

The Asymptotic Behavior of the Free Boundary
in a One Phase Stefan Problem

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§1. Introduction

A one phase Stefan problem is a free boundary problem arising from a description of melting ice, where ice is maintained at zero degrees centigrade. In this note we are concerned with a Dirichlet type problem. The situation is roughly described as follows.

Let D be a bounded simply connected domain in R^n , $n \geq 3$, with smooth boundary ∂D . (D is the heat island, so to speak.) The exterior $R^n \setminus \bar{D}$ of D is occupied by water or ice. At $t = 0$ water occupies a domain G bounded by ∂D and another bounded connected smooth hypersurface Γ_0 lying outside D . We suppose that the temperature of water on ∂D for all $t > 0$ is given and positive. Let us denote this boundary value by $g(x, t)$, $t > 0$. Let $W(t)$ be the region occupied by water at time t . As time goes on, $W(t)$ will increase.

In this note we are interested in the following question :

Supposing that $g(x, t)$ satisfies the inequality

$$(1.1) \quad c_1(1+t)^\beta \leq g(x, t) \leq c_2(1+t)^\beta, \quad ,$$

can one prove the estimate of the type

$$(1.2) \quad \{x \mid |x| \leq R_1(1+t)^b\} \subset W(t) \cup D \subset \{x \mid |x| \leq R_2(1+t)^b\}$$

and, if so, for what value of b ?

A. Friedman [3],[4] proved (1.2) for the case $\beta = n/2 - 1$ with $b = 1/2$. Recently, S. Tokuda [11] proved the second relation for the case $\beta = 0$ with $b = 1/n$. In this note we extend these results for $0 \leq \beta \leq n/2 - 1$ with $b = (\beta + 1)/n$ (see Theorems 2 and 3). The second relation of (1.2) holds in fact for all $\beta \geq 0$ (Theorem 2) but we do not know if the first relation holds outside the mentioned range of β . Our result will be proved for weak solutions. The proof is based on a simple comparison argument. We compare the solution with a function which may be called a quasi-stationary solution.

§ 2. Formulation of the one phase Stefan problem

2.1. One phase Stefan problem

Let G be a bounded domain in R^n , $n \geq 3$, whose boundary consists of two smooth connected hypersurfaces ∂D and Γ_0 , with $\hat{\partial D}$ lying inside Γ_0 and bounding a simply connected domain D . Further we assume that D contains the origin 0 . We put $\Omega = R^n \setminus \bar{D}$. Since $W(t)$, the region occupied by water at time t , increases with t in a one phase problem, we can assume

that $W(t)$ can be expressed as

$$W(t) = \{(x, t) \in \Omega \times (0, \infty) \mid t > s(x)\},$$

where $s(x)$ is a certain function defined on Ω . The free boundary Γ_t at time t is given by $\Gamma_t = \{x \in \Omega \mid s(x) = t\}$. Now, the Stefan problem we are concerned with is formulated as follows :

Problem (I) Given smooth functions $h(x) \geq 0$, $x \in G$, and $g(x, t) > 0$, $x \in \partial D$, $t > 0$, find (smooth) functions $s(x)$, $x \in \Omega$, and $\theta(x, t)$, $x \in \Omega$, $t \geq s(x)$, which satisfy

$$(2.1) \quad \left(-\Delta + \frac{\partial}{\partial t} \right) \theta(x, t) = 0, \quad t > s(x),$$

$$(2.2) \quad \theta(x, 0) = h(x), \quad x \in G,$$

$$(2.3) \quad \theta(x, t) = g(x, t), \quad (x, t) \in \partial D \times (0, \infty),$$

$$(2.4) \quad \theta(x, t) = 0, \quad t = s(x),$$

$$(2.5) \quad \text{grad}_x \theta \cdot \text{grad}_x s = -\kappa, \quad t = s(x),$$

where κ is a given positive constant.

In several space variables, the existence of a classical solution of Problem (I) is proved only for a small time interval (see E.-I. Hanzawa [6] and A.M. Meirmanov [9, 10]). In this note we shall prove (1.2) for weak solutions which have been proved to exist. Specifically, we follow the approach of G. Duvaut [2], A. Friedman and D. Kinderlehrer [5], and formulate the problem as a parabolic variational inequality.

2.2. Variational inequality

According to Duvaut [2], Friedman and Kinderlehrer [5], Problem (I) leads to the variational inequality, expressed in a pointwise form, of the following type.

Let $K = \{v \mid v \in H^1(D), v \geq 0\}$.

Problem (II) Given functions $f(x)$, $x \in \Omega$, and $\psi(x, t) \geq 0$, $x \in \partial D$, $t > 0$, find $u \in L^2_{loc}([0, \infty); H^2_{loc}(\bar{\Omega}))$ such that $u_t \in L^2_{loc}([0, \infty); L^2_{loc}(\bar{\Omega}))$ and

$$\begin{cases} (-\Delta u + u_t)(v-u) \geq f(v-u) \text{ a.e. for } v \in K, \\ u = \psi \text{ on } \partial D \times (0, \infty), \\ u = 0 \text{ on } D \times \{0\}. \end{cases}$$

Problems (I) and (II) are related as follows. Given g , h , and κ of Problem (I), define f and ψ as

$$(2.6) \begin{cases} \psi(x, t) = \int_0^t g(x, \tau) d\tau, \quad x \in \partial D, \quad t \geq 0, \\ f(x, t) = \begin{cases} h(x, t) & \text{if } x \in G, \\ -\kappa & \text{if } x \in \Omega \setminus G. \end{cases} \end{cases}$$

It was proved in [5] that Problem (II) has then a unique solution u and that u satisfies $u_t \geq 0$, a.e., and

$$u \in L^\infty_{loc}([0, \infty); H^{2,p}_{loc}(\bar{\Omega})) \quad \text{for } 1 \leq p < \infty,$$

$$u_t \in L^\infty_{loc}([0, \infty); L^\infty_{loc}(\bar{\Omega})) = L^\infty_{loc}(\bar{\Omega} \times [0, \infty)).$$

Thus $u \geq 0$, a.e.. Relate θ and u through

$$\theta(x, t) = u_t(x, t),$$

$$(2.7) \quad \begin{cases} u(x,t) = \int_{s(x)}^t \theta(x,\tau) d\tau & \text{if } x \in \Omega \setminus G, \quad s(x) \leq t, \\ u(x,t) = 0 & \text{if } x \in \Omega \setminus G, \quad 0 \leq t \leq s, \\ u(x,t) = \int_0^t \theta(x,\tau) d\tau & \text{if } x \in G, \quad 0 \leq t. \end{cases}$$

Then it was proved in [5] that u is a solution of Problem (II) if and only if θ is a weak solution of Problem (I) in the sense of [4]. We also remark that u was proved to be continuous in $\Omega \times (0, \infty)$.

The following comparison theorem, also due to [5], will be used as a basic tool in our proof.

Theorem 1. Let $u, \hat{u}$ be solutions of Problem (II) for f, ψ and $\hat{f}, \hat{\psi}$ respectively. Assume that $f \leq \hat{f}$ and $\psi \leq \hat{\psi}$. Then $u \leq \hat{u}$ in Ω .*)

§3. Main results

Let G and D be as at the beginning of §2 and let $\Omega = \mathbb{R}^n$. Given g, h , and κ as in Problem (II), let u be the solution of Problem (II) with f and ψ defined as (2.6). Put

$$(3.1) \quad W(t) = \{x \in \Omega \mid u(x,t) > 0, \quad t > 0\}.$$

Our main results can now be formulated as follows.

Theorem 2. If $g(x,t)$ is majorized by a smooth non-decreasing function $\eta(t)$, namely if

$$(3.2) \quad 0 < g(x,t) \leq \eta(t), \quad x \in \partial D, \quad t > 0; \quad \eta_t \geq 0$$

($\eta_t = \frac{\partial \eta}{\partial t}$), then there exist positive constants a_1

*) We remark that this theorem applies even if f and ψ of Problem (II) are not derived from Problem (I).

and a_2 such that

$$(3.3) \quad W(t) \subset \{x \in \mathbb{R}^n \mid |x|^n < a_1 \int_0^t \eta(\tau) d\tau + a_2\}, \quad t > 0.$$

Corollary If $\eta(t) = \alpha(1+t)^\beta$, $\alpha > 0$, $\beta \geq 0$,

we obtain $W(t) \subset \{x \in \mathbb{R}^n \mid |x| \leq c_3(1+t)^{(\beta+1)/n}\}$, $t > 0$,

where c_3 is a positive constant.

Theorem 3. If there exist constants β , $0 \leq \beta \leq n/2 - 1$, and $k > 0$ such that

$$(3.4) \quad k(1+t)^\beta \leq g(x,t),$$

then there exist positive constants k' and t_0 such that

$$(3.5) \quad W(t) \cup D \supset \{x \in \mathbb{R}^n \mid |x| \leq k'(1+t)^{(\beta+1)/n}\}, \quad t > t_0.$$

Remark 1. Theorem 3 and Corollary to Theorem 2 show that (1.1) implies (2.1) if $0 \leq \beta \leq n/2 - 1$.

Remark 2. Suppose that D is a ball with center O and $g(x,t) = c(1+t)^\beta$. The proof of Theorems 2 and 3 shows that $\partial\theta/\partial r|_{\partial D, r=|x|}$ is of $O(t^\beta)$. Thus, (1.2) shows that the amount of heat used to melt ice up to time t is of the same order as the amount of heat having flown in from D up to time t . In fact, the proof shows that the heat retained in water at time t is of lower order if $\beta \leq n/2 - 1$.

§4. Proof of Theorems 2 and 3

4.1. Upper bound for $W(t)$; the proof of Theorem 2

The proof is based on Theorem 1. In order to obtain $\tilde{u}$, with which u , the solution of Problem (II), is to be compared, we introduce a melting-ice situation with heat generation in the water region. It is a spherically symmetric problem, and for convenience we express the free boundary as $|x| = \rho(t)$. The function ρ and the temperature function $\Phi(r, t)$, $r = |x|$, are defined as follows.

$$(4.1) \quad \begin{cases} \Phi(r, t) = c(t) \{r^{2-n} - \rho(t)^{2-n}\}, & r \in (0, \rho(t)], \quad t \geq 0, \\ \rho(t)^n = \frac{n(n-2)}{\kappa} \int_0^t c(\tau) d\tau + \rho(0)^n, \end{cases}$$

where $c(t)$ is a positive function to be determined later.

Note that Φ is determined by $c(t)$ and $\rho(0)$. We use the notation

$$B = \{(x, t) \in (R^n \setminus \{0\}) \times R \mid |x| \leq \rho(t)\},$$

$$\tilde{s}(x) = \rho^{-1}(x),$$

where ρ^{-1} is the inverse function of ρ . The free boundary would then be described by $t = \tilde{s}(x)$.

It is now easy to prove the following properties of Φ .

Lemma 1.

$$\Delta \Phi(|x|, t) = 0, \quad (x, t) \in B, \quad t \geq 0,$$

$$\Phi(\rho(t), t) = 0, \quad t \geq 0,$$

$$\text{grad } \Phi(|x|, t) \cdot \text{grad } \tilde{s}(x) = -\kappa, \quad |x| = \rho(t), \quad t \geq 0.$$

Lemma 2. $c_t(t) \geq 0 \Rightarrow \Phi_t(r,t) \geq 0, r \leq \rho(t)$.

Proof of Theorem 2. We use function Φ with $c(t)$ equal to $c'\eta(t)$ where c' is a positive constant to be determined. We take $\rho(0)$ large enough so that the set $\{x \mid |x| \leq \rho(0)\}$ contains D . It follows from Lemmas 1 and 2 that $(\Phi, \tilde{s})$ is a solution of Problem (I) if we make the following replacement. Firstly, θ, s and G are replaced by $\Phi, \tilde{s}$ and $\tilde{G} = \{x \in \Omega \mid |x| < \rho(0)\}$, respectively. Secondly, $h(x)$ and $g(x,t)$ are replaced by the values of $\Phi(|x|,t)$ on the respective set. Finally, equation (2.1) is replaced by an inhomogeneous equation that

$$(-\Delta + \frac{\partial}{\partial t}) \Phi(|x|,t) = q(x,t) , t > \tilde{s}(x) ,$$

where

$$(4.2) \quad q(x,t) = -\frac{\partial}{\partial t} \Phi(|x|,t) \geq 0 .$$

Define now $\tilde{u}$ by (2.7) with obvious replacements. Then, it is seen that $\tilde{u}$ is a solution of Problem (II) with Ψ and f replaced respectively by $\tilde{\Psi}$ and $\tilde{f}$ defined as follows :

$$\begin{aligned} \tilde{\Psi}(x,t) &= \int_0^t \Phi(|x|,\tau) d\tau , x \in \partial D , t > 0 , \\ \tilde{f}(x,t) &= \begin{cases} \Phi(|x|,0) + Q(x,t) & \text{if } x \in \tilde{G} , t > 0 , \\ -\kappa + Q(x,t) & \text{if } x \in \Omega \setminus \tilde{G} , t > 0 , \end{cases} \end{aligned}$$

where we put

$$Q(x, t) = \begin{cases} \int_{\tilde{S}(x)}^t q(x, \tau) d\tau, & x \in \Omega \setminus \tilde{G}, t > 0, \\ \int_0^t q(x, \tau) d\tau, & x \in \tilde{G}, t > 0. \end{cases}$$

It is clear that by choosing c' large enough and $\rho(0)$ still larger, if necessary, we can make $\tilde{G} \supset G$, $\tilde{\Psi} \geq \Psi$, and $\tilde{f} \geq f$. Applying Theorem 1, we then obtain $\tilde{u} \geq u$. Therefore, u is zero whenever $\tilde{u}$ is zero. Thus, we get

$$\begin{aligned} W(t) &\subset \{x \mid |x| < \rho(t)\} \\ &= \{x \mid |x|^n < \rho(t)^n\} \\ &= \{x \mid |x|^n < c_4 \int_0^t \eta(\tau) d\tau + c_5\} \end{aligned}$$

where $c_4 = \frac{n(n-2)c'}{\kappa}$, $c_5 = \rho(0)^n$. Q.E.D.

4.2. Lower bound for $W(t)$; the proof of Theorem 3

We first reduce the problem to a spherically symmetric one.

Let us denote by B_R the open ball of radius R with the center 0 . Since D is a domain containing 0 , we can take R_1 such that $\overline{B_{R_1}} \subset D$. We fix such an R_1 and take R_0 so that $0 < R_0 < R_1$. Letting k and β be constants appearing in (3.4), we introduce a function of the type Φ defined by (4.1). Namely, we put

$$(4.3) \quad \hat{\Phi}(r, t) = k R_0^{n-2} (1+t)^\beta \{r^{2-n} - \hat{\rho}(t)^{2-n}\}, \quad r \in (0, \hat{\rho}(t)], t > 0$$

$$(4.4) \quad \hat{\rho}(t)^n = \frac{n(n-2)kR_0^{n-2}}{\kappa} \int_0^t (1+\tau)^\beta d\tau + \hat{\rho}(0)^n, \quad \hat{\rho}(0) = R_1.$$

We now consider the following problem :

Problem (III) This is Problem (II) with D , G , etc. replaced by the following $\hat{D}$, $\hat{G}$, etc.

$$\begin{cases} \hat{D} = B_{R_0}, \quad \hat{G} = B_{R_1} \setminus \overline{B_{R_0}}, \\ \hat{h}(x) = \hat{\phi}(|x|, 0) \quad ; \quad \hat{g}(x, t) = \hat{\phi}(R_0, t), \quad |x| = R_0. \end{cases}$$

This problem has a classical solution $\hat{\theta}(x, t)$ and $\hat{s}(x)$. The pair $(\hat{\theta}, \hat{s})$ gives rise to a solution $\hat{u}$ of the corresponding variational inequality in $(R^n \setminus \hat{D}) \times (0, \infty)$. Since $R^n \setminus \hat{D} \supset \Omega$, $\hat{u}$ may be regarded as a solution of the variational inequality in $\Omega \times (0, \infty)$. Thus, we can compare the solution u of the original problem with $\hat{u}$. Since the boundary value $\hat{g}$ of Problem (III) is non-decreasing and majorizes the initial value, it is clear that $\hat{\theta}(x, t)|_{x \in \partial D} \leq \max_{|x|=R_0} \hat{\theta}(x, t) \leq k(1+t)^\beta$, where the last inequality follows from the definition of $\hat{\phi}$. Therefore, it follows from Theorem 1 that $\hat{u} \leq u$. Thus, we have seen that it suffices to prove Theorem 3 for the solution $\hat{\theta}$ (or $\hat{u}$) of Problem (III).

Since $\hat{\theta}$ and $\hat{u}$ are spherically symmetric, there exists a smooth function $\hat{r}(t)$ such that

$$\hat{W}(t) = \{x \in R^n \setminus \hat{D} \mid \hat{u}(x, t) > 0\} = B_{\hat{r}(t)} \setminus \overline{B_{R_0}}.$$

The following lemma, which expresses the conservative law for heat flow, can be proved easily by integrating the heat equation by parts.

Lemma 3. Denoting by $|A|$ the volume of the set $A \subset \mathbb{R}^n$, we have $(\theta_r = \partial\theta/\partial r)$

$$(4.5) \quad \int_{\hat{W}(t)} \hat{\theta}(\xi, t) d\xi - \int_{\hat{G}} \hat{h}(\xi) d\xi + \kappa \{ |\hat{W}(t)| - |\hat{G}| \} \\ + \int_0^t d\tau \int_{\partial\hat{D}} \hat{\theta}_r(\xi, \tau) d\xi = 0$$

On the other hand, we see from the proof of Theorem 2 that

$$(4.6) \quad \hat{\theta}(x, t) \leq \hat{\Phi}(|x|, t), \quad x \in \hat{W}(t), \quad t > 0.$$

Since $\hat{\theta}$ and $\hat{\Phi}$ take the same boundary values on $\partial\hat{D}$, the following lemma holds. This lemma is crucial, as it gives a lower bound of the amount of heat flowing in through $\partial\hat{D}$.

Lemma 4. We have

$$(4.7) \quad -\hat{\theta}_r(x, t) \geq -\hat{\Phi}_r(|x|, t), \quad x \in \partial\hat{D}, \quad t > 0.$$

In what follows we denote by $c_1, c_2, \dots$ various constants which do not depend on t nor R_0 . Using (4.7), we see that

$$(4.8) \quad - \int_0^t d\tau \int_{\partial\hat{D}} \hat{\theta}_r(\xi, \tau) d\xi \geq - \int_0^t d\tau \int_{\partial\hat{D}} \hat{\Phi}_r(|\xi|, \tau) d\xi \\ = c_1 R_0^{n-2} (1+t)^{\beta+1}.$$

Using (4.7), we also obtain

$$(4.9) \quad \int_{\hat{W}(t)} \hat{\theta}(\xi, t) d\xi \leq \int_{\hat{W}(t)} \hat{\Phi}(|\xi|, t) d\xi \leq \int_{B_{\hat{\rho}(t)}} \hat{\Phi}(|\xi|, t) d\xi \\ = c_2 R_0^{n-2} (1+t)^\beta \int_0^{\hat{\rho}(t)} \{ r - \hat{\rho}(t) \}^{2-n} r^{n-1} dr$$

$$\begin{aligned}
&\leq c_3 R_0^{n-2} (1+t)^\beta \{c_4 R_0^{2(n-2)/n} (1+t)^{2(\beta+1)/n} + c_5\} \\
&= c_6 R_0^{n-2+p} (1+t)^{(1+2/n)\beta+2/n} + c_7 R_0^{n-2} (1+t)^\beta,
\end{aligned}$$

where $p = \frac{2(n-2)}{n} > 0$.

If $0 \leq \beta < n/2 - 1$, the right side of (4.9) is of $o((1+t)^{\beta+1})$.

Thus, by using (4.8) and (4.9) in (4.5) and recalling that $\hat{h}(x) \geq 0$, we obtain

$$(4.10) \quad \{|\hat{W}(t)| - |\hat{G}|\} \geq c_8 R_0^{n-2} (1+t)^{\beta+1} + o((1+t)^{\beta+1}).$$

If $\beta = n/2 - 1$, the first term on the right side of (4.9) is of order $(1+t)^{\beta+1}$. However, if we take R_0 sufficiently small, which is permissible, so that $c_1 - c_6 R_0^p > 0$, we can obtain

(4.10) also for $\beta = n/2 - 1$. (3.5) for $\hat{W}(t)$ follows from (4.10)

at once. Q.E.D.

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