

Extreme points of an intersection of operator intervals

北星学園大・経 安藤 毅 (Tsuyoshi Ando)

Though the title of my talk at the RIMS meeting was *Extreme points of an intersection of matrix intervals*, here the problem is treated in the Hilbert space setting with title *Extreme points of an intersection of operator intervals*. The detail with full proof will appear in the **Proceedings of International Mathematics Conference '94, Kaohsiung, Taiwan**, World Scientific, 1996.

1. Introduction

Let $\mathcal{H}$ be a (complex) Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and $\mathcal{B}(\mathcal{H})$ the space of (bounded linear) operators on $\mathcal{H}$. When $\dim(\mathcal{H}) = n < \infty$, with respect to a suitably chosen orthonormal basis, $\mathcal{B}(\mathcal{H})$ is identified with the space of $n \times n$ matrices.

Recall that an operator A is *selfadjoint* if $A^* = A$ where A^* is the *adjoint* of A , that is, $\langle A^*x, y \rangle = \langle x, Ay \rangle$ ($x, y \in \mathcal{H}$). Selfadjointness of A is characterized by that $\langle x, Ax \rangle$ is real for all $x \in \mathcal{H}$. Recall further that A is *positive (semidefinite)* if $\langle x, Ax \rangle \geq 0$ for all $x \in \mathcal{H}$. Order relation $A \geq B$ for a pair of selfadjoint operators A, B is defined as $A - B$ is positive. Therefore $A \geq 0$ means positivity of A . We write $A > B$ to mean that $A \geq B$ and $A - B$ is invertible, which is equivalent to $A - B \geq \varepsilon I$ for some $\varepsilon > 0$ where I is the identity operator.

Given $A_1, A_2, \dots, A_m \geq 0$, let us observe the set

$$\Delta(A_1, A_2, \dots, A_m) \stackrel{\text{def}}{=} \{X \mid 0 \leq X \leq A_j \text{ } j = 1, 2, \dots, m\}.$$

In particular, $\Delta(A)$ for $A \geq 0$ is the *operator interval* $\{X \mid 0 \leq X \leq A\}$, so that $\Delta(A_1, A_2, \dots, A_m)$ is an intersection of operator intervals;

$$\Delta(A_1, A_2, \dots, A_m) = \bigcap_{j=1}^m \Delta(A_j).$$

Since $\Delta \equiv \Delta(A_1, A_2, \dots, A_m)$ is a convex set, compact with respect to the weak operator topology, according to the Krein-Milman theorem it has plenty many extreme points. Recall that $X \in \Delta$ is *extreme* if $X = (Y + Z)/2$ for $Y, Z \in \Delta$ is possible only when $Y = Z = X$. Let us denote by $\text{ex}\Delta \equiv \text{ex}\Delta(A_1, A_2, \dots, A_m)$ the set of extreme points of Δ .

Our aim is a detailed study of the extreme points of Δ . In Section 3 we give several necessary and sufficient conditions for $X \in \Delta$ to be an extreme point. In Section 4 we present labelling of extreme points and in Section 5, under the assumption $\dim(\mathcal{H}) < \infty$ and $m = 2$, we give a complete parametrization of extreme points of Δ . The final part, Section 6, contains an algorithm of construction of an extreme point, associated with arbitrarily given $X \in \Delta$. In the preliminary part, Section 2, we recall various properties of two basic operations, necessary for our study; parallel sum and short.

This paper enlarges and extends the content of an unpublished manuscript [Ao 89], a part of which has been made public in [P 91].

2. Preliminaries

Given $A, B > 0$, the operator

$$A : B \stackrel{\text{def}}{=} (A^{-1} + B^{-1})^{-1}$$

is called the *parallel sum* of A and B . This notion was first introduced in [AD 69] as mathematical description of the impedance of parallel connection of electrical networks. From the standpoint of quadratic forms the following variational description is more useful (see [AT 75]) and shows that parallel addition corresponds to the so-called *inf-convolution* of two positive quadratic forms (see [M 88]);

$$\langle x, (A : B)x \rangle = \inf \{ \langle y, Ay \rangle + \langle z, Bz \rangle ; y + z = x \} \quad (x \in \mathcal{H}). \quad (2.1)$$

Via (2.1) we can extend the definition of parallel sum to any pair of positive operators. This extended operation enjoys the following properties (see [AT 75], [NA 76], [PS 76] and [EL 89]);

$$A, B \geq A : B = B : A, \quad (2.2)$$

$$(A : B) : C = A : (B : C), \quad (2.3)$$

$$(\lambda A) : (\mu A) = \frac{\lambda\mu}{\lambda + \mu} A \quad (\lambda, \mu > 0), \quad (2.4)$$

$$A_1 \geq A_2, B_1 \geq B_2 \implies A_1 : B_1 \geq A_2 : B_2, \quad (2.5)$$

and more generally

$$A_k \downarrow A, B_k \downarrow B \implies A_k : B_k \downarrow A : B, \quad (2.6)$$

where $A_k \downarrow A$ say, means that A_k decreasingly converges to A in strong operator-topology as $k \rightarrow \infty$, and

$$(S^*AS) : (S^*BS) = S^*(A : B)S \quad \text{for invertible } S \in \mathcal{B}(\mathcal{H}). \quad (2.7)$$

In view of commutativity (2.2) and associativity (2.3) there is no confusion to use

$$\prod_{j=1}^m : A_j \equiv (\dots((A_1 : A_2) : A_3) \cdots : A_m).$$

Another notion we need is short (operation). For $A, X \geq 0$ the sequence $(kX) : A$ increases as $k \rightarrow \infty$. Following [Ao 76] let us define $[X]A$ as its strong limit;

$$[X]A \stackrel{\text{def}}{=} \lim_{k \rightarrow \infty} (kX) : A. \quad (2.8)$$

By (2.5) the following are clear from definition (2.8):

$$A \geq [X]A, \quad (2.9)$$

$$A_1 \geq A_2 \geq 0, X_1 \geq X_2 \implies [X_1]A_1 \geq [X_2]A_2, \quad (2.10)$$

and by (2.3) and (2.4)

$$\alpha X_1 \geq X_2 \geq \beta X_1 \text{ for some } \alpha, \beta > 0 \implies [X_1]A = [X_2]A. \quad (2.11)$$

Every positive operator X admits a unique *positive square-root* $X^{1/2}$, that is, $X^{1/2} \geq 0$ and $(X^{1/2})^2 = X$. For general $X \in \mathcal{B}(\mathcal{H})$ the positive square root of X^*X is called the *modulus* of X and denoted by $|X|$. When X is selfadjoint, $|X|$ commutes with X .

We use $\text{ran}(X)$ and $\ker(X)$ to denote the *range* and the *kernel* of $X \in \mathcal{B}(\mathcal{H})$ respectively. Then obviously

$$\text{ran}(X)^\perp = \ker(X^*) \quad \text{and} \quad \ker(X)^\perp = \overline{\text{ran}(X^*)}, \quad (2.12)$$

where $\{\}^\perp$ denotes the orthogonal complement. For $X \geq 0$ we have

$$\ker(X) = \ker(X^{1/2}) \quad \text{and} \quad \text{ran}(X) \subset \text{ran}(X^{1/2}) \subset \overline{\text{ran}(X)}. \quad (2.13)$$

When $\dim(\mathcal{H}) < \infty$, every subspace is closed, so that positive A is invertible if $\text{ran}(A) = \mathcal{H}$, or equivalently $\ker(A) = \{0\}$.

For $A \geq 0$ and selfadjoint $X \in \mathcal{B}(\mathcal{H})$

$$A \geq X \geq -A \iff X = A^{1/2}CA^{1/2}, \quad (2.14)$$

where C is a (selfadjoint) *contraction*, that is, $I \geq C \geq -I$. If in addition $\text{ran}(C) \subset \overline{\text{ran}(A)}$ is required, C is uniquely determined. In a similar way $A \geq X \geq 0$ is characterized by $I \geq C \geq 0$ in (2.14).

Range inclusion is characterized by an operator inequality (see [D 66]); for $X, Y \in \mathcal{B}(\mathcal{H})$

$$\text{ran}(X) \supset \text{ran}(Y) \iff \alpha XX^* \geq YY^* \quad \text{for some } \alpha \geq 0. \quad (2.15)$$

A consequence of (2.14) is that the operation $A \mapsto [X]A$ is determined only by the range space $\mathcal{L} = \text{ran}(X^{1/2})$. Therefore $[X]A$ will be called the *short* of A to the operator range $\mathcal{L}$.

If $0 \leq C, D \leq I$ and $\text{ran}(C), \text{ran}(D) \subset \overline{\text{ran}(A)}$ then we have

$$(A^{1/2}CA^{1/2}) : (A^{1/2}DA^{1/2}) = A^{1/2}(C : D)A^{1/2}. \quad (2.16)$$

(see [PS 76]).

If $\text{ran}(X)$ is closed, then by (2.13) $[X]A = [P]A$ where P is the projection to $\text{ran}(X) = \text{ran}(X^{1/2})$. The shorted operator $[P]A$ is written in the block operator

matrix with respect to the decomposition $\mathcal{H} = \text{ran}(P) \oplus \text{ker}(P)$ in the following form (see [A 71]):

$$[P]A = \begin{bmatrix} A_{11} - A_{12} \cdot A_{22}^{-1} \cdot A_{21} & 0 \\ 0 & 0 \end{bmatrix} \quad (2.17)$$

for

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \geq 0.$$

Here we remark that positivity of A ensures that there is an operator K from $\text{ker}(P)$ to $\text{ran}(P)$ such that $\|K\| \leq 1$ and $A_{12} = A_{11}^{1/2} K A_{22}^{1/2}$, and $A_{12} \cdot A_{22}^{-1} \cdot A_{21}$ should be understood as

$$A_{12} \cdot A_{22}^{-1} \cdot A_{21} \stackrel{\text{def}}{=} A_{11}^{1/2} K K^* A_{11}^{1/2}.$$

The shorted operator $[P]A$ for a projection P admits the following maximum characterization, originally due to M. Krein (see [AT 75]);

$$[P]A = \max\{Y \mid 0 \leq Y \leq A, \text{ran}(Y) \subset \text{ran}(P)\}. \quad (2.18)$$

For general $X \geq 0$ we have

$$[X]A = A^{1/2} Q A^{1/2}, \quad (2.19)$$

where Q is the projection to the closure of $A^{-1/2} \text{ran}(X) \equiv \{x \in \mathcal{H} \mid A^{1/2} x \in \text{ran}(X)\}$ (see [P 78] and [K 84]). Conversely for any projection Q with $\text{ran}(Q) \subset \overline{\text{ran}(A)}$

$$A^{1/2} Q A^{1/2} = [X]A \quad \text{with } X = A^{1/2} Q A^{1/2}. \quad (2.20)$$

It is known (see [Ao 76], [P 78] and [EL 86]) that

$$[X]A = A \iff A = \sup\{Y \mid 0 \leq Y \leq A \text{ and } Y \leq \lambda X \text{ for some } \lambda = \lambda(Y) \geq 0\} \quad (2.21)$$

Sum and intersection of operator ranges are well determined (see [FW 71]): for $A, B \geq 0$

$$\text{ran}(A) + \text{ran}(B) = \text{ran}(A^2 + B^2)^{1/2}, \quad (2.22)$$

and

$$\text{ran}(A) \cap \text{ran}(B) = \text{ran}(A^2 : B^2)^{1/2}. \quad (2.23)$$

The range of a short to a closed subspace is also well determined (see [AT 75]): for projection P

$$\text{ran}([P]A)^{1/2} = \text{ran}(P) \cap \text{ran}(A^{1/2}). \quad (2.24)$$

It follows from (2.8) and (2.24) that for $A, B \geq 0$

$$\text{ran}(A^{1/2}) \cap \text{ran}(B^{1/2}) = \{0\} \iff A : B = 0. \quad (2.25)$$

3. Characterizations

It is well known that $ex\Delta(I)$ for identity operator I consists of all projections (see [Sa 71, p.12]). Remark that a projection X is characterized as a selfadjoint idempotent, that is, $X(I-X) = 0$, and also that when $I \geq X \geq 0$ then $X(I-X) = X : (I-X)$. By definition (2.8) $[X]I$ for every $X \geq 0$ coincides with the projection to $\overline{ran(X)}$. These together can be formulated as various characterizations of $ex\Delta(I)$ in the following Lemma (see [AT 88] and [EL 86]):

Lemma 1.

$$\begin{aligned} ex\Delta(I) &= \{P \mid \text{projection } P\} \\ &= \{[X]I \mid X \geq 0\} \\ &= \{X \in \Delta(I) \mid X : (I-X) = 0\}. \end{aligned}$$

An easy consequence is that for a projection P

$$ex\Delta(P) = \{Q \mid \text{projection } Q \leq P\}.$$

Given $A \geq 0$ let P be the projection to $\overline{ran(A)}$. Then by (2.14) the affine map $X \mapsto A^{1/2}XA^{1/2}$ transforms bijectively $\Delta(P)$ to $\Delta(A)$, so that the affine structure of $ex\Delta(A)$ can be copied from Lemma 1 and its consequence via (2.16), by using (2.19) and (2.20). Therefore we have a complete answer for the case $m = 1$ (cf. [AT 88], [EL 86], [AMT 92] and [P 92]).

Theorem 2. For $A \geq 0$

$$\begin{aligned} ex\Delta(A) &= \{A^{1/2}PA^{1/2} \mid \text{projection } P \text{ such that } ran(P) \subset \overline{ran(A)}\} \\ &= \{[X]A \mid X \geq 0\} \\ &= \{X \in \Delta(A) \mid X : (A-X) = 0\}. \end{aligned}$$

Remark that, in general, not every extreme point of $\Delta(A)$ is of the form $[P]A$ for a projection P (see [P 91] and [GM 94]).

The case $m > 1$ is more delicate, but still can be derived from Lemma 1 (cf. [Ao 89] and [P 91])

Theorem 3. Let $A_j \geq 0$ ($j = 1, 2, \dots, m$) and $X \in \Delta(A_1, \dots, A_m)$. Then the following conditions on X are mutually equivalent.

- (i) $X \in ex\Delta(A_1, \dots, A_m)$.
- (ii) $X : \{\prod_{j=1}^m (A_j - X)\} = 0$.
- (iii) $ran(X^{1/2}) \cap \bigcap_{j=1}^m ran(A_j - X)^{1/2} = 0$.

Also each of (i), (ii) and (iii) is equivalent to any of the following.

- (i') $X \in ex\Delta([X]A_1, \dots, [X]A_m)$.

$$(ii') \quad X : \{\prod_{j=1}^m : ([X]A_j - X)\} = 0.$$

$$(iii') \quad \text{ran}(X^{1/2}) \cap \bigcap_{j=1}^m \text{ran}([X]A_j - X)^{1/2} = 0.$$

Theorem 4. Let $A_j \geq 0$ ($j = 1, 2, \dots, m$) and $X \in \Delta(A_1, \dots, A_m)$. Then each of the following conditions implies any one (and all) in Theorem 3.

$$(iv) \quad \prod_{j=1}^m : ([X]A_j - X) = 0.$$

$$(v) \quad \ker(X) + \sum_{j=1}^m \ker(A_j - X) = \mathcal{H}.$$

(vi) There are mutually commuting (not necessarily selfadjoint) idempotents $Q_j \in \mathcal{B}(\mathcal{H})$ ($j = 0, 1, 2, \dots, m$) such that

$$Q_i Q_j = \delta_{ij} Q_j \quad (i, j = 0, 1, 2, \dots, m), \quad \sum_{j=0}^m Q_j = I, \quad \text{and} \quad X = \sum_{j=1}^m A_j Q_j.$$

When $\dim(\mathcal{H}) < \infty$, each of (iv), (v) and (v) is equivalent to any one (and all) in Theorem 3.

4. Labelling

All characterizations of extremality in Theorem 3 and Theorem 4 are of qualitative nature except (vi). Representation (vi), however, has a defect in using non-selfadjoint summands. In fact, there is no guarantee of selfadjointness of $A_j Q_j$ ($j = 1, 2, \dots, m$).

In the finite dimensional case, however, the following theorem gives a labelling of every extreme point X of $\Delta(A_1, A_2, \dots, A_m)$ in terms of those in $\Delta(Y_j)$ ($j = 1, 2, \dots, m$) where Y_j ($j = 1, 2, \dots, m$) are defined recursively from X (cf. [Ao 89] and [P 91]).

Theorem 5. Let $A_j \geq 0$ ($j = 1, 2, \dots, m$) and $X \in \Delta(A_1, \dots, A_m)$. Then if there are $X_k \geq 0$ ($k = 1, 2, \dots, m$) such that

$$X = \sum_{j=1}^m X_j \quad \text{and} \quad X_k \in \text{ex}\Delta(A_k - \sum_{j=1}^{k-1} X_j) \quad (k = 1, 2, \dots, m), \quad (4.1)$$

where $\sum_{j=1}^{k-1} X_j \equiv 0$ for $k = 1$, then X is an extreme point of $\Delta(A_1, \dots, A_m)$.

Those X_k ($k = 1, 2, \dots, m$), satisfying (4.1) and the additional condition that for some $1 > \varepsilon > 0$

$$(1 - \varepsilon)(A_k - \sum_{j=1}^k X_j) \geq \sum_{j=k+1}^m X_j \equiv X - \sum_{j=1}^k X_j \quad (k = 1, 2, \dots, m) \quad (4.2)$$

are unique for X if exist.

When $\dim(\mathcal{H}) < \infty$, existence of those X_k ($k = 1, 2, \dots, m$), satisfying (4.1) and (4.2), is always guaranteed for every $X \in \text{ex}\Delta(A_1, A_2, \dots, A_m)$.

5. Parametrization

When $\dim(\mathcal{H}) < \infty$ and $m = 2$, in Theorem 5 X_1 is considered as a free parameter for $X \in \text{ex}\Delta(A_1, A_2)$ because $X_2 = X - X_1$. But the requirements $(1 - \varepsilon)(A_1 - X_1) \geq X - X_1$ for some $0 < \varepsilon < 1$ and $X_2 \in \text{ex}\Delta(A_2 - X_1)$ are still restrictive. In this section, using a result from the theory of indefinite inner product spaces, we shall present a more transparent parametrization of $\text{ex}\Delta(A_1, A_2)$ along an idea of [Ao 93].

Let $\dim(\mathcal{H}) < \infty$ and $m = 2$. For brevity let us write

$$A \stackrel{\text{def}}{=} A_1 \quad \text{and} \quad B \stackrel{\text{def}}{=} A_2.$$

According to Theorem 4 extremity of $X \in \Delta(A, B)$ is characterized by

$$([X]A - X) : ([X]B - X) = 0.$$

When restricted to $\text{ran}(X) = \text{ran}(X^{1/2})$, each of $[X]A$, $[X]B$ and X is positive invertible by (2.26). Therefore assuming $A, B > 0$ we shall consider parametrization of all invertible extreme points of $\Delta(A, B)$. In this case requirement for extremity of X becomes

$$(A - X) : (B - X) = 0. \quad (5.1)$$

Let $A, B > 0$, and

$$C \stackrel{\text{def}}{=} \frac{1}{2}(A - B).$$

If $A \geq B$ or $A \leq B$ then $\Delta(A, B)$ reduces to either $\Delta(B)$ or $\Delta(A)$, and a parametrization of $\Delta(A, B)$ is already known by Theorem 2. Therefore we shall assume that C is indefinite, that is, $C \not\geq 0$, and $C \not\leq 0$. The space $\mathcal{H}$ is written as an orthogonal sum

$$\mathcal{H} = \text{ran}(C) \oplus \ker(C). \quad (5.2)$$

Since

$$\text{ran}(C) = \text{ran}(|C| - C) \oplus \text{ran}(|C| + C),$$

we can also write

$$\mathcal{H} = \text{ran}(|C| - C) \oplus \text{ran}(|C| + C) \oplus \ker(C). \quad (5.3)$$

Let

$$n_+ = \dim(|C| - C) \quad \text{and} \quad n_- = \dim(|C| + C).$$

Then $n_{\pm} > 0$ by indefiniteness of C . The triple $\{n_+, n_-, n_0\}$ with $n_0 \stackrel{\text{def}}{=} \dim \ker(C)$ is usually called the *inertia* of selfadjoint C . Fix an invertible operator $V \in \mathcal{B}(\mathcal{H})$ such that with respect to decompositions (5.2) and (5.3)

$$V = \begin{bmatrix} V_{11} & 0 \\ 0 & I_0 \end{bmatrix} \quad \text{and} \quad C = V^* \cdot \begin{bmatrix} I_+ & 0 & 0 \\ 0 & -I_- & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot V, \quad (5.4)$$

where I_0 and $I_{\pm}$ are the identity matrices of order n_0 and $n_{\pm}$ respectively.

Theorem 5. *Let $\dim(\mathcal{H}) < \infty$ and let $A, B > 0$, and let C and V be as in (5.4). Then every invertible extreme point X of $\Delta(A, B)$ is uniquely written in the form*

$$X = \frac{1}{2}(A + B) - V^* \cdot \begin{bmatrix} D(K) & 0 \\ 0 & 0 \end{bmatrix} \cdot V, \quad (5.5)$$

where K is an $n_- \times n_+$ matrix with $K^*K < I_+$ and

$$D(K) \stackrel{\text{def}}{=} \begin{bmatrix} (I_+ + K^*K)(I_+ - K^*K)^{-1} & 2(I_+ - K^*K)^{-1}K^* \\ 2K(I_+ - K^*K)^{-1} & (I_- + KK^*)(I_- - KK^*)^{-1} \end{bmatrix}. \quad (5.6)$$

Conversely each $n_- \times n_+$ matrix K with $K^*K < I_+$ gives rise to an invertible extreme point X of $\Delta(A, B)$ by (5.5) and (5.6).

6. Construction

Let $A_j \geq 0$ ($j = 1, 2, \dots, m$). Obviously 0 is an extreme point, which is $\leq X$ for all $X \in \Delta(A_1, A_2, \dots, A_m)$. In this section along an idea of [Ao 93] we shall present an algorithm of obtaining an extreme point $\tilde{X}$ such that $X \leq \tilde{X}$ for given $X \in \Delta(A_1, A_2, \dots, A_m)$.

Theorem 6. *Let $X \in \Delta(A_1, \dots, A_m)$. Starting with $X_0 \stackrel{\text{def}}{=} X$, define successively*

$$X_{k+1} \stackrel{\text{def}}{=} X_k + \prod_{j=1}^k : (A_j - X_k) \quad (k = 1, 2, \dots). \quad (6.1)$$

Then $\{X_k \mid k = 1, 2, \dots\}$ is an increasing sequence in $\Delta(A_1, \dots, A_m)$ and its (strong) limit $X_\infty \stackrel{\text{def}}{=} \lim_{k \rightarrow \infty} X_k$ is an extreme point of $\Delta(A_1, \dots, A_m)$ such that

$$X \leq X_\infty \quad \text{and} \quad \prod_{j=1}^m : (A_j - X_\infty) = 0. \quad (6.2)$$

Remark that the algorithm in Theorem 6 produces all extreme points X of $\Delta(A_1, A_2, \dots, A_m)$ such that

$$\prod_{j=1}^m : (A_j - X) = 0,$$

because for such X it is immediate to see that all X_k coincide with X for all k , and $X_\infty = X$.

Theorem 7. Let $\dim(\mathcal{H}) < \infty$ and $X \in \Delta(A_1, A_2, \dots, A_m)$. Starting with $X_0 \stackrel{\text{def}}{=} X$, define successively

$$X_{k+1} \stackrel{\text{def}}{=} X_k + \prod_{j=1}^k : ([X]A_j - X_k) \quad (k = 1, 2, \dots). \quad (6.4)$$

Then $\{X_k \mid k = 1, 2, \dots\}$ is an increasing sequence in $\Delta(A_1, \dots, A_m)$ and its limit $X_\infty \stackrel{\text{def}}{=} \lim_{k \rightarrow \infty} X_k$ is an extreme point of $\Delta(A_1, \dots, A_m)$ such that

$$X \leq X_\infty = [X]X_\infty \quad \text{and} \quad \prod_{j=1}^m : ([X_\infty]A_j - X_\infty) = 0. \quad (6.5)$$

Conversely all extreme points are obtained in this way.

REFERENCES

- [A 71] W.N. Anderson, *Shorted operators*, SIAM J. Appl. Math. **20** (1971), 520–525.
- [A 90] ———, *Structure of extreme points of a set of positive semidefinite matrices*, Preprint (1990).
- [AD 69] W.N. Anderson and R.J. Duffin, *Series and parallel addition of matrices*, J. Math. Anal. Appl. **26** (1969), 576–594.
- [AT 75] W.N. Anderson and G.E. Trapp, *Shorted operators II*, SIAM J. Appl. Math. **28** (1988), 60–71.
- [AT 88] ———, *The extreme points of a set of positive semidefinite operators*, Linear Alg. Appl. **106** (1988), 209–218.
- [AMT 92] W.N. Anderson, T.D. Morley and G.E. Trapp, *A Dixie cup visualizing order intervals of matrices*, Linear Alg. Appl. **164** (1992), 207–260.
- [Ao 76] T. Ando, *Lebesgue type decomposition of positive operators*, Acta Sci. Math. (Szeged) **38** (1976), 253–260.
- [Ao 89] ———, *Structure of extreme points of a set of positive semidefinite matrices* (1989), unpublished manuscript.
- [Ao 93] ———, *Paraemtrization of minimal points of some convex sets of matrices*, Acta Sci. Math. (Szeged) **57** (1993), 3–10.
- [AI 89] T.Ya. Azizov and I.S. Iokhvidov, *Linear operators in spaces with an indefinite metric*, John Wiley & Sons, Chichester, 1989.
- [D 66] R.G. Douglas, *On the majorization, factorization and range inclusion of operators*, Proc. Amer. Math. Soc. **17** (1966), 413–416.
- [EL 86] S.L. Eriksson and H. Leutwiller, *A potential theoretic approach to parallel addition*, Math. Ann. **274** (1986), 301–317.
- [EL 89] ———, *A generalization of parallel addition*, Aequationes Math. **38** (1989), 99–110.
- [FW 71] P.E. Fillmore and J.P. Williams, *On operator ranges*, Adv. in Math. **7** (1971), 254–281.
- [GM 94] W.L. Green and T.D. Morley, *The extreme points of order intervals of positive operators*, Adv. in Appl. Math. **15** (1994), 360–370.
- [IKL 82] I.S. Iokhvidov, M.G. Krein and H. Langer, *Introduction to the spectral theory of operators in spaces with an indefinite metric*, Akademie-Verlag, Berlin, 1982.
- [K 84] H. Kosaki, *Remark on Lebesgue-type decomposition of positive operators*, J. Operator Theory **11** (1984), 137–142.
- [Kr 47] M.G. Krein, *The theory of selfadjoint extensions of semibounded operators and its applications (Russian)*, Math. Sb. **20(62)** (1947), 431–495.
- [M 88] M.-L. Mazure, *Shorted operators through convex analysis*, Intern. Ser. Numer. Math. **84** (1988), 139–152.

- [N 80] K. Nishio, *Characterization of Lebesgue-type decomposition of positive operators*, Acta Sci. Math. (Szeged) **42** (1980), 145–152.
- [NA 76] K. Nishio and T. Ando, *Characterization of operations derived from network connections*, J. Math. Anal. Appl. **53** (1976), 538–549.
- [P 78] E.L. Pekarev, *Short to operator range*, Funktsional. Anal. i Prilozhen. **12** (1978), 84–85; English translation in Functional Anal. Appl. **12** (1978), 230–231 (1979).
- [P 91] ———, *A note on extreme points of the intersection of operator segments*, Integral Equa. Operator Theory **14** (1991), 458–463.
- [P 92] ———, *Shorts to operators and some extremal problems*, Acta Sci. Math. (Szeged) **56** (1992), 147–163.
- [PS 76] E.L. Pekarev and Ju.L. Smul'yan, *Parallel addition and parallel subtraction of operators*, Izv. AN SSSR **40** (1976), 366–387; English translation in Math. USSR Izv. **10** (1976), 289–337.
- [Sa 71] S. Sakai, *C^* -Algebras and W^* -algebras*, Springer-Verlag, Berlin-Heidelberg-New York, 1971.