{R} \cong \prod_{k=1}^{\infty} S^k(\mathbb{R}^{2n}, \omega)$$

$$\mathfrak{ham}_{2n}^0 = \mathfrak{ham}_{2n} \text{ without constant term}$$

Kontsevich: transversely symplectic codimension $2n$ foliated manifold $(X, \mathcal{F})$

$$H_{GF}^*(\mathfrak{ham}_{2n}^0, U(n)) \longrightarrow H_{\mathcal{F}}^*(X) \xrightarrow{\wedge \omega^n} H^{*+2n}(X; \mathbb{R})$$

$$H_{GF}^*(\mathfrak{ham}_{2n}^0, U(n)) \longrightarrow H^{*+2n}(\mathfrak{ham}_{2n}, U(n)) \longrightarrow H^{*+2n}(B\Gamma\omega_{2n}; \mathbb{R})$$

$\Rightarrow$ for any symplectic manifold (M, ω)

$$H_{GF}^*(\mathfrak{ham}_{2n}^0, U(n)) \longrightarrow H^*(\text{Symp}^\delta(M, \omega); \mathbb{R})$$

Furthermore

$$\mathfrak{ham}_{2n}^1 = \mathfrak{ham}_{2n}^0 \text{ without linear term} \Rightarrow H_{GF}^*(\mathfrak{ham}_{2n}^0, \text{Sp}(2n, \mathbb{R})) \cong H_{GF}^*(\mathfrak{ham}_{2n}^1; \mathbb{R})^{\text{Sp}}$$

$\Rightarrow$ obtain homomorphisms

$$H_{GF}^*(\mathfrak{ham}_{2n}^1; \mathbb{R})^{\text{Sp}} \longrightarrow H^{*+2n}(\mathfrak{ham}_{2n}, \text{Sp}(2n, \mathbb{R}))$$

$$H_{GF}^*(\mathfrak{ham}_{2n}^1; \mathbb{R})^{\text{Sp}} \longrightarrow H^*(\text{Symp}(M, \omega); \mathbb{R})$$

in the case of (Σ_g, ω) , what is

$$H_{GF}^*(\mathfrak{ham}_2^1)^{\text{Sp}} \longrightarrow H_{GF}^{*+2}(\mathfrak{ham}_2, \text{Sp}(2, \mathbb{R}))?$$

$$H_{GF}^*(\mathfrak{ham}_2^1)^{\text{Sp}} \longrightarrow H^*(\text{Symp}(\Sigma_g, \omega); \mathbb{R})?$$

detects Gel'fand-Kalinin-Fuks and Metoki classes?

Answer: yes for GKF class

$\exists$ unique element $\eta \in H^5(\mathfrak{ham}_2^1)^{\text{Sp}}$ s.t.

GKF class = $\eta \wedge \omega \in H^7(\mathfrak{ham}_2, \text{Sp}(2, \mathbb{R}))$

merits of this approach:

(*) the stable cohomology

$$\lim_{n \rightarrow \infty} H_{GF}^*(\mathfrak{ham}_{2n}) \cong \mathbb{R}[\omega]$$

is uninteresting (Guillemin-Shnider), but the stable cohomology

$$\lim_{n \rightarrow \infty} H_{GF}^*(\mathfrak{ham}_{2n}^1)^{\text{Sp}}$$

seems to be highly non-trivial!

(**) $H_{GF}^*(\mathfrak{ham}_{2n}^1)^{\text{Sp}}$ is easier than $H_{GF}^*(\mathfrak{ham}_{2n})$ we are trying to decompose **Metoki class**,

and possibly define new exotic classes

(***) similarly for the case of $\mathfrak{div}_n$

$$\exists \eta \wedge \text{volume form} \in H^{5+n}(\mathfrak{div}_n, \text{SL}(n, \mathbb{R}))?$$

(****) we could try to prove

$$\dim H_{GF}^*(\mathfrak{ham}_{2n}^1)^{\text{Sp}} = \infty$$

6 The GV classes and foliated cohomology

The Godbillon-Vey classes

$$\text{GV} = h_1 c_1^n \in H_{GF}^{2n+1}(\mathfrak{a}_n, \text{O}(n))$$

are **NOT** stable classes, but they can be decomposed as

$$h_1 c_1^n = h_1 \wedge c_1^n \quad \text{where}$$

$$h_1 \in H_{GF}^1(\mathfrak{a}_n^0, \mathcal{O}(n))$$

is a **stable leaf invariant!** coming from the determinant of the holonomy

$$\mathfrak{a}_n^0 = \mathfrak{a}_n \quad \text{without constant term}$$

$$(X, \mathcal{F}): \text{codimension } n \text{ foliated manifold} \Rightarrow H_{\mathcal{F}}^1(X) \ni h_1(\mathcal{F}) \xrightarrow{\wedge c_1^n} GV(\mathcal{F}) \in H^{2n+1}(X; \mathbb{R})$$

So if $h_1(\mathcal{F}) = 0$, then $GV(\mathcal{F}) = 0$ as well.

We have proved that the original GKF class can be decomposed into
a leaf invariant η times ω

7 Sketch of proof

Metoki gave a formula for GKF class , 15 lines (with about 100 terms!)

10 lines are **multiple of the symplectic form** ω others are not, but we expected that

the remaining terms will also be multiple of ω by adding suitable coboundaries

$\mathfrak{ham}_{2n}^1$ is a graded Lie algebra $\Rightarrow$

$H_{GF}^*(\mathfrak{ham}_{2n}^1)^{\text{Sp}}$ is bigraded and furthermore $C_{GF}^*(\mathfrak{ham}_{2n}^1)^{\text{Sp}}$ is a direct sum of

finite subcomplexes

We looked at the part $(H := (\mathbb{R}^{2n}, \omega))$

$$C_{11}[32] = 0 \xrightarrow{\partial} C_{10}[30] \cong (\Lambda^{10} S^3 H)^{\text{Sp}} \xrightarrow{\partial} C_9[28] \cong (\Lambda^8 S^3 H \otimes S^4 H)^{\text{Sp}} \xrightarrow{\partial}$$

$$C_8[26] \cong ((\Lambda^7 S^3 H \otimes S^5 H) \oplus (\Lambda^6 S^3 H \otimes \Lambda^2 S^4 H))^{\text{Sp}} \xrightarrow{\partial}$$

... more and more complicated terms ... $\xrightarrow{\partial}$

$$C_2[14] \cong ((\Lambda^2 S^7 H) \oplus (S^3 H \otimes S^{11} H) \oplus \dots)^{\text{Sp}} \xrightarrow{\partial} C_1[12] \cong (S^{12} H)^{\text{Sp}} \xrightarrow{\partial} C_0[10] = 0.$$

$$\Rightarrow \text{compute } H_{GF}^5(\mathfrak{ham}_{2n}^1)^{\text{Sp}}_{(20)}$$

For $n = 1$, we found that rank is given by 0, 0, 0, 0, 4, 12, 9, 3, 1, 0

$$\Rightarrow \chi = -1 \text{ and } H_{GF}^5(\mathfrak{ham}_2^1)^{\text{Sp}}_{(20)} \cong \mathbb{R}$$

8 Big open problem

Problem 6 *Prove that the GKL class is geometrically non-trivial as an element of*

$$H^5(\text{Symp}^\delta(\Sigma_g, \omega); \mathbb{R})$$

or as an element of

$$H^7(\text{B}\Gamma_2^\omega; \mathbb{R}).$$

What kind of geometry should we use ? Probably hyperbolic geometry would be

insufficient