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Algorithmic approach to Uchida's theorem for one-dimensional function fields over finite fields

By

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Abstract

Uchida proved that the isomorphism class of a one-dimensional function field over a finite field is completely determined by (a suitable quotient of) its absolute Galois group. But his proof of this theorem essentially gives a group-theoretic reconstruction algorithm for one-dimensional function fields over finite fields. In this article, we discuss the group-theoretic reconstruction algorithm.

§ 1. Introduction

In [10], Uchida proved the following theorem:

Theorem A (Uchida). *For $i \in \{1, 2\}$, let K_i be a one-dimensional function field over a finite field* (*i.e., a finitely generated field of transcendence degree one over a finite field*) *and Ω_i a solvably closed Galois extension of K_i* (*i.e., Galois extension of K_i which has no nontrivial abelian extension*)*. For $i \in \{1, 2\}$, write $G_i := \mathrm{Gal}(\Omega_i/K_i)$. Moreover, write $\mathrm{Isom}(\Omega_2/K_2, \Omega_1/K_1)$ for the set of isomorphisms $\Omega_2 \xrightarrow{\sim} \Omega_1$ of fields such that the image of K_2 coincides with K_1 and $\mathrm{Isom}(G_1, G_2)$ for the set of isomorphisms $G_1 \xrightarrow{\sim} G_2$ of profinite groups. Then the natural map $\mathrm{Isom}(\Omega_2/K_2, \Omega_1/K_1) \to \mathrm{Isom}(G_1, G_2)$ is bijective.*

In particular, the following corollary follows immediately from Theorem A:

Corollary B. *The isomorphism class of a one-dimensional function field over a finite field is completely determined by* (*a suitable quotient of*) *its absolute Galois group.*

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Proof. Let K_i, Ω_i be as in Theorem A. Suppose that $G_1 = \mathrm{Gal}(\Omega_1/K_1)$ is isomorphic to $G_2 = \mathrm{Gal}(\Omega_2/K_2)$. Then, since $\mathrm{Isom}(G_1, G_2) \neq \emptyset$, it follows from the surjectivity of the map $\mathrm{Isom}(\Omega_2/K_2, \Omega_1/K_1) \to \mathrm{Isom}(G_1, G_2)$ (cf. Theorem A) that $\mathrm{Isom}(\Omega_2/K_2, \Omega_1/K_1) \neq \emptyset$, which implies that $K_1 \cong K_2$. $\square$

Theorem A gives a *bi-anabelian* reconstruction (i.e., involving two fields, cf. [4] Remark 1.9.8) of one-dimensional function fields over finite fields.

But in fact, the proof of Theorem A in [10] essentially gives a *mono-anabelian* reconstruction (i.e., involving only one field, cf. [4] Remark 1.9.8). In other words, the argument in [10] implies that a one-dimensional function field over a finite field can be reconstructed from (a suitable quotient of) its absolute Galois group by a functorial group-theoretic reconstruction algorithm.

We shall say that a profinite group G is of *PGF-type* if there exist a one-dimensional function field K over a finite field and a solvably closed Galois extension Ω of K such that G is isomorphic to the Galois group $\mathrm{Gal}(\Omega/K)$ (cf. Definition 3.1(iv)). Let us express more precisely the statement of a mono-anabelian version of Theorem A:

Theorem C. *There exists a functorial group-theoretic algorithm $G \mapsto K(G)$ for constructing a field $K(G)$ from a profinite group G of PGF-type such that the following hold: an isomorphism $\alpha : \mathrm{Gal}(\Omega/K) \xrightarrow{\sim} G$ (where K is a one-dimensional function field over a finite field and Ω is a solvably closed Galois extension of K) induces a natural isomorphism $K \xrightarrow{\sim} K(G)$ of fields.*

The purpose of this article is to explain in detail this mono-anabelian reconstruction algorithm. More specifically, in §3, we reconstruct the multiplicative group of $K(G)$ (cf. Theorem 3.11). The main ingredients of §3 are the reconstruction of various objects from the absolute Galois group of a positive characteristic local field in §2 (cf. Theorem 2.6) and global class field theory. In §4, we reconstruct the additive structure on $K(G)$ from the multiplicative structure of $K(G)$ (cf. Definition 4.17).

Remark 1. Uchida also proved the bi-anabelian results for number fields (cf. [9], [11]). However, in this case, the proofs of [9], [11] do not give a mono-anabelian reconstruction. A mono-anabelian reconstruction algorithm of number fields is given in [2].

Remark 2. Some notations and discussions in this article are based on those of [2].

§ 2. Local Theory

In this section, we discuss generalities of the absolute Galois group of positive characteristic local fields, and review mono-anabelian reconstructions of various objects.

Definition 2.1.

(i) We shall refer to a field which is isomorphic to a finite extension of $\mathbb{F}_p((t))$ for some prime number p as a *PLF* (Positive characteristic Local Field).

(ii) Let k be a PLF and k^{sep} a separable closure of k. Then we shall write

- $p_k := \mathrm{char}(k)(> 0)$ for the characteristic of k,
- $\mathcal{O}_k \subset k$ for the ring of integers of k,
- $\mathcal{O}_k^{\triangleright} := \mathcal{O}_k \setminus \{0\}$ for the multiplicative monoid of nonzero integers of k,
- $\mathfrak{m}_k \subset \mathcal{O}_k$ for the maximal ideal of $\mathcal{O}_k$,
- $U_k^{(1)} := 1 + \mathfrak{m}_k \subset \mathcal{O}_k^{\times}$ for the multiplicative group of principal units of k,
- $\kappa_k := \mathcal{O}_k/\mathfrak{m}_k$ for the residue field of $\mathcal{O}_k$,
- $\overline{\kappa}_k$ for the residue field of (the ring of integers of) k^{sep} (note that $\overline{\kappa}_k$ is an algebraic closure of κ_k),
- $f_k := [\kappa_k : \mathbb{F}_{p_k}]$ for the extension degree of κ_k over the prime field contained in κ_k,
- $G_k := \mathrm{Gal}(k^{\mathrm{sep}}/k)$ for the absolute Galois group of k,
- $I_k \subset G_k$ for the inertia subgroup of G_k,
- $P_k \subset I_k$ for the wild inertia subgroup of G_k, and
- $\mathrm{Frob}_{\kappa_k} \in \mathrm{Gal}(\overline{\kappa}_k/\kappa_k)$ for the ($\sharp\kappa_k$-th power) Frobenius element of $\mathrm{Gal}(\overline{\kappa}_k/\kappa_k)$.

(iii) Let G be a profinite group. Then we shall refer to a collection of data

$$(k, k^{\mathrm{sep}}, \alpha : G_k \xrightarrow{\sim} G)$$

consisting of a PLF k, a separable closure k^{sep} of k, and an isomorphism of profinite groups $\alpha : G_k \xrightarrow{\sim} G$ as a *PLF-envelope* for G. We shall say that the profinite group G is *of PLF-type* if there exists a PLF-envelope for G.

Remark 3. An open subgroup of a profinite group of PLF-type is of PLF-type.

Lemma 2.2 (Local class field theory)**.** *Let k be a PLF. Let us write $(k^{\times})^{\wedge} := \varprojlim_{n\geq 1} k^{\times}/(k^{\times})^n$. Then there exists a commutative diagram*

$$\begin{tikzcd}
1 \arrow[r] & \mathcal{O}_k^{\times} \arrow[r] \arrow[d, "\wr"] & (k^{\times})^{\wedge} \arrow[r] \arrow[d, "\wr"] & \hat{\mathbb{Z}} \arrow[r] \arrow[d, "\wr"] & 1 \\
1 \arrow[r] & \mathrm{Im}(I_k \hookrightarrow G_k \twoheadrightarrow G_k^{\mathrm{ab}}) \arrow[r] & G_k^{\mathrm{ab}} \arrow[r] & G_k/I_k \arrow[r] & 1,
\end{tikzcd}$$

where the horizontal sequences are exact, the middle vertical arrow $(k^\times)^\wedge \xrightarrow{\sim} G_k^{\mathrm{ab}}$ *is the homomorphism induced by the reciprocity homomorphism* $k^\times \to G_k^{\mathrm{ab}}$ *in local class field theory, and the right-hand vertical arrow maps* $1 \in \mathbb{Z}$ *to* $\mathrm{Frob}_{\kappa_k} \in \mathrm{Gal}(\overline{\kappa}_k/\kappa_k) \xleftarrow{\sim} G_k/I_k$.

Lemma 2.3. *Let* k *be a PLF and* $\pi \in \mathcal{O}_k$ *a prime element of* $\mathcal{O}_k$. *Then it holds that* $k^\times \cong \langle\pi\rangle \times \mathcal{O}_k^\times \cong \langle\pi\rangle \times (k^\times)_{\mathrm{tor}} \times U_k^{(1)}$.

Proof. Well-known (cf. e.g., [7] Chapter II, Proposition (5.3)). □

Lemma 2.4. *Let* k *be a PLF. Then the following hold:*

(i) p_k *is the unique prime number* l *such that* $l \mid \sharp(G_k^{\mathrm{ab}})_{\mathrm{tor}} + 1$.

(ii) *It holds that* $f_k = \log_{p_k}(\sharp(G_k^{\mathrm{ab}})_{\mathrm{tor}} + 1)$.

(iii) *It holds that* $I_k = \mathrm{Gal}(k^{\mathrm{sep}}/k^{\mathrm{ur}}) = \bigcap_{k'} \mathrm{Gal}(k^{\mathrm{sep}}/k')$, *where* $k' \subset k^{\mathrm{sep}}$ *runs over all finite unramified extensions of* k *contained in* k^{sep}.

(iv) *Let* $k' \subset k^{\mathrm{sep}}$ *be a finite extension of* k *contained in* k^{sep}. *Then* k' *is unramified over* k *if and only if it holds that* $[\mathrm{Gal}(k^{\mathrm{sep}}/k) : \mathrm{Gal}(k^{\mathrm{sep}}/k')] = f_{k'}/f_k$.

(v) P_k *is the unique Sylow pro-*p_k*-subgroup of* I_k.

(vi) $\mathrm{Frob}_{\kappa_k} \in \mathrm{Gal}(\overline{\kappa}_k/\kappa_k) \xleftarrow{\sim} G_k/I_k$ *is the unique element of* G_k/I_k *which acts by conjugation on* I_k/P_k *by multiplication by* $p_k^{f_k}$.

(vii) *It holds that* $\mathrm{Im}(\mathcal{O}_k^\times \hookrightarrow k^\times \hookrightarrow G_k^{\mathrm{ab}}) = \mathrm{Im}(I_k \to G_k^{\mathrm{ab}})$.

(viii) $\mathrm{Im}(k^\times \hookrightarrow G_k^{\mathrm{ab}})$ *coincides with a subgroup of* G_k^{ab} *generated by* $\mathrm{Im}(\mathcal{O}_k^\times \hookrightarrow G_k^{\mathrm{ab}})$ *and* (*a lifting of*) Frob_{κ_k} (*in* G_k^{ab}). *In other words,* $\mathrm{Im}(k^\times \hookrightarrow G_k^{\mathrm{ab}}) = G_k^{\mathrm{ab}} \times_{G_k/I_k} \mathrm{Frob}_{\kappa_k}^{\mathbb{Z}}$, *where we write* $\mathrm{Frob}_{\kappa_k}^{\mathbb{Z}}$ *for the discrete subgroup of* G_k/I_k *generated by* Frob_{κ_k}.

(ix) $\mathrm{Im}(\mathcal{O}_k^\triangleright \hookrightarrow k^\times \hookrightarrow G_k^{\mathrm{ab}})$ *coincides with a submonoid of* G_k^{ab} *generated by* $\mathrm{Im}(\mathcal{O}_k^\times \hookrightarrow G_k^{\mathrm{ab}})$ *and* (*a lifting of*) Frob_{κ_k} (*in* G_k^{ab}). *In other words,* $\mathrm{Im}(\mathcal{O}_k^\triangleright \hookrightarrow k^\times \hookrightarrow G_k^{\mathrm{ab}}) = G_k^{\mathrm{ab}} \times_{G_k/I_k} \mathrm{Frob}_{\kappa_k}^{\mathbb{Z}_{\geq 0}}$, *where we write* $\mathrm{Frob}_{\kappa_k}^{\mathbb{Z}_{\geq 0}}$ *for the discrete submonoid of* G_k/I_k *generated by* Frob_{κ_k}.

(x) $U_k^{(1)}$ *is the unique Sylow pro-*p_k*-subgroup of* $\mathcal{O}_k^\times$.

Proof. Assertions (i), (ii) follow from Lemmas 2.2, 2.3. Assertions (iii)-(v), (vii)-(x) are immediate. Assertion (vi) follows from [8] Proposition (7.5.2). □

Definition 2.5. Let G be a profinite group of PLF-type.

(i) It follows from Lemma 2.4(i) that there exists a unique prime number l such that $l \mid \sharp(G^{\mathrm{ab}})_{\mathrm{tor}} + 1$. Write $p(G)$ for this prime number.

(ii) Write $f(G) := \log_{p(G)}(\sharp(G^{\mathrm{ab}})_{\mathrm{tor}} + 1)$.

(iii) Write $I(G) := \bigcap_{G'} G'$, where G' runs over all open subgroups of G such that $[G : G'] = f(G')/f(G)$ (cf. Remark 3).

(iv) It follows from Lemma 2.4(v), together with Theorem 2.6(i), (ii) below, that there exists a unique Sylow pro-$p(G)$-subgroup of $I(G)$. Write $P(G)$ for this subgroup of $I(G)$.

(v) It follows from Lemma 2.4(vi), together with Theorem 2.6(i)-(iii) below, that there exists a unique element of $G/I(G)$ which acts by conjugation on $I(G)/P(G)$ by multiplication by $p(G)^{f(G)}$. Write $\mathrm{Frob}(G) \in G/I(G)$ for this element of $G/I(G)$.

(vi) Write $\mathcal{O}^{\times}(G) := \mathrm{Im}(I(G) \to G^{\mathrm{ab}})$.

(vii) Write $k^{\times}(G) := G^{\mathrm{ab}} \times_{G/I(G)} \mathrm{Frob}(G)^{\mathbb{Z}} \subset G^{\mathrm{ab}}$, where we write $\mathrm{Frob}(G)^{\mathbb{Z}}$ for the discrete subgroup of $G/I(G)$ generated by $\mathrm{Frob}(G)$.

(viii) Write $\mathcal{O}^{\triangleright}(G) := G^{\mathrm{ab}} \times_{G/I(G)} \mathrm{Frob}(G)^{\mathbb{Z}_{\geq 0}} \subset G^{\mathrm{ab}}$, where we write $\mathrm{Frob}(G)^{\mathbb{Z}_{\geq 0}}$ for the discrete submonoid of $G/I(G)$ generated by $\mathrm{Frob}(G)$.

(ix) Since $\mathcal{O}^{\times}(G) \subset G^{\mathrm{ab}}$ is abelian, there exists a unique Sylow pro-$p(G)$-subgroup of $\mathcal{O}^{\times}(G)$. Write $U^{(1)}(G)$ for this subgroup of $\mathcal{O}^{\times}(G)$.

Theorem 2.6. *Let G be a profinite group of PLF-type and $(k, k^{\mathrm{sep}}, \alpha : G_k \xrightarrow{\sim} G)$ a PLF-envelope for G. Then the following hold:*

(i) *It holds that $p_k = p(G)$, $f_k = f(G)$.*

(ii) *It holds that $\alpha(I_k) = I(G)$.*

(iii) *It holds that $\alpha(P_k) = P(G)$.*

(iv) *The image of* $\mathrm{Frob}_{\kappa_k} \in G_k/I_k$ *under the isomorphism $G_k/I_k \xrightarrow{\sim} G/I(G)$ determined by α (cf.* (ii)*) coincides with* $\mathrm{Frob}(G) \in G/I(G)$.

(v) *The reciprocity homomorphism $k^{\times} \to G_k^{\mathrm{ab}}$ and the isomorphism α determine an isomorphism $k^{\times} \xrightarrow{\sim} k^{\times}(G)$. Moreover, the image of $\mathcal{O}_k^{\triangleright}$ (resp. $\mathcal{O}_k^{\times}$, $U_k^{(1)}$) under the isomorphism $k^{\times} \xrightarrow{\sim} k^{\times}(G)$ coincides with $\mathcal{O}^{\triangleright}(G)$ (resp. $\mathcal{O}^{\times}(G)$, $U^{(1)}(G)$).*

Proof. These assertions follow from Lemma 2.4. □

Theorem 2.7. *Let G be a profinite group of PLF-type, $(k, k^{\mathrm{sep}}, \alpha : G_k \overset{\sim}{\to} G)$ a PLF-envelope for G, and $H \subset G$ an open subgroup of G. Write k' for the finite extension of k contained in k^{sep} which corresponds to the open subgroup $\alpha^{-1}(H) \subset G_k$ of G_k. Then we obtain a commutative diagram*

$$\begin{array}{ccc} k^\times & \xrightarrow{\sim} & k^\times(G) \\ \downarrow & & \downarrow \\ (k')^\times & \xrightarrow{\sim} & k^\times(H), \end{array}$$

where the horizontal arrows are the isomorphisms appearing in Theorem 2.6(v), *and the right-hand vertical arrow is the homomorphism induced by the transfer homomorphism $G^{\mathrm{ab}} \to H^{\mathrm{ab}}$.*

Proof. This follows from Theorem 2.6(v), together with [12] Chapter XII, Theorem 6. $\square$

§ 3. Multiplicative Structure of One-dimensional Function Fields over Finite Fields

In this section, we reconstruct the multiplicative structure of one-dimensional function fields over finite fields.

Definition 3.1.

(i) We shall refer to a field which is isomorphic to a one-dimensional function field over a finite field as a *PGF* (Positive characteristic Global Field).

(ii) Let K be an algebraic extension (not necessarily finite) of a PGF. Then we shall write $\mathcal{V}_K$ for the set of all places of K.

(iii) Let K be a PGF and $v \in \mathcal{V}_K$ a place of K. Then we shall write

- K_v for the PLF obtained by forming the completion of K at v,
- $\mathrm{ord}_v : K^\times \to \mathbb{Z}$ for the uniquely determined surjective valuation associated to v,
- $\mathcal{O}_v := \{a \in K \mid \mathrm{ord}_v(a) \geq 0\} \subset K$ for the discrete valuation ring at v,
- $\mathcal{O}_v^\triangleright := \mathcal{O}_v \setminus \{0\}$ for the multiplicative monoid of nonzero elements of $\mathcal{O}_v$,
- $\mathfrak{m}_v \subset \mathcal{O}_v$ for the maximal ideal of $\mathcal{O}_v$,

- $U_v^{(1)} := 1 + \mathfrak{m}_v \subset \mathcal{O}_v^\times$, and
- $J_K := \varinjlim_S (\prod_{v \in S} K_v^\times) \times (\prod_{v \in \mathcal{V}_K \setminus S} \mathcal{O}_{K_v}^\times)$ for the idèle group of K, where S runs over all finite subsets of $\mathcal{V}_K$.

(iv) Let G be a profinite group. Then we shall refer to a collection of data

$$(K, \Omega, \alpha : \mathrm{Gal}(\Omega/K) \xrightarrow{\sim} G)$$

consisting of a PGF K, a solvably closed Galois extension Ω of K, and an isomorphism of profinite groups $\alpha : \mathrm{Gal}(\Omega/K) \xrightarrow{\sim} G$ as a *PGF-envelope* for G. We shall say that the profinite group G is *of PGF-type* if there exists a PGF-envelope for G.

Remark 4. An open subgroup of a profinite group of PGF-type is of PGF-type.

Lemma 3.2 (PGF-analogue of [3] Proposition 2.1(i)). *Let K be a PGF, K^{sep} a separable closure of K, Ω a solvably closed Galois extension of K contained in K^{sep}, and A a continuous discrete torsion* $\mathrm{Gal}(\Omega/K)$*-module. Then, for each integer $i \geq 0$, the natural surjection $G_K = \mathrm{Gal}(K^{\mathrm{sep}}/K) \twoheadrightarrow \mathrm{Gal}(\Omega/K)$ induces an isomorphism*

$$H^i(\mathrm{Gal}(\Omega/K), A) \xrightarrow{\sim} H^i(G_K, A).$$

In particular, $\mathrm{cd}_p(\mathrm{Gal}(\Omega/K)) = \begin{cases} 2 & (p \neq \mathrm{char}(K)) \\ 1 & (p = \mathrm{char}(K)). \end{cases}$

Proof. It is well-known that $\mathrm{cd}_p(G_K) = \begin{cases} 2 & (p \neq \mathrm{char}(K)) \\ 1 & (p = \mathrm{char}(K)) \end{cases}$ (cf. e.g., [8] Proposition (6.5.10), Theorem (7.1.8)(i), Theorem (8.3.17)). Thus, the second assertion follows from the first assertion. We verify the first assertion. Write $J := \ker(G_K \twoheadrightarrow \mathrm{Gal}(\Omega/K))$. It suffices to prove that $H^i(J, A) = 0$ for $i \geq 1$. We may assume that A is finite and p-primary for some prime number p. Since $\mathrm{cd}_p(J) \leq \mathrm{cd}_p(G_K)$, we may assume that $1 \leq i \leq \mathrm{cd}_p(G_K)$. If $i = 2$ (hence $p \neq \mathrm{char}(K)$), then it follows from an argument similar to the argument in [3] Proposition 2.1(i), that $H^2(J, A) = 0$. Moreover, if $i = 1$, then, since Ω is solvably closed, we obtain $H^1(J, A) = \mathrm{Hom}_{\mathrm{cts}}(J, A) = \{0\}$, where we write $\mathrm{Hom}_{\mathrm{cts}}(J, A)$ for the set of continuous homomorphisms from J to A. This completes the proof of Lemma 3.2. $\square$

Lemma 3.3 (PGF-analogue of [3] Proposition 2.3(iii)-(v)). *Let K be a PGF, K^{sep} a separable closure of K, Ω a solvably closed Galois extension of K contained in K^{sep}, and $v, w \in \mathcal{V}_\Omega$ places of Ω. Suppose that $v \neq w$. Write $D_v, D_w \subset$* $\mathrm{Gal}(\Omega/K)$ *for the decomposition subgroups associated to v, w, respectively. Then the following hold:*

(i) *The natural surjection* $\mathrm{Gal}(K^{\mathrm{sep}}/K) \twoheadrightarrow \mathrm{Gal}(\Omega/K)$ *induces an isomorphism of* D_v *with the decomposition subgroup associated to a lifting of* v *in* $\mathcal{V}_{K^{\mathrm{sep}}}$*.*

(ii) *It holds that* $D_v \cap D_w = \{1\}$*.*

(iii) D_v *is its own commensurator in* $\mathrm{Gal}(\Omega/K)$*, i.e., for* $g \in \mathrm{Gal}(\Omega/K)$*,* g *lies in* D_v *if and only if* $D_v \cap gD_vg^{-1}$ *is of finite index in both* D_v *and* gD_vg^{-1}*.*

Proof. Assertion (i) follows from an argument similar to [3] Proposition 2.3(iii). Assertion (iii) follows from assertion (ii). We verify assertion (ii). Since $v \neq w$, there exists a finite extension L of K contained in Ω such that v and w are not equivalent over L. Then it follows from an argument similar to [3] Proposition 2.3(iv) that $(D_v \cap \mathrm{Gal}(\Omega/L)) \cap (D_w \cap \mathrm{Gal}(\Omega/L)) = \{1\}$, which implies that $D_v \cap D_w$ is finite. Since $\mathrm{Gal}(\Omega/K)$ is of finite cohomological dimension (cf. Lemma 3.2), hence torsion-free, we conclude that $D_v \cap D_w = \{1\}$. This completes the proof of assertion (ii), hence also of Lemma 3.3. □

Lemma 3.4. *Let* K *be a PGF,* Ω *a solvably closed Galois extension of* K*,* $H \subset \mathrm{Gal}(\Omega/K)$ *a closed subgroup of* $\mathrm{Gal}(\Omega/K)$*, and* l *a prime number different from* $\mathrm{char}(K)$*. Then the following hold:*

(i) *The natural map* $\mathcal{V}_\Omega \to \mathcal{V}_K$ *and the natural action of* $\mathrm{Gal}(\Omega/K)$ *on* $\mathcal{V}_\Omega$ *determine a bijection* $\mathcal{V}_\Omega/\mathrm{Gal}(\Omega/K) \xrightarrow{\sim} \mathcal{V}_K$*.*

(ii) *Consider the following conditions:*

  (1) H *is an open subgroup of the decomposition subgroup of* $\mathrm{Gal}(\Omega/K)$ *associated to some* $v \in \mathcal{V}_\Omega$*.*

  (2) H *is of PLF-type.*

  (3) *There exists an open subgroup* V *of* H *such that, for any open subgroup* $U \subset V$ *of* V*, it holds that* $\dim_{\mathbb{F}_l} H^2(U, \mathbb{F}_l) = 1$*, where the action of* U *on* $\mathbb{F}_l$ *is trivial.*

  (4) H *is a closed subgroup of the decomposition subgroup of* $\mathrm{Gal}(\Omega/K)$ *associated to some* $v \in \mathcal{V}_\Omega$*.*

Then we have implications $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$*.*

Proof. Assertion (i) is immediate. We verify assertion (ii). The implication (1) $\Rightarrow$ (2) follows from Lemma 3.3(i), together with Remark 3. Next, we verify the implication (2) $\Rightarrow$ (3). Suppose that condition (2) is satisfied. Let $(k, k^{\mathrm{sep}}, \alpha : G_k \xrightarrow{\sim} H)$ be a PLF-envelope for H. Write $V := \mathrm{Gal}(k^{\mathrm{sep}}/k(\mu_l)) \subset H$. Then V is an open subgroup of H, and, moreover, for any open subgroup $U \subset V$ of V, it holds that $H^2(U, \mathbb{F}_l) \cong H^2(U, \mu_l)$.

On the other hand, it follows from Hilbert's theorem 90, together with the well-known fact that $\mathrm{cd}_l\, U = 2$ (cf. e.g., [8] Theorem (7.1.8)(i)), that the exact sequence

$$1 \to \mu_l \to (k^{\mathrm{sep}})^\times \xrightarrow{l} (k^{\mathrm{sep}})^\times \to 1$$

induces an exact sequence

$$0 \to H^2(U, \mu_l) \to H^2(U, (k^{\mathrm{sep}})^\times) \xrightarrow{l} H^2(U, (k^{\mathrm{sep}})^\times) \to 0.$$

Since (it is well-known that) $H^2(U, (k^{\mathrm{sep}})^\times) \cong \mathrm{Br}((k^{\mathrm{sep}})^U)$ is isomorphic to $\mathbb{Q}/\mathbb{Z}$, it holds that $\dim_{\mathbb{F}_l} H^2(U, \mu_l) = 1$. This completes the proof of the implication (2) $\Rightarrow$ (3).

Finally, we verify the implication (3) $\Rightarrow$ (4). Suppose that condition (3) is satisfied. Let $v \in \mathcal{V}_\Omega$. By abuse of notation, let us write K_v for the "K_v", where we take "$v \in \mathcal{V}_K$" to be the image of $v \in \mathcal{V}_\Omega$ under the natural surjection $\mathcal{V}_\Omega \twoheadrightarrow \mathcal{V}_K$. Then we can consider Ω as a subfield of a separable closure of K_v. For any intermediate field L of K and Ω, write $L_v := L \cdot K_v$. Let V be as in condition (3) and F a finite extension of $(\Omega^V)(\mu_l)$ contained in Ω. Write $U := \mathrm{Gal}(\Omega/F) \subset V$. Then it follows from condition (3) that $\dim_{\mathbb{F}_l} H^2(U, \mathbb{F}_l) = 1$. Moreover, it holds that $H^2(U, \mathbb{F}_l) \cong H^2(U, \mu_l)$. Thus, it follows from Hilbert's theorem 90, together with Lemma 3.2, that the exact sequence

$$1 \to \mu_l \to \Omega^\times \xrightarrow{l} \Omega^\times \to 1,$$

induces an exact sequence

$$0 \to H^2(U, \mu_l) \to H^2(U, \Omega^\times) \xrightarrow{l} H^2(U, \Omega^\times) \to 0,$$

which implies that the l-primary part $H^2(U, \Omega^\times)(l)$ of $H^2(U, \Omega^\times)$ is of corank 1. It follows from [10] Lemma 1 that there exists a unique $v(F) \in \mathcal{V}_F$ such that, for any extension $v \in \mathcal{V}_\Omega$ of $v(F)$ in Ω, it holds that $H^2(\mathrm{Gal}(\Omega_v/F_v), \Omega_v^\times)(l) \neq \{0\}$. Moreover, it follows from the uniqueness of $v(F')$ for any finite extension of F contained in Ω, that $v(F)$ has a unique extension in Ω.

Now let us write $E := \Omega^H \subset F$ and $v(E) \in \mathcal{V}_E$ for the restriction of $v(F) \in \mathcal{V}_F$ to E. Then, since F is finite over E, it follows from [10] Lemma 1, together with the (already verified) fact that $H^2(U, \Omega^\times)(l)$ is of corank 1, that $v(F)$ is the unique extension of $v(E)$. Thus, we conclude that $v(E)$ has a unique extension $v \in \mathcal{V}_\Omega$ in Ω, which implies that H is contained in the decomposition subgroup of $\mathrm{Gal}(\Omega/K)$ associated to $v \in \mathcal{V}_\Omega$. This completes the proof of the implication (3) $\Rightarrow$ (4), hence also of Lemma 3.4. $\square$

Definition 3.5. Let G be a profinite group of PGF-type.

(i) Write $\overline{\mathcal{V}}(G)$ for the set of maximal elements of the set of all closed subgroups $H \subset G$ satisfying the following condition:

there exist a prime number l and an open subgroup V of H such that, for any open subgroup $U \subset V$ of V, it holds that $\dim_{\mathbb{F}_l} H^2(U, \mathbb{F}_l) = 1$, where the action of U on $\mathbb{F}_l$ is trivial.

Let us define the action of G on $\overline{\mathcal{V}}(G)$ by conjugation.

(ii) Write $\mathcal{V}(G) := \overline{\mathcal{V}}(G)/G$.

Theorem 3.6. *Let G be a profinite group of PGF-type and $(K, \Omega, \alpha : \mathrm{Gal}(\Omega/K) \xrightarrow{\sim} G)$ a PGF-envelope for G.*

(i) *The isomorphism α determines a bijection $\mathcal{V}_\Omega \xrightarrow{\sim} \overline{\mathcal{V}}(G)$, which is compatible with the actions of $\mathrm{Gal}(\Omega/K)$ and G. In particular, any $D \in \overline{\mathcal{V}}(G)$ is of PLF-type. Moreover, the above bijection induces a bijection $\mathcal{V}_K \xrightarrow{\sim} \mathcal{V}(G)$.*

(ii) *Let $H \subset G$ be an open subgroup of G. Write L for the finite extension of K contained in Ω which corresponds to the open subgroup $\alpha^{-1}(H) \subset \mathrm{Gal}(\Omega/K)$ of $\mathrm{Gal}(\Omega/K)$. Then we obtain a commutative diagram*

$$\begin{tikzcd} \mathcal{V}_\Omega \arrow[r, "\sim"] \arrow[d, equal] & \overline{\mathcal{V}}(G) \arrow[d, "\wr"] \\ \mathcal{V}_\Omega \arrow[r, "\sim"] & \overline{\mathcal{V}}(H), \end{tikzcd}$$

where the horizontal arrows are the bijections appearing in (i), and the right-hand vertical arrow is the bijection which maps $D \in \overline{\mathcal{V}}(G)$ to $D \cap H \in \overline{\mathcal{V}}(H)$. Moreover, the inverse map of the right-hand vertical arrow of this diagram determines a commutative diagram

$$\begin{tikzcd} \mathcal{V}_L \arrow[r, "\sim"] \arrow[d, two heads] & \mathcal{V}(H) \arrow[d, two heads] \\ \mathcal{V}_K \arrow[r, "\sim"] & \mathcal{V}(G). \end{tikzcd}$$

(*Note that it follows from Lemma 3.3(iii) that the inverse map $\overline{\mathcal{V}}(H) \xrightarrow{\sim} \overline{\mathcal{V}}(G)$ maps $D \in \overline{\mathcal{V}}(H)$ to the commensurator of D in G.*)

Proof. Assertion (i) follows from Lemma 3.4, together with Lemma 3.2 and Lemma 3.3(ii). Assertion (ii) follows from assertion (i). $\square$

Remark 5. The reconstruction of $\mathcal{V}_\Omega$ is essentially due to J. Neukirch [5], [6].

Lemma 3.7 (Global class field theory)**.** *Let K be a PGF and Ω a solvably closed Galois extension of K. Let us consider the homomorphism $J_K \to \mathrm{Gal}(\Omega/K)^{\mathrm{ab}}$ determined by the reciprocity homomorphisms $K_v^\times \to D_v^{\mathrm{ab}}$, where $D_v \subset \mathrm{Gal}(\Omega/K)$ is the decomposition subgroup associated to a lifting of $v \in \mathcal{V}_K$ in $\mathcal{V}_\Omega$ (note that, since D_v is well-defined up to conjugation, $J_K \to \mathrm{Gal}(\Omega/K)^{\mathrm{ab}}$ is well-defined). Then it holds that $K^\times = \ker(J_K \to \mathrm{Gal}(\Omega/K)^{\mathrm{ab}})$.*

Lemma 3.8. *Let G be a profinite group of PGF-type and $v \in \mathcal{V}(G) = \overline{\mathcal{V}}(G)/G$.*

(i) *There exists a unique submodule M of $\prod_{D\in v} k^\times(D)$ (cf. Theorem 3.6(i)) which satisfies the following conditions:*

  (1) *The action of G on $\prod_{D\in v} k^\times(D)$ by conjugation induces the identity automorphism on M.*

  (2) *For any $D_0 \in v$, the composite $M \hookrightarrow \prod_{D\in v} k^\times(D) \twoheadrightarrow k^\times(D_0)$ is an isomorphism of modules.*

(ii) *The inverse image of $\mathcal{O}^\triangleright(D_0)$ (resp. $\mathcal{O}^\times(D_0)$, $U^{(1)}(D_0)$) under the isomorphism $M \xrightarrow{\sim} k^\times(D_0)$ of condition (2) of assertion (i) does not depend on the choice of $D_0 \in v$.*

Proof. In light of Theorem 2.6(v) and Theorem 3.3(iii), it is clear that the "diagonal" of $\prod_{D\in v} k^\times(D)$ is the unique submodule satisfying the conditions of (i). Assertion (ii) is immediate. □

Lemma 3.9. *Let K be a PGF and $v \in \mathcal{V}_K$. Then the inverse image of $\mathcal{O}_{K_v}^\triangleright$ (resp. $\mathcal{O}_{K_v}^\times$, $U_{K_v}^{(1)}$) under the natural inclusion $K^\times \hookrightarrow K_v^\times$ coincides with $\mathcal{O}_v^\triangleright$ (resp. $\mathcal{O}_v^\times$, $U_v^{(1)}$).*

Proof. Trivial. □

Definition 3.10. Let G be a profinite group of PGF-type and $v \in \mathcal{V}(G)$.

(i) Write $k^\times(v)$ for the unique submodule M of $\prod_{D\in v} k^\times(D)$ (cf. Theorem 3.6(i)) satisfying the conditions of Lemma 3.8(i).

(ii) It follows from Lemma 3.8(ii) that the inverse image of $\mathcal{O}^\triangleright(D_0)$ (resp. $\mathcal{O}^\times(D_0)$, $U^{(1)}(D_0)$) under the isomorphism $k^\times(v) \xrightarrow{\sim} k^\times(D_0)$ of condition (2) of Lemma 3.8(i) does not depend on the choice of $D_0 \in v$. Write $\mathcal{O}^\triangleright(v)$ (resp. $\mathcal{O}^\times(v)$, $U^{(1)}(v)$) for this inverse image in $k^\times(v)$.

(iii) Write $J(G) := \varinjlim_S (\prod_{w \in S} k^\times(w)) \times (\prod_{w \in \mathcal{V}(G) \setminus S} \mathcal{O}^\times(w))$, where S runs over all finite subsets of $\mathcal{V}(G)$. Note that $J(G) \subset \prod_{w \in \mathcal{V}(G)} \prod_{D \in w} D^{\mathrm{ab}} = \prod_{D \in \overline{\mathcal{V}}(G)} D^{\mathrm{ab}}$.

(iv) It follows from Lemma 3.7, together with Theorem 3.11(i), (ii) below, that the inclusions $D \hookrightarrow G$ $(D \in \overline{\mathcal{V}}(G))$ determine a homomorphism $J(G) \to G^{\mathrm{ab}}$. Write $K^\times(G) := \ker(J(G) \to G^{\mathrm{ab}})$.

(v) Write $\mathcal{O}_v^{\triangleright}(G)$ (resp. $\mathcal{O}_v^\times(G)$, $U_v^{(1)}(G)$) for the inverse image of $\mathcal{O}^{\triangleright}(v)$ (resp. $\mathcal{O}^\times(v)$, $U^{(1)}(v)$) under the composite of the inclusion $K^\times(G) \hookrightarrow J(G)$ and the projection $J(G) \to k^\times(v)$.

Theorem 3.11. *Let G be a profinite group of PGF-type, $(K, \Omega, \alpha : \mathrm{Gal}(\Omega/K) \xrightarrow{\sim} G)$ a PGF-envelope for G, and $v \in \mathcal{V}_K$. Write $v_G \in \mathcal{V}(G)$ for the image of $v \in \mathcal{V}_K$ under the bijection $\mathcal{V}_K \xrightarrow{\sim} \mathcal{V}(G)$ appearing in Theorem* 3.6(i)*.*

(i) *The isomorphism α determines an isomorphism $K_v^\times \xrightarrow{\sim} k^\times(v_G)$.*

(ii) *The image of $\mathcal{O}_{K_v}^{\triangleright}$ (resp. $\mathcal{O}_{K_v}^\times$, $U_{K_v}^{(1)}$) under the isomorphism $K_v^\times \xrightarrow{\sim} k^\times(v_G)$ appearing in* (i) *coincides with $\mathcal{O}^{\triangleright}(v_G)$ (resp. $\mathcal{O}^\times(v_G)$, $U^{(1)}(v_G)$).*

(iii) *The isomorphism α and various isomorphisms appearing in* (i) *determine a commutative diagram of groups*

$$\begin{array}{ccc} J_K & \longrightarrow & \mathrm{Gal}(\Omega/K)^{\mathrm{ab}} \\ \downarrow\wr & & \downarrow\wr \\ J(G) & \longrightarrow & G^{\mathrm{ab}}, \end{array}$$

where the lower horizontal arrow is the homomorphism appearing in Definition 3.10(iv)*. Moreover, this diagram determines an isomorphism of groups $K^\times \xrightarrow{\sim} K^\times(G)$.*

(iv) *The image of $\mathcal{O}_v^{\triangleright}$ (resp. $\mathcal{O}_v^\times$, $U_v^{(1)}$) under the isomorphism $K^\times \xrightarrow{\sim} K^\times(G)$ appearing in* (iii) *coincides with $\mathcal{O}_{v_G}^{\triangleright}(G)$ (resp. $\mathcal{O}_{v_G}^\times(G)$, $U_{v_G}^{(1)}(G)$).*

Proof. These assertions follow from Lemmas 3.7, 3.8, 3.9, together with Theorems 2.6, 3.6. ☐

Theorem 3.12. *Let G be a profinite group of PGF-type, $(K, \Omega, \alpha : \mathrm{Gal}(\Omega/K) \xrightarrow{\sim} G)$ a PGF-envelope for G, $H \subset G$ an open subgroup of G, and $w \in \mathcal{V}(H)$. Write $v \in \mathcal{V}(G)$ for the image of w under the surjection $\mathcal{V}(H) \twoheadrightarrow \mathcal{V}(G)$ appearing in Theorem*

3.6(ii) *and* L *for the finite extension of* K *contained in* Ω *which corresponds to the open subgroup* $\alpha^{-1}(H) \subset \mathrm{Gal}(\Omega/K)$ *of* $\mathrm{Gal}(\Omega/K)$*. Then we obtain a commutative diagram*

$$\begin{array}{ccc} K^\times & \xrightarrow{\sim} & K^\times(G) \\ \big\downarrow & & \big\downarrow \\ L^\times & \xrightarrow{\sim} & K^\times(H), \end{array}$$

where the horizontal arrows are the isomorphisms appearing in Theorem 3.11(iii)*, and the right-hand vertical arrow is an injection determined by various injections "*$k^\times(v) \hookrightarrow k^\times(w)$*" induced by the right-hand vertical arrow of the commutative diagram appearing in Theorem* 2.7*. In particular, the inverse image of* $\mathcal{O}_w^\triangleright(H)$ *(resp.* $\mathcal{O}_w^\times(H), U_w^{(1)}(H)$*) under the injection* $K^\times(G) \hookrightarrow K^\times(H)$ *coincides with* $\mathcal{O}_v^\triangleright(G)$ *(resp.* $\mathcal{O}_v^\times(G), U_v^{(1)}(G)$*).*

Proof. This follows from Theorems 2.7, 3.11. □

§ 4. Additive Structure of One-dimensional Function Fields over Finite Fields

In this section, we reconstruct the additive structure of one-dimensional function fields over finite fields.

Definition 4.1. Let K be a PGF.

(i) We shall write

- $F_K \subset K$ for the constant field of K,
- $\tilde{K} := K \otimes_{F_K} \overline{F}_K$, where $\overline{F}_K$ is an algebraic closure of F_K,
- $C_{\tilde{K}}$ for a nonsingular projective curve whose function field is $\tilde{K}$ (which is unique up to isomorphism, cf. e.g., [1] Chapter I, Corollary 6.12), and
- $\mathrm{Div}(\tilde{K})$ for the group of divisors of $C_{\tilde{K}}$.

(ii) Let $v \in \mathcal{V}_{\tilde{K}}$. Then we shall write

- $\mathrm{ord}_v : \tilde{K}^\times \to \mathbb{Z}$ for the uniquely determined surjective valuation associated to v,
- $\tilde{\mathcal{O}}_v := \{a \in \tilde{K} \mid \mathrm{ord}_v(a) \geq 0\} \subset \tilde{K}$ for the discrete valuation ring at v,
- $\tilde{\mathcal{O}}_v^\triangleright := \tilde{\mathcal{O}}_v \setminus \{0\}$ for the multiplicative monoid of nonzero elements of $\tilde{\mathcal{O}}_v$,

- $\tilde{\mathfrak{m}}_v \subset \tilde{\mathcal{O}}_v$ for the maximal ideal of $\tilde{\mathcal{O}}_v$,
- $\tilde{U}_v^{(1)} := 1 + \tilde{\mathfrak{m}}_v \subset \tilde{\mathcal{O}}_v^\times$, and
- $\tilde{\kappa}_v := \tilde{\mathcal{O}}_v/\tilde{\mathfrak{m}}_v$ for the residue field of $\tilde{\mathcal{O}}_v$.

(iii) Let $v \in \mathcal{V}_{\tilde{K}}$ and $s \in \tilde{\mathcal{O}}_v$. Then we shall write $s(v) \in \tilde{\kappa}_v$ for the image of $s \in \tilde{\mathcal{O}}_v$ under the natural surjection $\tilde{\mathcal{O}}_v \twoheadrightarrow \tilde{\kappa}_v$.

(iv) Let $D \in \mathrm{Div}(\tilde{K})$. Then we shall write

- $H^0(D) := H^0(C_{\tilde{K}}, \mathcal{L}(D))$, where $\mathcal{L}(D)$ is the invertible sheaf associated to D, and
- $l(D) := \dim_{\overline{F}_K} H^0(D)$.

Lemma 4.2. *Let K be a PGF, Ω a solvably closed Galois extension of K, and $H \subset \mathrm{Gal}(\Omega/K)$ an open subgroup of $\mathrm{Gal}(\Omega/K)$. We regard $\overline{F}_K$ and $\tilde{K}$ as subfields of Ω in a natural way (i.e., $\overline{F}_K$ is the algebraic closure of F_K in Ω). Write L for the finite extension of K contained in Ω associated to $H \subset \mathrm{Gal}(\Omega/K)$. Then the following hold:*

(i) *It holds that $F_K^\times = \bigcap_{v \in \mathcal{V}_K} \mathcal{O}_v^\times$.*

(ii) *H contains $\ker(\mathrm{Gal}(\Omega/K) \twoheadrightarrow \mathrm{Gal}(\overline{F}_K/F_K))$ if and only if $[G : H] = [F_L : F_K] = \log_{\sharp F_K}(\sharp F_L)$ (in this case, $L = K \otimes_{F_K} F_L$).*

Proof. Trivial. □

Definition 4.3. Let M be a monoid. Then let us write $M^{\circledast} := M \cup \{*_M\}$. We regard $M^{\circledast}$ as a monoid by $a \cdot *_M = *_M \cdot a = *_M$ for every $a \in M^{\circledast}$. If $N \subset M$ is a submonoid of M, then we regard $N^{\circledast} \subset M^{\circledast}$ by identifying $*_N$ by $*_M$. We always write $*$ instead of $*_M$ for simplicity.

Remark 6. In our reconstruction algorithm, $*$ eventually corresponds to the additive identity element (multiplicative absorbing element) 0 of various fields.

Definition 4.4. Let G be a profinite group of PGF-type.

(i) Write $K(G) := (K^\times(G))^{\circledast}$.

(ii) Write $F^\times(G) := \bigcap_{v \in \mathcal{V}(G)} \mathcal{O}_v^\times(G) \subset K^\times(G)$.

(iii) It follows from Theorem 4.5(i) below that $\sharp F^\times(G)$ is finite and nonzero. Write $\tilde{G} := \bigcap_H H$, where H runs over all open subgroups of G such that $[G : H] = \log_{\sharp F^\times(G)+1}(\sharp F^\times(H) + 1)$ (cf. Remark 4).

(iv) Write $\tilde{K}^\times(G) := \varinjlim_H K^\times(H)$, $\overline{\tilde{\mathcal{V}}}(G) := \varinjlim_H \overline{\mathcal{V}}(H)$ (cf. Remark 4), where H runs over all open subgroups of G containing $\tilde{G}$, and the transition maps are the maps appearing in Theorem 3.12, Theorem 3.6(ii). Note that the actions of "H"s on "$\overline{\mathcal{V}}(H)$"s determine an action of $\tilde{G}$ on $\overline{\tilde{\mathcal{V}}}(G)$.

(v) Write $\tilde{\mathcal{V}}(G) := \overline{\tilde{\mathcal{V}}}(G)/\tilde{G}$, $\mathrm{Div}(G) := \bigoplus_{v \in \tilde{\mathcal{V}}(G)} \mathbb{Z} \cdot v$.

(vi) It follows from Theorem 4.5(ii) below that any open subgroup of G containing $\tilde{G}$ is normal in G. We define an action of G on $\tilde{K}^\times(G)$ by conjugation.

(vii) Write $\tilde{K}(G) := (\tilde{K}^\times(G))^{\circledast}$, and define an action of G on $\tilde{K}(G)$ by the natural action determined by the action of G on $\tilde{K}^\times(G)$ appearing in (vi) and the trivial action of G on $\{*\} \subset \tilde{K}(G)$.

Theorem 4.5. *Let G be a profinite group of PGF-type and $(K, \Omega, \alpha : \mathrm{Gal}(\Omega/K) \xrightarrow{\sim} G)$ a PGF-envelope for G. We regard $\overline{F}_K$ and $\tilde{K}$ as subfields of Ω in a natural way. Then the following hold:*

(i) *The isomorphism of groups $K^\times \xrightarrow{\sim} K^\times(G)$ appearing in Theorem 3.11(iii) determines an isomorphism of monoids $K \xrightarrow{\sim} K(G)$. Moreover, the image of $F_K^\times \subset K$ under the isomorphism $K \xrightarrow{\sim} K(G)$ coincides with $F^\times(G)$.*

(ii) *It holds that $\tilde{G} = \alpha(\ker(\mathrm{Gal}(\Omega/K) \twoheadrightarrow \mathrm{Gal}(\overline{F}_K/F_K)))$.*

(iii) *The isomorphisms of groups "$K^\times \xrightarrow{\sim} K^\times(G)$" appearing in Theorem 3.11(iii) for various open subgroups of G containing $\tilde{G}$ determine an isomorphism of groups $\tilde{K}^\times \xrightarrow{\sim} \tilde{K}^\times(G)$, which is compatible with the actions of $\mathrm{Gal}(\Omega/K)$ and G with respect to α. In particular, the above isomorphism induces an isomorphism of monoids $\tilde{K} \xrightarrow{\sim} \tilde{K}(G)$, which is compatible with the actions of $\mathrm{Gal}(\Omega/K)$ and G.*

(iv) *Let $H \subset G$ be an open subgroup of G containing $\tilde{G}$. Write L for the finite extension of K contained in Ω which corresponds to the open subgroup $\alpha^{-1}(H) \subset \mathrm{Gal}(\Omega/K)$ of $\mathrm{Gal}(\Omega/K)$. Then the natural map $\overline{\mathcal{V}}(H) \to \overline{\tilde{\mathcal{V}}}(G)$ is bijective. Moreover, the inverse map of this bijection and the bijection $\mathcal{V}_\Omega \xrightarrow{\sim} \overline{\mathcal{V}}(G)$ appearing in Theorem 3.6(i) determine a commutative diagram*

$$\begin{tikzcd} \mathcal{V}_{\tilde{K}} \arrow[r, "\sim"] \arrow[d, two heads] & \tilde{\mathcal{V}}(G) \arrow[d, two heads] \\ \mathcal{V}_L \arrow[r, "\sim"] & \mathcal{V}(H). \end{tikzcd}$$

In particular, the bijection $\mathcal{V}_{\tilde{K}} \xrightarrow{\sim} \tilde{\mathcal{V}}(G)$ determines an isomorphism $\mathrm{Div}(\tilde{K}) \xrightarrow{\sim} \mathrm{Div}(G)$.

Proof. Assertion (i) follows from Theorem 3.11(ii) and Lemma 4.2(i). Assertion (ii) follows from assertion (i) and Lemma 4.2(ii). Assertion (iii) follows from Theorem 3.12. Assertion (iv) follows from Theorem 3.6. □

Lemma 4.6. *Let K be a PGF and $v \in \mathcal{V}_{\tilde{K}}$. For any finite extension L of K contained in $\tilde{K}$, write $v_L \in \mathcal{V}_L$ for the image of $v \in \mathcal{V}_{\tilde{K}}$ under the natural surjection $\mathcal{V}_{\tilde{K}} \twoheadrightarrow \mathcal{V}_L$. Then the following hold:*

(i) *It holds that $\tilde{\mathcal{O}}_v^{\triangleright} = \bigcup_L \mathcal{O}_{v_L}^{\triangleright}$, $\tilde{\mathcal{O}}_v^{\times} = \bigcup_L \mathcal{O}_{v_L}^{\times}$, $\tilde{U}_v^{(1)} = \bigcup_L U_{v_L}^{(1)}$, where L runs over all finite extensions of K contained in $\tilde{K}$.*

(ii) *The natural surjection $\tilde{\mathcal{O}}_v \twoheadrightarrow \tilde{\kappa}_v$ induces an isomorphism of groups $\tilde{\mathcal{O}}_v^{\times}/\tilde{U}_v^{(1)} \xrightarrow{\sim} \tilde{\kappa}_v^{\times}$.*

Proof. Trivial. □

Definition 4.7. Let G be a profinite group of PGF-type and $v \in \tilde{\mathcal{V}}(G)$. For any open subgroup $H \subset G$ of G containing $\tilde{G}$, write $v_H \in \mathcal{V}(H)$ for the image of $v \in \tilde{\mathcal{V}}(G)$ under the surjection $\tilde{\mathcal{V}}(G) \twoheadrightarrow \mathcal{V}(H)$ appearing in Theorem 4.5(iv).

(i) Write $\tilde{\mathcal{O}}_v^{\triangleright}(G) := \varinjlim_H \mathcal{O}_{v_H}^{\triangleright}(H)$, $\tilde{\mathcal{O}}_v^{\times}(G) := \varinjlim_H \mathcal{O}_{v_H}^{\times}(H)$, $\tilde{U}_v^{(1)}(G) := \varinjlim_H U_{v_H}^{(1)}(H)$ (cf. Remark 4), where H runs over all open subgroups of G containing $\tilde{G}$, and the transition maps are the maps induced by the map "$K^{\times}(G) \hookrightarrow K^{\times}(H)$" appearing in Theorem 3.12.

(ii) Write $\tilde{\kappa}_v^{\times}(G) := \tilde{\mathcal{O}}_v^{\times}(G)/\tilde{U}_v^{(1)}(G)$.

(iii) Write $\tilde{\mathcal{O}}_v(G) := (\tilde{\mathcal{O}}_v^{\triangleright}(G))^{\circledast}$, $\tilde{\kappa}_v(G) := (\tilde{\kappa}_v^{\times}(G))^{\circledast}$.

Theorem 4.8. *Let G be a profinite group of PGF-type, $(K, \Omega, \alpha : \mathrm{Gal}(\Omega/K) \xrightarrow{\sim} G)$ a PGF-envelope for G, and $v \in \mathcal{V}_{\tilde{K}}$. Write $v_G \in \tilde{\mathcal{V}}(G)$ for the image of $v \in \mathcal{V}_{\tilde{K}}$ under the bijection $\mathcal{V}_{\tilde{K}} \xrightarrow{\sim} \tilde{\mathcal{V}}(G)$ appearing in Theorem 4.5(iv). Then the following hold:*

(i) *The image of $\tilde{\mathcal{O}}_v$ (resp. $\tilde{\mathcal{O}}_v^{\triangleright}$, $\tilde{\mathcal{O}}_v^{\times}$, $\tilde{U}_v^{(1)}$) under the isomorphism $\tilde{K} \xrightarrow{\sim} \tilde{K}(G)$ appearing in Theorem 4.5(iii) coincides with $\tilde{\mathcal{O}}_{v_G}(G)$ (resp. $\tilde{\mathcal{O}}_{v_G}^{\triangleright}(G)$, $\tilde{\mathcal{O}}_{v_G}^{\times}(G)$, $\tilde{U}_{v_G}^{(1)}(G)$).*

(ii) *The isomorphisms of groups $\tilde{\mathcal{O}}_v^{\times} \xrightarrow{\sim} \tilde{\mathcal{O}}_{v_G}^{\times}(G)$, $\tilde{U}_v^{(1)} \xrightarrow{\sim} \tilde{U}_{v_G}^{(1)}(G)$ obtained in (i) determine an isomorphism of groups $\tilde{\kappa}_v^{\times} \xrightarrow{\sim} \tilde{\kappa}_{v_G}^{\times}(G)$. In particular, we obtain an isomorphism of monoids $\tilde{\kappa}_v \xrightarrow{\sim} \tilde{\kappa}_{v_G}(G)$.*

Proof. Assertion (i) follows from Theorem 3.11(iv), Theorem 3.12, Theorem 4.5(iii), (iv), Lemma 4.6(i). Assertion (ii) follows from assertion (i) and Lemma 4.6(ii). □

Definition 4.9. Let G be a profinite group of PGF-type, $v \in \tilde{\mathcal{V}}(G)$, and $s \in \tilde{\mathcal{O}}_v(G)$. If $s \in \tilde{\mathcal{O}}_v^\times(G)$, then we shall write $s(v) \in \tilde{\kappa}_v(G)$ for the image of s under the composite $\tilde{\mathcal{O}}_v^\times(G) \twoheadrightarrow \tilde{\kappa}_v^\times(G) \hookrightarrow \tilde{\kappa}_v(G)$. If $s \notin \tilde{\mathcal{O}}_v^\times(G)$, then we shall write $s(v) := * \in \tilde{\kappa}_v(G)$.

Theorem 4.10. *Let G be a profinite group of PGF-type, $(K, \Omega, \alpha : \mathrm{Gal}(\Omega/K) \xrightarrow{\sim} G)$ a PGF-envelope for G, $v \in \mathcal{V}_{\tilde{K}}$, and $s \in \tilde{\mathcal{O}}_v$. Write $v_G \in \tilde{\mathcal{V}}(G)$ for the image of $v \in \mathcal{V}_{\tilde{K}}$ under the bijection $\mathcal{V}_{\tilde{K}} \xrightarrow{\sim} \tilde{\mathcal{V}}(G)$ appearing in Theorem* 4.5(iv)*, and $s_G \in \tilde{\mathcal{O}}_{v_G}(G)$ for the image of $s \in \tilde{\mathcal{O}}_v$ under the isomorphism $\tilde{\mathcal{O}}_v \xrightarrow{\sim} \tilde{\mathcal{O}}_{v_G}(G)$ obtained in Theorem* 4.8(i)*. Then the image of $s(v) \in \tilde{\kappa}_v$ under the isomorphism $\tilde{\kappa}_v \xrightarrow{\sim} \tilde{\kappa}_{v_G}(G)$ appearing in Theorem* 4.8(ii) *coincides with $s_G(v_G) \in \tilde{\kappa}_{v_G}(G)$.*

Proof. This follows from Theorem 4.8(ii). ☐

Lemma 4.11. *Let K be a PGF, $v \in \mathcal{V}_{\tilde{K}}$ and $s \in \tilde{K}^\times$. Then the following hold:*

(i) $\mathrm{ord}_v(s) = 1$ *if and only if $\mathcal{O}_v^\triangleright \subset \tilde{K}^\times$ is generated by $\tilde{\mathcal{O}}_v^\times$ and s as a monoid.*

(ii) *Let $t \in \tilde{K}^\times$ such that* $\mathrm{ord}_v(t) = 1$. *Then* $\mathrm{ord}_v(s)$ *is the unique integer n such that $s \cdot t^{-n} \in \tilde{\mathcal{O}}_v^\times$.*

Proof. Trivial. ☐

Definition 4.12. Let G be a profinite group of PGF-type, $v \in \tilde{\mathcal{V}}(G)$, and $s \in \tilde{K}^\times(G)$. Let us define $\mathrm{ord}_v^G(s) \in \mathbb{Z}$ as follows:

(i) If $\tilde{\mathcal{O}}_v^\triangleright(G) \subset \tilde{K}^\times(G)$ is generated by $\tilde{\mathcal{O}}_v^\times(G) \subset \tilde{\mathcal{O}}_v^\triangleright(G)$ and s as a monoid, then let us write $\mathrm{ord}_v^G(s) := 1$.

(ii) It follows from Theorem 4.5(iii), Theorem 4.8(i), Lemma 4.11 that there exists $t \in \tilde{K}^\times(G)$ such that $\mathrm{ord}_v^G(t) = 1$ (in the sense of (i)), and, moreover, there exists a unique integer n such that $s \cdot t^{-n} \in \tilde{\mathcal{O}}_v^\times(G)$. Let us write $\mathrm{ord}_v^G(s)$ for this integer n.

Note that it follows from Theorem 4.5(iii), Theorem 4.8(i), Lemma 4.11 that $\mathrm{ord}_v^G(s)$ is well-defined, i.e.,

- the condition "$s \cdot t^{-n} \in \tilde{\mathcal{O}}_v^\times(G)$" does not depend on the choice of t, and
- "$\mathrm{ord}_v^G(s) = 1$" in the sense of (i) if and only if "$\mathrm{ord}_v^G(s) = 1$" in the sense of (ii).

Theorem 4.13. *Let G be a profinite group of PGF-type, $(K, \Omega, \alpha : \mathrm{Gal}(\Omega/K) \xrightarrow{\sim} G)$ a PGF-envelope for G, and $v \in \mathcal{V}_{\tilde{K}}$. Write v_G for the image of $v \in \mathcal{V}_{\tilde{K}}$ under the bijection $\mathcal{V}_{\tilde{K}} \xrightarrow{\sim} \tilde{\mathcal{V}}(G)$ appearing in Theorem* 4.5(iv)*. Then the composite of $\mathrm{ord}_{v_G}^G : \tilde{K}^\times(G) \to \mathbb{Z}$ and the isomorphism $\tilde{K}^\times \xrightarrow{\sim} \tilde{K}^\times(G)$ appearing in Theorem* 4.5(iii) *coincides with* $\mathrm{ord}_v : \tilde{K}^\times \twoheadrightarrow \mathbb{Z}$.

Proof. This follows from Theorem 4.5(iii), Theorem 4.8(i), Lemma 4.11. $\square$

Lemma 4.14. *Let K be a PGF. Then the following hold:*

(i) *Let $D = \sum_{v \in \mathcal{V}_{\tilde{K}}} n_v \cdot v \in \mathrm{Div}(\tilde{K})$. Then it holds that*

$$H^0(D) = \{s \in \tilde{K}^\times \mid \mathrm{ord}_v(s) + n_v \geq 0 \textit{ for all } v \in \mathcal{V}_{\tilde{K}}\} \cup \{0\},$$

$$l(D) = \min\{n \in \mathbb{Z}_{\geq 0} \mid \textit{there exist } v_1, \ldots, v_n \in \mathcal{V}_{\tilde{K}} \textit{ such that } H^0(D - \sum_{m=1}^{n} v_m) = \{0\}\}.$$

(ii) *For any $v \in \mathcal{V}_{\tilde{K}}$, there exist $D = \sum_{w \in \mathcal{V}_{\tilde{K}}} n_w \cdot w \in \mathrm{Div}(\tilde{K})$ and $w_1, w_2 \in \mathcal{V}_{\tilde{K}}$ such that v, w_1, w_2 are distinct, $n_v = n_{w_1} = n_{w_2} = 0$, $l(D) = 2$, $l(D - v - w_1) = l(D - v - w_2) = l(D - w_1 - w_2) = 0$.*

(iii) *Let $v, w_1, w_2 \in \mathcal{V}_{\tilde{K}}$, $D \in \mathrm{Div}(\tilde{K})$ be as in* (ii). *Moreover, let $\zeta, \lambda \in \tilde{\kappa}_v^\times$. Then there exists a unique element s (resp. t) of $H^0(D)$ such that $s(v) = \zeta$, $s(w_1) = 0$, $s(w_2) \neq 0$ (resp. $t(v) = \lambda$, $t(w_1) \neq 0$, $t(w_2) = 0$). Moreover, $s + t \in H^0(D)$ is the unique element u of $H^0(D)$ such that $u(w_1) = t(w_1)$, $u(w_2) = s(w_2)$.*

Proof. Let us observe that, it follows from the proof of [1] Chapter IV, Proposition 3.1 that, for any $D \in \mathrm{Div}(\tilde{K})$ and $w \in \mathcal{V}_{\tilde{K}}$, it holds that $l(D - w) \geq l(D) - 1$, and, moreover, if $l(D) > 0$, then for all but finitely many $w \in \mathcal{V}_{\tilde{K}}$, it holds that $l(D - w) = l(D) - 1$. In particular, assertion (i) holds. Next, we verify assertion (ii). Let W be a canonical divisor (cf. [1] Chapter IV, §1). Write g for the genus of $C_{\tilde{K}}$. Then it follows from Riemann-Roch theorem (cf. [1] Chapter IV, Theorem 1.3) that $l(W + v) = \deg(W + v) + 1 - g + l(-v) = g < g + 1$. Thus, it follows from the observation above that there exists a divisor $D = \sum_{w \in \mathcal{V}_{\tilde{K}}} n_w \cdot w \in \mathrm{Div}(\tilde{K})$ such that $n_v = 0$, $\deg D = g + 1$ and $l(W + v - D) = 0$. Then, since $l(W - D) \leq l(W + v - D) = 0$ (hence $l(W - D) = 0$), it follows from Riemann-Roch theorem that $l(D) = \deg D + 1 - g - l(W - D) = 2$, $l(D - v) = \deg(D - v) + 1 - g - l(W + v - D) = 1$. Moreover, it follows from the observation above that there exist $w_1, w_2 \in \mathcal{V}_{\tilde{K}}$ such that v, w_1, w_2 are distinct, $n_v = n_{w_1} = n_{w_2} = 0$, $l(D - v - w_1) = l(D - v) - 1$, $l(D - v - w_2) = l(D - v) - 1$, $l(D - w_1 - w_2) = l(D - w_1) - 1$. Now we can easily check that D, w_1, w_2 satisfy the condition of assertion (ii). This completes the proof of assertion (ii). Finally, we verify assertion (iii). The existence of s (resp. t) follows from the fact that $l(D - w_1) = 1$ (resp. $l(D - w_2) = 1$), and the uniqueness of s (resp. t, u) follows from the assumption that $l(D - v - w_1) = 0$ (resp. $l(D - v - w_2) = 0, l(D - w_1 - w_2) = 0$). This completes the proof of assertion (iii), hence also of Lemma 4.14. $\square$

Definition 4.15. Let G be a profinite group of PGF-type and $D = \sum_{v \in \tilde{\mathcal{V}}(G)} n_v \cdot v \in \mathrm{Div}(G)$.

(i) Write $H^0_G(D) = \{s \in \tilde{K}^\times(G) \mid \mathrm{ord}^G_v(s) + n_v \geq 0 \text{ for all } v \in \tilde{\mathcal{V}}(G)\} \cup \{*\} \subset \tilde{K}(G)$.

(ii) It follows from Theorem 4.5(iii), (iv), Theorem 4.13, Lemma 4.14(i), together with Theorem 4.16(i) below, that the set $\{n \in \mathbb{Z}_{\geq 0} \mid \text{there exist } v_1, \ldots, v_n \in \tilde{\mathcal{V}}(G) \text{ such that } H^0_G(D - \sum_{m=1}^n v_m) = \{*\}\}$ is not empty. Write $l_G(D)$ for the smallest integer in this set.

Theorem 4.16. *Let G be a profinite group of PGF-type, $(K, \Omega, \alpha : \mathrm{Gal}(\Omega/K) \xrightarrow{\sim} G)$ a PGF-envelope for G, and $D \in \mathrm{Div}(\tilde{K})$. Write $D_G \in \mathrm{Div}(G)$ for the image of $D \in \mathrm{Div}(\tilde{K})$ under the isomorphism $\mathrm{Div}(\tilde{K}) \xrightarrow{\sim} \mathrm{Div}(G)$ appearing in Theorem 4.5(iv). Then the following hold:*

(i) *The image of $H^0(D) \subset \tilde{K}$ under the isomorphism of monoids $\tilde{K} \xrightarrow{\sim} \tilde{K}(G)$ appearing in Theorem 4.5(iii) coincides with $H^0_G(D_G) \subset \tilde{K}(G)$.*

(ii) *It holds that $l(D) = l_G(D_G)$.*

Proof. These assertions follow from Theorem 4.13, Lemma 4.14(i). □

Definition 4.17. Let G be a profinite group of PGF-type, $v \in \tilde{\mathcal{V}}(G)$, and $\zeta, \lambda \in \tilde{\kappa}_v(G)$. Let us define $\zeta \boxplus_v \lambda \in \tilde{\kappa}_v(G)$ as follows:

(i) If $\zeta = *$, then $\zeta \boxplus_v \lambda := \lambda$.

(ii) If $\lambda = *$, then $\zeta \boxplus_v \lambda := \zeta$.

(iii) Suppose that $\zeta \neq *,\ \lambda \neq *$. Then it follows from Theorem 4.5(iv), Lemma 4.14(ii), Theorem 4.16(ii) that there exist $D = \sum_{w \in \tilde{\mathcal{V}}(G)} n_w \cdot w \in \mathrm{Div}(G)$ and $w_1, w_2 \in \tilde{\mathcal{V}}(G)$ such that v, w_1, w_2 are distinct, $n_v = n_{w_1} = n_{w_2} = 0$, $l_G(D) = 2$, $l_G(D - v - w_1) = l_G(D - v - w_2) = l_G(D - w_1 - w_2) = 0$. Moreover, it follows from Theorem 4.5(iv), Theorem 4.10, Lemma 4.14(iii), Theorem 4.16 that there exists a unique element s (resp. t, u) of $H^0_G(D)$ such that $s(v) = \zeta,\ s(w_1) = *,\ s(w_2) \neq *$ (resp. $t(v) = \lambda,\ t(w_1) \neq *,\ t(w_2) = *;\ u(w_1) = t(w_1),\ u(w_2) = s(w_2)$). Then we shall write $\zeta \boxplus_v \lambda := u(v)$.

Theorem 4.18. *Let G be a profinite group of PGF-type, $(K, \Omega, \alpha : \mathrm{Gal}(\Omega/K) \xrightarrow{\sim} G)$ a PGF-envelope for G, and $v \in \mathcal{V}_{\tilde{K}}$. Write $v_G \in \tilde{\mathcal{V}}(G)$ for the image of $v \in \mathcal{V}_{\tilde{K}}$ under the bijection $\mathcal{V}_{\tilde{K}} \xrightarrow{\sim} \tilde{\mathcal{V}}(G)$ appearing in Theorem 4.5(iv). Then $\boxplus_{v_G} : \tilde{\kappa}_{v_G}(G) \times \tilde{\kappa}_{v_G}(G) \to \tilde{\kappa}_{v_G}(G)$ is well-defined (i.e., does not depend on the choice of D, w_1, w_2) and determines a structure of field on $\tilde{\kappa}_{v_G}(G)$. Moreover, the isomorphism of monoids $\tilde{\kappa}_v \xrightarrow{\sim} \tilde{\kappa}_{v_G}(G)$ appearing in Theorem 4.8(ii) is an isomorphism of fields.*

Proof. This follows from Theorem 4.5(iv), Theorem 4.10, Lemma 4.14(iii), Theorem 4.16. □

Lemma 4.19. *Let $x, y \in \tilde{K}$. Then $x + y \in \tilde{K}$ is the unique element z of $\tilde{K}$ such that, for any $v \in \mathcal{V}_{\tilde{K}}$, if $x, y, z \in \tilde{\mathcal{O}}_v$, then it holds that $x(v) + y(v) = z(v)$.*

Proof. Trivial. □

Definition 4.20. Let G be a profinite group of PGF-type and $x, y \in \tilde{K}(G)$. Then it follows from Theorem 4.5(iii), (iv), Theorem 4.10, Theorem 4.18, Lemma 4.19 that there exists a unique element z of $\tilde{K}(G)$ such that, for any $v \in \tilde{\mathcal{V}}(G)$, if $x, y, z \in \tilde{\mathcal{O}}_v(G)$, then it holds that $x(v) \boxplus_v y(v) = z(v)$. Write $x \boxplus y \in \tilde{K}(G)$ for this element z.

Theorem 4.21. *Let G be a profinite group of PGF-type and $(K, \Omega, \alpha : \mathrm{Gal}(\Omega/K) \xrightarrow{\sim} G)$ a PGF-envelope for G.*

(i) $\boxplus : \tilde{K}(G) \times \tilde{K}(G) \to \tilde{K}(G)$ *determines a structure of field on $\tilde{K}(G)$. Moreover, the isomorphism of monoids $\tilde{K} \xrightarrow{\sim} \tilde{K}(G)$ appearing in Theorem* 4.5(iii) *is an isomorphism of fields.*

(ii) *It holds that $(\tilde{K}(G))^G = K(G)$, hence $\boxplus$ determines a structure of field on $K(G)$. In particular, the isomorphism of monoids $K \xrightarrow{\sim} K(G)$ appearing in Theorem* 4.5(i) *is an isomorphism of fields.*

Proof. Assertion (i) follows from Theorem 4.5(iii), (iv), Theorem 4.10, Theorem 4.18, Lemma 4.19. Assertion (ii) follows from assertion (i) and Theorem 4.5(iii). □

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