

Sasakian Exact Solutions for Spinning Black Holes in Superstring Inspired Gravities



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Abstract

In this thesis, we study the gravitational solution with Sasakian structure and its novel relation to higher-dimensional black hole physics. As long as superstring theory is valid in some sense, higher-dimensional spacetime is a necessity framework. Therefore, we need to study higher-dimensional gravitational theories, which may result in its low-energy limit.

We have successfully found new solutions in two supergravity (SUGRA) theories. First, in the 5-dimensional gauged minimal SUGRA, we have obtained a new (reduced) solution, which corresponds to a nontrivial limit of a known charged rotating black hole. Although this solution violates the Sasakian condition slightly, we have used this as means of extending the Sasakian condition in any odd-dimensional spacetime; it may be used as a guiding principle for finding other reduced solutions whose original black hole has not been found yet. This results in allowing the presence of torsion, which deforms the original Sasakian condition. Applying this modification, we have found a new solution in the 11-dimensional SUGRA.

We have also obtained new exact solution in Einstein–Gauss–Bonnet gravity in every odd dimensions higher than 3, using the highly isometric ansatz that admits Sasakian structure. These solutions are novel in the sense that (i) their angular momentum is finite (they are rotating), (ii) they are asymptotically AdS at least locally, (iii) they couple to the Maxwell-field (they are charged). In regard to (i), we note that this is the very first case of successful evaluation of conserved charges (angular momentum and energy), made possible by the detail study of the Abbott-Deser formulation. We have clarified the difference of our solutions from other known solutions: some of the 2-rotating solution with zero angular momentum can be reproduced as a special case of our solution, and some of the topological static black holes are also obtained similarly.

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Chap. 1

Introduction

Background and Motivation

Since the suggestion of superstring theory, it has been eagerly desired to pioneer the higher-dimensional physics. Superstring theory insists the necessity of the 10- or 11-dimensional spacetime originally, yielding a disagreement to our visible spacetime. As long as superstring theory is valid in some sense, higher-dimensional spacetime is an indispensable framework. Therefore, we need to study higher-dimensional gravitational theories, which may result in its low-energy limit.

Here, there is a peculiar difficulty to investigate the higher-dimensional gravity theories: whereas they are capable of having wider diversity of gravitational backgrounds than the usual 4-dimensional theories, it is quite hard to find out such solutions in exact form, especially lower-isometric ones. One difficulty can be seen in diverse extensions of the Kerr solution, the rotating black hole in 4 dimensions. At 1986, R. C. Myers and M. J. Perry achieved its higher-dimensional generalization which preserves the flat boundary condition [1]. Nevertheless, this accomplishment has left an unclear problem how to extend this solution exactly by coupling any object. Actually, until 2004 the extension to add the cosmological constant was not succeeded in all dimensions [2], and even now the one to uplift the Kerr–Newman solution has not been constructed in exact form. While already-known technique, called solution-generating technique, yielded some matter-charged black holes in un-gauged supergravity theories [3, 4, 5], this technique is known as so delicate that it is unavailable for the asymptotic (A)dS case.

On the other hand, a mathematical subject has contributed to explore other class of gravitational solutions: this is what is called Sasakian geometry. As its name suggests, the founder of this geometry is one of the greatest mathematicians in Japan, Shigeo Sasaki [6]. During its long history since 1960s, at one time the stimulating study had been done especially by Japanese researchers [7, 8], gradually declining due to the undiscovery of the nontrivial example for the time being. In recent the intense investigation has revived

because of the disclosure in theoretical physics. Represented by AdS/CFT correspondence [9, 10, 11, 12], superstrings provide various scenarios of high-energy physics, which respectively requires to cope with extra dimensions. Among them, Sasakian geometry was found out its ability as a precise prescription, so that nontrivial examples were hunted by theoretical physicists [13, 14, 15, 16, 17]. These works also confirmed a credible worth of Sasakian geometry itself, inclusion of nontrivial examples.

In this development, a study revealed an interesting fact: that is, taking a certain limit to a sort of the Kerr-AdS solution, $Y^{p,q}$ can be reproduced [18]. $Y^{p,q}$ is the 5-dimensional Sasaki–Einstein manifold which was found firstly as the nontrivial Sasakian geometry [13]. The similar approach had been established previously in 4 dimensions for bringing the Euclidean Kerr-dS solution to a gravitational Kähler instanton [19], but this odd-dimensional matching was revealed to have the advantage of global aspects as regularity or signature. Owing to this achievement, further concrete examples of Sasaki–Einstein manifolds were obtained by using this relation to more general class of rotating solutions, e.g., the ones with the lower isometry and higher dimensions [14, 15, 16, 17]. In this sense, such relationship has succeeded in Einstein gravity, that is, these Sasaki–Einstein spaces can be regarded as the reduced limit of rotating black hole solutions. Then, it seems natural to suspect that this relationship can extend beyond the pure Einstein gravity.

A naive generalization has been discussed from the viewpoint of the mathematical beauty or the tensorial symmetry, called hidden symmetry [20]: The clue fact is that both-side solutions over the limit possess a number of Killing–Stäckel tensor fields enough to separate the variables of some equations of motion [21, 22]. Considering this property, the metrics of both sides can be generalized not to impose the Einstein equation but to preserve the tensorial symmetry, which are called *off-shell metrics*. However, this extension might not be suitable to research the other physical theories. Indeed, in the both off-shell classes it was found few solutions except for the Einstein space. Moreover, at the off-shell Kerr-NUT side the matter-solution only known previously was doubted to be absent from its partner in the class, and supergravitational solutions were constructed slightly out of the class. Since the way of extension should not be necessarily relevant to such tensorial symmetry, we anticipate the ample of improvement for the relationship. If this relationship is found to upgrade toward a sort of physical universality, it would provide a new method to construct analytic solutions of any physical theories. Therefore, we attach great importance to exploring this relationship, and to that end, gathering more samples of exact solutions.

Subject and Contents

In this thesis, we study the gravitational exact solutions with Sasakian structure and their novel relationship to rotating black holes. In accordance with the prediction of the

higher-dimensional spacetime, our analysis proceeds in several gravitational theories which are inspired superstring theory: the candidates of the correct low-energy limit in this theory of everything.

The one class we investigate is supergravity (SUGRA) [23]. There are wide varieties to select matter-fields obeying the supersymmetric requirement while the gravitational sector is commonly adopted to the Einstein–Hilbert term. Thus, not the all but a part of this variety should originate from superstring theory as the low-energy effective theory. We successfully obtain exact results in the theories included in such low-energy limits, that is, the 5-dimensional (un-)gauged minimal SUGRA and the 11-dimensional SUGRA. Among a number of SUGRAs, they have respectively established their place for theoretical interest. First, the gauged minimal supergravity of 5 dimensions is known as the simplest higher-dimensional theory with supersymmetry (SUSY), so that it was well confirmed the embedding property in higher-energy theories, e.g. the consistent Kaluza-Klein reduction of type IIB SUGRA [24, 25, 26]. Many of interesting researches have been done, especially constructing a charged rotating black hole solutions in exact form [27]. In this theory we obtain a new exact solution, which corresponds to the reduced limit from this black hole. In contrast to the Einstein gravity case, this reduced solution slightly violates the Sasakian condition. We consider that it should indicate an extension of Sasakian geometry, which may be used as a guiding principle for finding other reduced solutions whose original black hole has not been found yet. Therefore, we discuss the generalization of Sasakian geometry suited to this insight. In this argument, we adopt the deformation of Sasakian structure by the presence of torsion, which is expected from the feature of the present black hole solution before the reduced limit [28]. This argument ensures a concrete class of any odd-dimensional metrics possessing rich tensorial symmetry, so that we find exact solutions newly, i.e., the 5-dimensional un-gauged extension and the 11-dimensional one. The 11-dimensional SUGRA is manifested as the low-energy limit of M-theory, linked to superstring theory with a string duality.

We also succeed in the other theory called Einstein–Gauss–Bonnet gravity (EGB gravity) [29]. This gravitational theory belongs to a sort of the generalization beyond Einstein gravity in higher dimensions, possessing several advantage in the higher-derivative theories e.g. the ghost-free renormalizability [30, 31] and the quasi-linear property [32]. In particular EGB gravity is often seen as a superstring inspired gravity, that is, the effective theory with the quantum correction originating from super string theory: The α' -expansion indicates a possibility of the Gauss-Bonnet term appearing as the first curvature correction to Einstein gravity [33, 34, 35]. In the case of pioneering new solutions in this theory, a much more trouble confronts us than the ordinary gravitational theory. For one thing, less exact solutions have been established than general relativity with the interference of the higher power term. A strange fact also has been confirmed that somewhat various solutions

happen to appear [36, 37, 38], only when the Gauss-Bonnet coupling constant is chosen to a special value [39]. Such difficulty especially disturbs to find rotating black hole solutions. These solutions could be realized as a generalization of the Myers–Perry solution for some particular meaning, but such exact solutions have not been found yet. On the other hand, recent numerical analysis has indicated the existence of the rotating black hole in this gravitational theory, constructing the numerical solution in the 5-dimensional spacetime [40, 41]. Hence, investigating exact stationary solutions beyond static has been an important theme in EGB gravity. In our analysis, we obtain new exact solutions in EGB gravity of every odd dimension higher than 3. To find out these solutions, we assume the highly isometric ansatz that admits Sasakian structure. These solutions are novel in the sense that (i) their angular momentum is finite (they are rotating), (ii) they are asymptotically AdS at least locally, (iii) they couple to the Maxwell-field (they are charged). Despite their physical importance, such analytic solutions have not been known so far in EGB gravity. In regard to (i), we note that this is the very first case of successful evaluation of conserved charges (angular momentum and energy), made possible by the detail study of the Abbott-Deser formulation [42]. The location of them in the gravitational-solution space is so nontrivial that we convince that they should develop several fields of higher-dimensional physics.

This thesis is organized as follows: In chapter 2, we look around the basic property of Sasakian geometry. We start at the mathematical description with the convenience of theoretical physicists. It is also explained that how Sasakian geometry has been incorporated to AdS/CFT correspondence. In chapter 3 we provide a review of the novel relationship between Sasakian geometry and the Kerr-NUT-AdS metrics. We elucidate the explicit formula of the reduced limit in Einstein gravity and one of its generalization. In chapter 4 our investigation in SUGRAs is presented especially in 5 dimensions and 11 dimensions. Firstly, we obtain the new exact solution which properly links to a known black hole with the reduced limit. Next, we discuss a generalization of Sasakian structure deformed by the presence of torsion, ensuring the rich tensorial symmetry in a concrete class of metrics. Using this, we obtain more solutions. Its global behavior is confirmed in Euclidean regime. In chapter 5, we investigate other superstring inspired gravity, EGB gravity. We obtain new exact samples of Sasakian geometry. This results in overcoming one difficulty of this theory. We explain this difficulty from the unbalance of explored solution in this theory: Constructing the rotational solution exactly has been awaited so far. Owing to Sasakian structure, we obtain new exact solutions as unexpectedly large class. We finally evaluate the conserved charges by AD-formulation, properly acquiring the finite angular momentum, energy and Maxwell charge. In chapter 6, the thesis ends with the conclusion.

Chap. 2

Sasakian geometry for higher-dimensional physics

In this chapter, we provide a review for the basic property of Sasakian geometry. We start at Sec. 2.1 with explaining its mathematical description from the convenient viewpoint of theoretical physicists. A specific relation to Kähler geometry will help us to clarify our work and its potential. In Sec. 2.2, the sequence of events is presented how Sasakian geometry has been incorporated to one of the stories in superstring theory, i.e., AdS/CFT correspondence.

2.1 Definition and properties

There are several definitions of Sasaki manifold, whose equivalence has been well confirmed (see, e.g., reviews [43, 44, 45] and references therein). The following one is preferred by many of theoretical physicists:

Definition 2.1.1 *Let (M, g) be a Riemannian manifold. Then, (M, g) is called a Sasaki manifold if its metric cone $(C(M), \mathring{g} = dr^2 + r^2g)$ admits a Kähler structure.*

Owing to the brevity of this description, it is free from the mathematical complexity, otherwise other definitions with global restrictions. This description extracts the properties of Sasakian space from the one higher-dimensional space, adopting the metric cone for the lift-up manner. A simple conical uplift reads from the polar coordinate expression with a radial direction: The flat line element corresponds to the uplift from the spherical space. The Kähler structure on the uplift space brings several restrictions to the original manifold. Among them, it is indicated that Sasaki manifold must be the odd-dimensional space. Thus, the simplest example of Sasaki manifolds is the $(2n+1)$ -dimensional spherical space S^{2n+1} with the positive integer n .

The equivalence of other definitions has been established for Sasakian structure [46, 47]. Some of them also possess important physical meanings.

Proposition 2.1.2 *Let (M, g) be a Riemannian manifold. Then the following conditions are equivalent:*

- (1) (M, g) is a Sasaki manifold.
- (2) There exists a Killing vector field ζ of unit length on M so that the dual 1-form η satisfies

$$\nabla_X(d\eta) = -2X^\flat \wedge \eta, \quad (2.1)$$

for any vector field X , where $X^\flat = g(X, -)$.

- (3) There exists a Killing vector field ζ of unit length on M so that the $(1,1)$ -tensor field Φ defined by $\Phi(X) = \nabla_X \zeta$ satisfies

$$(\nabla_X \Phi)(Y) = g(\zeta, Y)X - g(X, Y)\zeta, \quad (2.2)$$

for any pair of vector fields X, Y .

- (4) There exists a Killing vector field ζ of unit length on M so that the curvature satisfies

$$R(X, Y)\zeta = g(\zeta, Y)X - g(\zeta, X)Y + \Phi([X, Y]). \quad (2.3)$$

where the Killing vector field ζ is called the Reeb vector field and the dual 1-form η called the contact 1-form. The unit-length condition is the restriction of its norm to be constant. This vector field ζ is relevant to the supersymmetry (SUSY) in the specific setup, as will explain at the latter of this section. One of the interest feature is a descendant relation, which is induced by the above proposition. This is often called the Kähler sandwich structure, as is shown in the figure 2.1, combining the conical uplift description. It is well confirmed fact that the $(1,1)$ -tensor field Φ assures the relation to the 1-lower dimensional space: The $(2n + 1)$ -dimensional metric with Sasakian structure can be written down the Kaluza-Klein metric describing a $U(1)$ -bundle over the $2n$ -dimensional base space B that is also Kähler at least locally. Therefore, this property forms a hierarchical structure over the different dimensional spaces.

Considering the simplest case of Sasakian geometry, the base space of the $(2n + 1)$ -dimensional round sphere is the complex projective space CP_n . Then, the local metric can be written by the complex n -dimensional Fubini-Study metric, which fortunately possesses global regularity over the descendance. However, for preserving the regularity to the $(2n + 1)$ -dimensional metric, it is not required that the 1-dimensional higher or lower space must

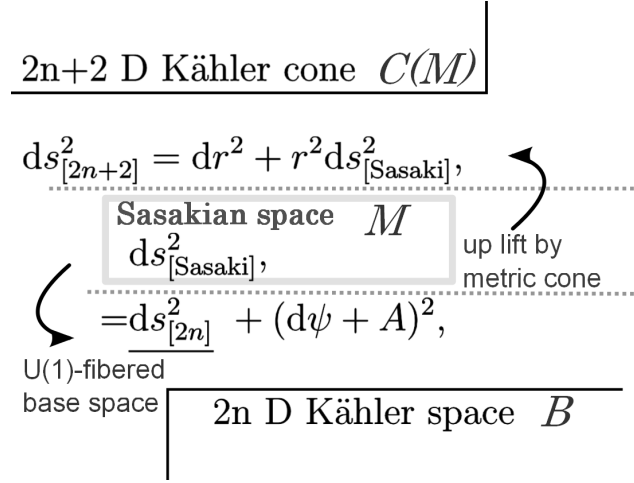


Figure 2.1 the description of Kähler sandwich structure

be regular globally; such one is rather so clean to find out obviously. Hence, Kähler sandwich is useful enough as a local description.

This property provides a bottom-up construction of Sasakian space, that is, making from a known $2n$ -dimensional metric of the Kähler manifold. For instance, there is another 5-dimensional Sasakian example constituted from the four-dimensional base space $S^2 \times S^2$, so-called $T^{1,1}$. It was also been confirmed that the higher-dimensional analogues can be obtained by the adopting the direct product of the low-dimensional Fubini-Study metric as the base space in early days of Sasakian history. Whereas their 1-lower metrics are globally well-defined, 1-higher spaces are conical singular unless the spherical case.

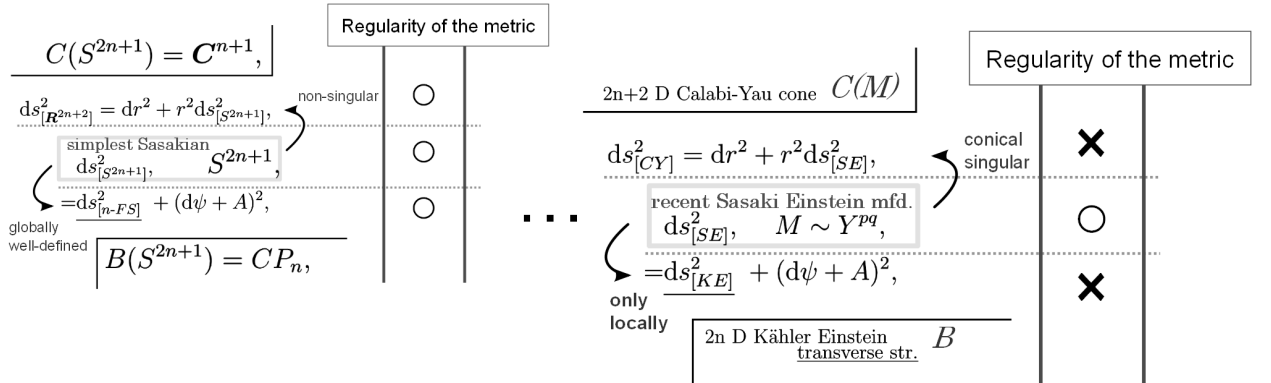


Figure 2.2 the contrast of Kähler sandwich structure

It is worth mentioned that a large class of Sasaki manifolds had been undiscovered until 2004: the class whose decomposed metrics cannot extend globally to $2n$ -dimensional man-

ifolds. During the time of this undiscovery, a classification of these objects was established from an abstract viewpoint by the behavior of ζ . If the the orbits of ζ cannot close, Sasaki manifolds are said to be *irregular*. If the orbits close, a $U(1)$ action can be defined. The fact ζ is nowhere vanishing assures that the isotropy groups of this action are all finite. Thus, the space of the leave foliation will be a $2n$ -dimensional Kähler orbifold. Such Sasaki manifolds are called *quasi-regular*. The space of leaves is actually a Kähler manifold only when the $U(1)$ action is free, then such Sasaki manifolds are said to be *regular*. In this classification, the regular examples were well recognized, such as S^5 , $T^{1,1}$, *etc.* Due to the doubt for the other classes containing concrete examples, the study of Sasakian geometry has declined once.

Here, a Sasaki manifold is upgraded to the Sasaki–Einstein manifold when it satisfies the Einstein condition of D dimensions:

$$R_{\mu\nu} = (D - 1)\lambda g_{\mu\nu}, \quad (2.4)$$

with λ a constant parameter*. This condition, just as it sounds, often appears in the gravitational equation of motion (EOM). The simplest case is the cosmological Einstein gravity (called Einstein gravity for simplicity), constructed from the Einstein–Hilbert action and the cosmological constant relevant to λ . In case of the AdS/CFT, the reduction of the Freund–Rubin flux plays the roll of the cosmological constant or λ on the reduced internal EOM [48].

Among the solutions of Eq. (2.4), the Sasakian condition corresponds to the supersymmetric condition. Sasakian structure assures the existence of the Killing spinor for the Einstein EOM. Intuitively, this is because the Reeb vector ζ is composed by the spinor bilinear, which fulfills the KS equation. Here, the cone of this space has been investigated; It becomes the Ricci flat Kähler cone, such that the Calabi-Yau cone. The Calabi-Yau manifold is so familiar as the supersymmetric object; Calabi-Yau 3-folds are the 6-dimensional compact manifolds that keep one SUSY at the other setup, the $R^{1,3} \times M_6$ background. The above Sasakian definition indicates the uplift space corresponds to the conical deformation of this SUSY object. Several investigation has also suggested the relation to its conicality and the addition of Freund–Rubin flux in the context of superstring theory [49, 50].

2.2 Sasaki–Einstein space in AdS/CFT correspondance

Sasaki–Einstein manifold has been well accepted by the AdS/CFT correspondance. The AdS/CFT correspondence is one of the conjecture in superstring theory, insisting the du-

*Adjusted to the traditional Sasakian definition, this scale factor is fixed to $1/2$. However, in this thesis that parameter sometimes plays an important roll. This is permitted by a slightly modification, i.e. scaling of (2.1.1).

ality between the gauge theory and the gravitational theory. This conjecture reads from the brane physics. It was found that the $(p + 1)$ -dimensional extended object exists in the $(9 + 1)$ -dimensional spacetime, originated from the non-perturbative effect in superstring theory. This is called the Dirichlet p -brane, for shortly Dp -brane or D -brane. As shown in the figure 2.3, a D -branes' configuration possesses two kinds of the physical effects appearing as the other descriptions in the low energy region: There are open strings on the

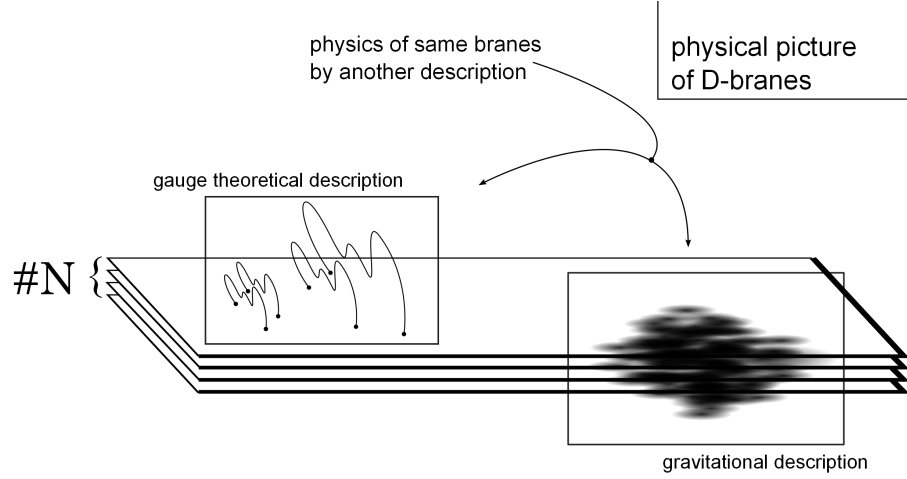


Figure 2.3 the picture of D -brane physics

brane fixing their endpoints by the Dirichlet boundary condition, composing field-contents of the gauge theory. Simultaneously, it gives the description from the gravitational effect that the existence of the D -brane distort the 10-dimensional spacetime. These difference at the low energy region is based on the choice of limit taking from the brane configuration; The number of Dp -branes N appears at the gauge theory side to the rank of gauge group. On the other side, due to increasing the total mass, taking larger N accompanies with the better gravitational description. Many of indirect evidences have supported that they should relate in each other as a strong-weak dual. It means that the classical gravitational theory should describe the strong coupling physics of the dual field theory, and vice versa.

Among such dualities, the AdS_5/CFT_4 correspondence has been investigated most successfully. Juan Maldacena suggested the duality between the $\mathcal{N} = 4$ super Yang-Mills gauge theory and the type IIB superstring or supergravity in $AdS_5 \times S^5$ background, extracting it from the $D3$ -brane system [9]. On the supergravity side, the static back reaction of Dp -branes is described by the black p -brane solution [51]. This is the exact solution in the supergravity system, possessing some matter flux. The Dp -branes' effect can read from the flux of the $(p + 2)$ -form field-strength F_{p+2} , due to the electric coupling as well as the gravitational one.

The figure 2.4 shows the present case, back-reacted the $D3$ -branes. If far from the

gravitational source, the 10-dimensional spacetime does not receive any deformation, being then asymptotically flat. As it gets closer, the black brane turns up its presence. At the near horizon region, the spacetime is well approximated as $AdS_5 \times S^5$, which is intuitively composed from the radius and the brane interior part, and compact co-dimensions. It should be remarked that the black p-branes possess $AdS_{p+2} \times S^{8-p}$ horizon geometry only when $p = 3$. In other case, the dilaton field must acquire a non-trivial configuration, suffering from a divergence at the near horizon region. On the other hand, M-theory is irrelevant to any scalar-field as the dilaton field. Then it is possible that the $AdS_4 \times M^7$ or $AdS_7 \times M^4$ is realized by membranes.

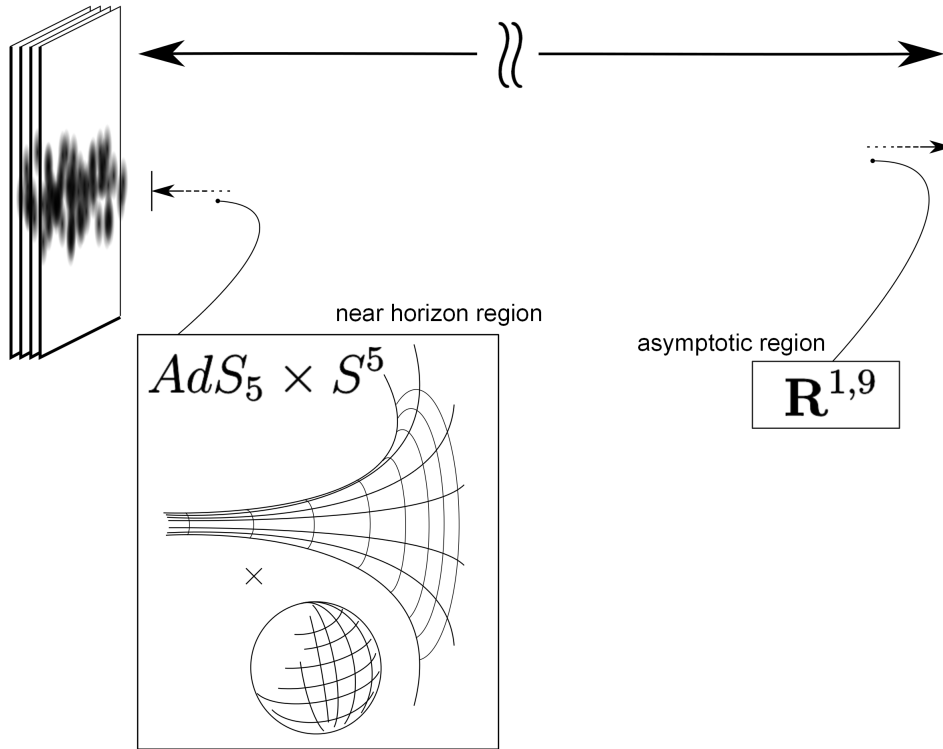


Figure 2.4 the black 3-brane spacetime

On such correspondence, the property of the compact extra-directions affects to the gauge theory side significantly. In case of the figure 2.4, the round S^5 space has the spherical isometry $SO(6)$, which is equivalent to $SU(4)$. This just coincides to the R-symmetry of $\mathcal{N} = 4$ SUSY. Moreover, at the gravity side, preserving all supersymmetry in the field theory reflects the presence of the maximum of Killing spinors in such a maximal symmetric space. Then it seems natural that the gravity dual of the low-SUSY gauge theory should be provided by suitably distorted compact spaces. Here is the successful place of Sasakian geometry in superstring theory: As explained the previous section, Sasakian structure

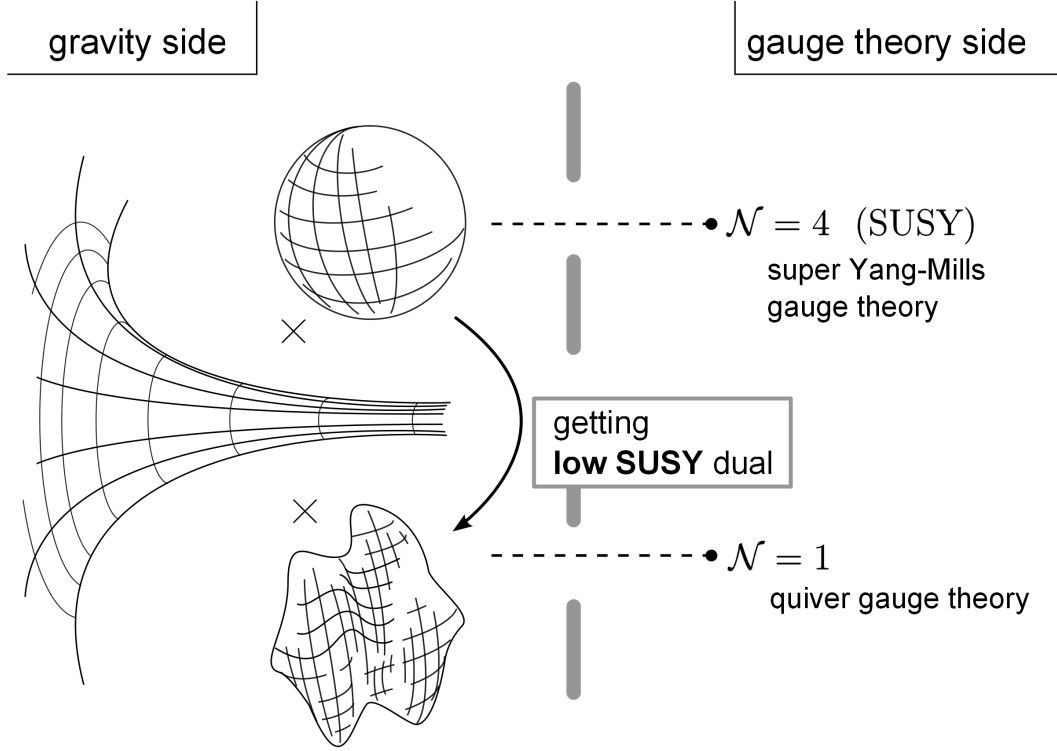


Figure 2.5 the place of Sasaki manifold in AdS/CFT

assures to exist a Killing spinor iff the EOM requests the Einstein condition. This situation is arranged at the figure 2.5. Hence, it was clarified that the new Sasaki–Einstein spaces found recently are the gravity duals to a class of $\mathcal{N} = 1$ gauge theories, i.e., quiver gauge theories [52].

Among them, the one called $Y^{p,q}$ is the first example discovered at 2004 [13]. This is the 5-dimensional one, which has $U(1) \times U(1) \times SU(2)$ isometry, and then inhomogeneous. The integer parameters p and q label the consistent parameter choice in the local metric for global condition, restricted then to be coprime. As is presented in the figure 2.2, this manifold possesses the non-trivial Kähler sandwich structure, that is, included in the irregular or quasi-regular class. Owing to the technical development, its alternatives of the lower isometry or higher dimensions also have been investigated [14, 15, 16, 17]. It is noteworthy that the 5-dimensional local metric of the gravity dual for $\mathcal{N} = 2$ has not been found: while the isometry of $Y^{p,q}$ contains $SU(2)$, it was insisted that it does not mean the R-symmetry but the flavor symmetry in the dual gauge theory.

Sasaki–Einstein space has been incorporated in AdS/CFT, not only 5 dimensions but also 7 dimensions. Recently, it was proposed by [53] that $\mathcal{N} = 6$ 3-dimensional Chern–Simons–matter theory is related to M2-brane physics on a background $AdS_4 \times S^7/\mathbb{Z}_k$ of M-theory. This motivates us to extend S^7 to general 7-dimensional Sasaki–Einstein

manifolds M_7 and study the backgrounds $\text{AdS}_4 \times M_7$ corresponding to $\mathcal{N} = 2$ Chern–Simons theories. Thanks to such proposals, we have now a number of concrete examples of Sasaki–Einstein manifolds. Until recent years, the only explicit examples of Sasaki–Einstein manifolds were $M^{3,2}$, $Q^{1,1,1}$ and $V^{5,2}$ in 7 dimensions. Gauntlett, Martelli, Sparks and Waldram constructed the higher-dimensional analogue of $Y^{p,q}$, the infinite families of inhomogeneous Sasaki–Einstein manifolds [14]. Further generalizations were achieved toward lower isometry [15, 16, 17].

In general, other than Sasaki–Einstein spaces, there can exist supersymmetric solutions which provide dual field theories still having $\mathcal{N} = 1$ supersymmetry. In this case, Sasakian condition does not necessarily coincident with the SUSY condition because the nontrivial matter flux deforms the Killing spinor equation itself. Here, this situation may be interpreted as the deformation of Sasakian structure: These backgrounds have some nontrivial fluxes which contribute to the energy-momentum tensor, so that the effect of the fluxes should deform Sasakian structure. Therefore, for finding out the gravitational solutions which can be regarded as generalizations of such Sasaki–Einstein manifolds, it might be useful to consider how we can deform Sasakian structure. In fact, Pilch and Warner [54] constructed a nontrivial supersymmetric background $\text{AdS}_5 \times M_5$, where M_5 is deformed from S^5 because a non-trivial 3-form field is present. Including this solution, one interesting generalized approach is known as Hitchin’s generalized geometry [55]. Its exploitation enables us to study the general feature of the $\text{AdS}_5/\text{CFT}_4$ correspondence [56, 57], introducing a notion of “generalized Sasaki–Einstein geometry”: It provides general supersymmetric AdS_5 solutions of type IIB supergravity theory with nontrivial fluxes by an abstracted description. Up to present, however, few explicit examples have been obtained with all its influence.

Chap. 3

Reduced limit from spinning black holes

In this chapter we elucidate the novel relationship of Sasakian geometry with a family of metrics including rotating black holes, called Kerr-NUT-AdS metrics. Thorough this review part, we aim to extract a plain formula and its understanding based on such plainness, decoupling the disuse difficulty or intricacy in prior researches. In Sec. 3.1, we provide the explicit procedure of the reduced limit in Einstein gravity. A profound of Kerr-NUT-AdS solution is also mentioned. In Sec. 3.2, a generalization of this relation is presented. Its utility for the other physical solutions has been still unclear. In Sec. 3.3, the difference from the even-dimensional limit is discussed.

3.1 Scaling limit in Einstein gravity

3.1.1 Kerr-NUT-AdS solutions

Sasakian geometry is related to a class of gravitational solutions containing the ordinary rotating black hole, which is called Kerr-NUT-AdS solutions. Preparing the explanation of this novel relationship, let us trace the extending procedure from the ordinary rotating solution to the Kerr-NUT-AdS solution. The higher-dimensional generalization of the Kerr black hole is known as the Myers–Perry solution [1], which possesses the flat boundary. Further extension to couple with the cosmological constant was obtained over 2004, which is often called the Kerr-(A)dS black hole solution [2].

In order to write down the Kerr-(A)dS solution in a general form with arbitrary dimensions and arbitrary angular momenta, it is necessary to introduce an abstracted expression.

In the D -dimensional spacetime with $D = 2n + \delta$, $n \in \mathbb{Z}^+$ and $\delta = 0, 1$, it is given by

$$\begin{aligned}
 ds^2 = & -W(1 - \lambda r^2)dt^2 + \frac{Udr^2}{V - 2M} + \frac{2M}{U} \left(dt + \sum_{i=1}^{n+\delta-1} \frac{a_i \rho_i^2 d\varphi_i}{1 + \lambda a_i^2} \right)^2 \\
 & + \sum_{i=1}^n \frac{r^2 + a_i^2}{1 + \lambda a_i^2} d\rho_i^2 + \sum_{i=1}^{n+\delta-1} \frac{r^2 + a_i^2}{1 + \lambda a_i^2} \rho_i^2 (d\varphi_i - \lambda a_i dt)^2 \\
 & + \frac{\lambda}{W(1 - \lambda r^2)} \left(\sum_{i=1}^n \frac{(r^2 + a_i)^2 \rho_i d\rho_i}{1 + \lambda a_i^2} \right)^2,
 \end{aligned} \tag{3.1}$$

where

$$\begin{aligned}
 W &= \sum_{i=1}^n \frac{\rho_i^2}{1 + \lambda a_i^2}, \quad V = \frac{1 - \lambda r^2}{r^{\delta+1}} \prod_{i=1}^{n+\delta-1} (r^2 + a_i^2), \\
 U &= r^\delta \sum_{i=1}^n \frac{\rho_i^2}{r^2 + a_i^2} \prod_{j=1}^{n+\delta-1} (r^2 + a_j^2),
 \end{aligned} \tag{3.2}$$

and M and a_i are constant parameters which indicate the mass and angular momenta of the spacetime. The index i runs 1 to $n + \delta - 1$, meaning the maximum degree of commutative rotational directions. In this expression it is clear that the radial or temporal direction reads respectively r or t , and similarly azimuthal directions $\{\varphi_i\}$. The residual part needs a careful treatment, called the latitudinal directions. Eq.(3.1) adopts this homogeneous coordinate $\{\rho_i \mid i = 1, \dots, n\}$, which is subject to the constraint,

$$\sum_{i=1}^n \rho_i^2 = 1. \tag{3.3}$$

With this coordinate, the unit $S^{2n+\delta-2}$ metric is obviously written as

$$d\Omega_{2n+\delta-2}^2 = \sum_{i=1}^n d\rho_i^2 + \sum_{i=1}^{n+\delta-1} \rho_i^2 d\varphi_i^2. \tag{3.4}$$

This metric is relevant to the spherical topology of the horizon in the present metric (3.1), because of embedding the $(2n + \delta - 2)$ -dimensional space into the section fixed r with some scaling deformation. While Eq. (3.1) describes the general black hole with the arbitrary values of angular momenta, n variables of the abstracted coordinate $\{\rho_i\}$ is sometimes unsuitable for the physical analysis. In such case, transferred to $(n - 1)$ unconstrained variables, it requires a prescription classified by the cohomogeneity or angular momentum values. The cohomogeneous degree means the number of the nontransitive direction with any isometry. Intuitively, this degree shows a distortion of the spacetime, e.g., the homogeneous space is the cohomogeneity-0, and the Schwarzschild–Tangherlini black hole the cohomogeneity-1.

When all of $\{a_i\}$ take different values in each other, the metric (3.1) raises the cohomogeneity to the maximum n . The coordinate system clarifying this property is realized by the Jacobi transformation,

$$\rho_i^2 = \frac{\prod_{\alpha=1}^{n-1} (a_i^2 - y_\alpha^2)}{\prod_{k \neq i}^n (a_i^2 - a_k^2)}. \quad (3.5)$$

Using the unconstrained latitudinal variables $\{y_\alpha \mid \alpha = 1, \dots, n-1\}$, the sphere metric (3.4) is

$$d\Omega_{2n+\delta-2}^2 = - \sum_{i=1}^n \frac{y_\alpha^2 \prod_{\beta \neq \alpha}^{n-1} (y_\alpha^2 - y_\beta^2)}{\prod_{k=1}^n (y_\alpha^2 - a_k^2)} dy_\alpha^2 + \sum_{i=1}^{n+\delta-1} \rho_i^2 d\varphi_i^2, \quad (3.6)$$

thus it acquires a diagonalized form. This diagonalization conceals the spherical symmetry $SO(2n+\delta-1)$ except for the Cartan subgroup, then embedding with the radial distortion results in the contraction of the isometry to $U(1)^{n+\delta-1} \times \mathbf{R}$.

The intuition that these variables are latitudinal is attributed to their bounded ranges; The global consistency of this spacetime requires $\{y_\alpha\}$ to fix the finite ranges, which depend on the values of rotational parameters. Without loss of generality, $\{a_i\}$ can arrange in order,

$$0 \leq a_n^2 < a_1^2 < a_2^2 \cdots < a_i^2 < \cdots < a_{n-2}^2 < a_{n-1}^2. \quad (3.7)$$

In every even dimensions a_n must vanish. Taking that order into account, a consistent choice of the latitudinal range reads

$$a_n^2 < y_1^2 < a_1^2 < y_2^2 < a_2^2 \cdots < y_i^2 < a_i^2 < \cdots < y_{n-1}^2 < a_{n-1}^2. \quad (3.8)$$

This is because the restriction above certainly satisfies the requests simultaneously, e.g. the Euclidean signature of Eq.(3.6) and the positivity of Eq.(3.5).

The latitudinal variables also diagonalize the black hole metric (3.1) as well as the spherical one. Owing to this simplification, the Einstein solution was achieved to one generalization [58], what is called, the NUT deformation. Seeing the radial and latitudinal part of Eq.(3.1),

$$\begin{aligned} ds^2 &= \frac{U dr^2}{V - 2M} + \sum_{i=1}^n \frac{r^2 + a_i^2}{1 + \lambda a_i^2} d\rho_i^2 + \frac{\lambda}{W(1 - \lambda r^2)} \left(\sum_{i=1}^n \frac{(r^2 + a_i)^2 \rho_i d\rho_i}{1 + \lambda a_i^2} \right)^2 + \cdots \quad (3.9) \\ &= \frac{r^{2\delta} \prod_{\beta=1}^{n-1} (r^2 + y_\beta^2) dr^2}{(1 - \lambda r^2) \prod_{k=1}^{n+\delta-1} (r^2 + a_k^2) - 2Mr^{1+\delta}} - \sum_{\alpha=1}^{n-1} \frac{y_\alpha^{2\delta} (y_\alpha^2 + r^2) \prod_{\beta \neq \alpha}^{n-1} (y_\alpha^2 - y_\beta^2) dy_\alpha^2}{(1 + \lambda y_\alpha^2) \prod_{k=1}^{n+\delta-1} (y_\alpha^2 - a_k^2)} \\ &\quad + (\dots \text{timelike or axial part}), \quad (3.10) \end{aligned}$$

a clue appears for that generalization. It can be seen that the radial and latitudinal component gets the similar form. Then, this indicates that there should be the deformation

for the latitudinal part, likewise the mass term M for the radial part: that is, changing the denominator in the summation to

$$\frac{1}{(1 + \lambda y_\alpha^2) \prod_{k=1}^{n+\delta-1} (y_\alpha^2 - a_k^2)} \longrightarrow \frac{1}{(1 + \lambda y_\alpha^2) \prod_{k=1}^{n+\delta-1} (y_\alpha^2 - a_k^2) - 2L_\alpha y_\alpha^{1+\delta}}, \quad (3.11)$$

with L_α $n-1$ constant parameters, so-called NUT charges. It satisfies the Einstein equation after taking the back reaction for the axial and timelike parts all into account. The way to close the back reaction is also similar to the mass-term case. There is a defect for the NUT charge otherwise the mass M : every NUT charge L_α causes the breaking of the asymptotic structure. This is attributed to the range of Eq.(3.8). The finite boundary does not allow infinitesimally fading out the charge effect like the mass.

A specific Euclideanization unveils the symmetric structure between the radius and latitudes more clearly and systematically. It is called the radial Wick rotation,

$$r \rightarrow i\tilde{x}_n, \quad (3.12)$$

$$\tilde{x}_\alpha \equiv y_\alpha, \quad \tilde{x}_v \equiv (\tilde{x}_\alpha, \tilde{x}_n). \quad (3.13)$$

Note that the Wick rotated coordinate is not the time direction at all. This transformation shows a specialty of such rotational spacetime, supported by the suitable range choice for $\tilde{x}_n$. Adopting it together with conventional redefinitions, the Euclid-Kerr-NUT-AdS solution with the cohomogeneity- n acquires a remarkably simpler form;

$$ds^2 = \sum_{v=1}^n \left\{ \frac{\tilde{U}_v}{\tilde{X}_v} d\tilde{x}_v^2 + \frac{\tilde{X}_v}{\tilde{U}_v} \left(\sum_{k=0}^{n-1} \tilde{\sigma}_k^{\hat{v}} d\tilde{\psi}_k \right)^2 \right\} + \delta \frac{\lambda \prod_{k=1}^{n+1} \tilde{\alpha}_k^2}{\tilde{\sigma}_n} \left(\sum_{k=0}^n \tilde{\sigma}_k d\tilde{\psi}_k \right)^2, \quad (3.14)$$

where

$$\tilde{X}_v = - \frac{\lambda}{\tilde{x}_v^{2\delta}} \prod_{k=1}^{n+\delta} (\tilde{x}_v^2 - \tilde{\alpha}_k^2) + \tilde{b}_v x_v^{1-\delta}, \quad (3.15)$$

$$\tilde{U}_v = \prod_{u \neq v}^n (\tilde{x}_v^2 - \tilde{x}_u^2), \quad (3.16)$$

$\tilde{\sigma}_k$ is the k -th elementary symmetric polynomial of $\{\tilde{x}_v^2\}$, and $\tilde{\sigma}_k^{\hat{v}}$ the k -th elementary symmetric polynomial of $\{\tilde{x}_u^2 | u \neq v\}$. $\hat{v}$ means the exception of the index- v variable in that polynomial. These explicit expressions are

$$\tilde{\sigma}_k = \sum_{0 \leq u_l \leq n} \tilde{x}_{u_1}^2 \cdots \tilde{x}_{u_k}^2, \quad \tilde{\sigma}_k^{\hat{v}} = \sum_{u_1 < \cdots < u_k}^{u_l \neq v} \tilde{x}_{u_1}^2 \cdots \tilde{x}_{u_k}^2. \quad (3.17)$$

$\{b_v\}$ denote free constants, which are the mass and NUT charges. $\{\tilde{\alpha}_k\}$ are also free constants, sorting out the relation between the angular momentum and the cosmological constant: in order to connect the Kerr-AdS solution it is necessary to impose a constraint

such as $\tilde{\alpha}_{n+\delta} = \frac{1}{\lambda}$ while the above metric function exactly satisfies the Einstein condition without any constraint.

It was discussed that the total count of nontrivial parameters in this solution is $2n - 1$, irrelevant to $\delta = 0$ or 1 [58]: Since the following scaling transformation leaves the metric invariant,

$$\tilde{x}_v \rightarrow C\tilde{x}_v, \quad \tilde{\psi}_k \rightarrow C^{-2k-1}\tilde{\psi}_k, \quad (3.18)$$

$$\tilde{\alpha}_k \rightarrow C\tilde{\alpha}_k, \quad \tilde{b}_v \rightarrow C^{2n-\delta}\tilde{b}_v, \quad (3.19)$$

then this absorbs one degree of non-trivial parameters. In even dimensions, all of $\{b_v\}$ are independent, resulting in $n - 1$ rotation, mass, and $n - 1$ NUT parameters. By contrast, the latitudinal power of b_v term provides one redundancy in odd dimension. This just cancels the one more rotational degree. In the literature, an equivalent expression is often adopted for the above metric function, which is

$$\tilde{X}_v = \sum_{i=\delta}^n \tilde{c}_i \tilde{x}_v^{2i} + b_v \tilde{x}_v^\delta + (-1)^n \frac{\tilde{c}_\delta}{\tilde{x}_v^2}, \quad (3.20)$$

with $\tilde{c}_n = -\lambda$. There faces a benefit of this expression at the local analysis as well as counting free degrees. A calculation shows that the constant parameters $\tilde{c}$ and $\tilde{c}_i$ ($i = \delta, \dots, n - 1$) never affects the local expression of the Riemann tensor, using a suitable orthonormal frame and the Vandermonde identity as in the App. A. The false choice of them rather affects the global condition, raising global pathology e.g. the bankruptcy of the metric signature.

If some angular momenta coincide, it requires to rerun a parallel procedure. In the case when some rotational parameters have same value in the metric (3.1), the coordinate transformation (3.5) cannot be done since it becomes singular. Even in such case, it has been known that the other transformations are available for each situations, which respects the enhancement of isometry and clarifies the possibility of NUT charging [59]. Consequently, Kerr-NUT-AdS solutions are classified by the cohomogeneity, or almost equivalently, the coincidence of angular momenta. The property of the classification is presented at the table 3.1. Eq. (3.14) places at the left-most column. The isometry is constituted from the axial symmetry and Euclideanized time-translation. In this class, there is the amplest scope for NUT charging, due to the largest cohomogeneity of this classification. It is also investigated that this admits n rank-2 Killing-Stäckel tensors (called Killing tensor for shortly), which provides n conserved quantities. Owing to the $(2n + \delta)$ conserved quantities totally, it has been confirmed that several EOMs brings to a separated form of the variables in arbitrary dimensions [21, 22]. As each coincidence occurs at the value of angular momenta, the cohomogeneity decreases one by one, raising the isometry enhancement. k -fold degeneration engages the enhancement to $U(k)$. While this

Euclid-Kerr-NUT-AdS					
cohomogeneity	n	$n - 1$	$\dots$	1	0
isometry	$U(1)^{n+\delta}$	$U(1)^{n+\delta-2} \times U(2)$	$\dots$	$U(1)^\delta \times U(n)$	$SO(2n + \delta + 1)$
# NUT charge	$n - 1$	$n - 2$	$\dots$	1 or 0	0
# Killing tensor	n	$n - 1$	$\dots$	1	n
homogeneous case	—				$S^{2n+\delta}$

Table 3.1 the coincident variation with rotations of Euclid-Kerr-NUT-AdS solutions

degeneration reduces the scope of the NUT deformation, it was explored that the other sort of the deformation becomes available, which also breaks the asymptotic structure and the part of enhanced isometry [60]. Hence, such parameter coincidence yields the rich variety in the gravitational-solution space. This classification can be understood more clearly from the viewpoint of tensorial symmetry, called hidden symmetry. They are classified by the presence of a closed rank-2 conformal Killing-Yano tensor and its eigenvalues.

Note that this classification contains a few non black hole solutions because of excessive tuning parameters. This should be seen to classify the local solution, rather than to focus on the physical request. For instance, the cohomogeneity-1 class with even dimensions contains not the rotating black hole but the pure NUT solution. $(n - 1)$ angular momenta cannot reach to the full degeneracy. On the other hand, in the odd-dimensional cohomogeneity-1 class the rotational black hole exists properly, owing to the one more rotation. In addition, the Kerr-NUT-AdS solutions have unexpectedly deep connections to the maximal symmetric space. If the metric (3.14) sets to all the b_v vanishing, it becomes the maximal symmetric space, at least locally. This must occur at all the class in the table 3.1. Furthermore, in this case there is no restriction between Wick rotating and inverting the cosmological constant, otherwise toric Sasaki or Kähler case as will explain below. Not only $S^{2n+\delta}$ but $AdS_{2n+\delta}$, $H^{2n+\delta}$, $dS_{2n+\delta}$, and $R^{2n+\delta}$ have the local expressions similar to Eq. (3.14).

3.1.2 Limit manipulation to Sasaki–Einstein

Let us show a limit procedure which brings the Kerr-NUT-AdS solution to a Sasaki–Einstein space. Then we set $\delta = 1$ here while the even dimensional analogue exists. Using Eq.(3.14), we can obtain the Sasaki–Einstein metric with the cohomogeneity- n . With an infinitesimal parameter ϵ , take the following transformation: The first half is the coordinate

transformation into

$$\begin{cases} \tilde{x}_v^2 \rightarrow C_L + \epsilon x_v, \\ \frac{1}{2} \sum_{l=0}^{n-1-k} \binom{n-1-k}{l} \epsilon^{k+1} C_L^{l-\frac{1}{2}} \tilde{\psi}_{k+l} \rightarrow \psi_{k+1}, & 0 \leq k \leq n-1, \\ -\frac{1}{2} \sum_{l=0}^n \binom{n}{l} C_L^{l+\frac{1}{2}} \tilde{\psi}_l \rightarrow \psi_0, \end{cases} \quad (3.21)$$

and the other is the reparametrization as

$$\begin{cases} \tilde{\alpha}_i^2 \rightarrow C_L + \epsilon \alpha_i, \\ \tilde{b}_v \rightarrow \frac{1}{4C_L} \epsilon^{n+1} b_v, \end{cases} \quad (3.22)$$

where C_L is a free parameter. Even if taking $\epsilon \rightarrow 0$, the EOM condition (2.4) is maintained. As a result, the metric becomes

$$ds^2 = \sum_{v=1}^n \left\{ \frac{U_v}{X_v} dx_v^2 + \frac{X_v}{U_v} \left(\sum_{k=1}^n \sigma_{k-1}^{\hat{v}} d\psi_k \right)^2 \right\} + 4\delta\lambda \left(d\psi_0 + \sum_{k=1}^n \sigma_k d\psi_k \right)^2, \quad (3.23)$$

with

$$X_v = -4\lambda \prod_{k=1}^{n+1} (x_v - \alpha_k) + b_v \quad (3.24)$$

$$= \sum_{i=1}^{n+1} c_i x_v^i + b_v, \quad (3.25)$$

$$U_v = \prod_{u \neq v}^n (x_v - x_u). \quad (3.26)$$

σ_k or $\sigma_k^{\hat{v}}$ is respectively the k -th elementary symmetric polynomial of $\{x_v\}$ or the excepted one for the v -direction, similar to Eq. (3.17). This metric cone assures that it is Sasakian space. So far as the X_v is any function of one variable x_v , Sasakian condition is preserved. The detail of metric functions (3.24) is attributed to the Einstein condition. The present metric belongs to the toric Sasaki–Einstein: the toric condition is a delicate object in mathematics (see e.g. [61, 62]), intuitively requiring the existence of $n+1$ Killing vectors that are independent and mutually commutative. Its Reeb vector is $\zeta = \partial_{\psi_n}$, and then the contact 1-form is

$$\eta = 2\sqrt{\lambda} \left(d\psi_0 + \sum_{k=1}^n \sigma_k d\psi_k \right). \quad (3.27)$$

By the foliation of this, the present coordinate provides the $2n$ -dimensional transverse structure on the Kähler sandwich, as explained in the previous chapter. Whereas it has a

close resemblance to Eq.(3.14), this solution admits the free parameters certainly one less than before the limit. C_L tells the reason that a gauge transformation has become possible to absorb one parameter freedom by the simultaneous shift $x_v \rightarrow x_v + C$. This indicates that it does not lose the information of $\{b_v\}$ because of each the index- v dependence.

Let us read the lines between the above manipulation. On this limit the NUT-like deformations are namely preserved, manifest at the map of the latitudinal directions;

$$\frac{\tilde{U}_v d\tilde{x}_v^2}{\tilde{X}_v} \longrightarrow \frac{U_v dx_v^2}{X_v}. \quad (3.28)$$

This map also respects the global condition, especially the range of latitudinal variables. The figure 3.1 shows the case before the limit, that is, a consistent range of the Euclid-Kerr-NUT-AdS solution (3.14) with assuming $n = 3$. Every metric function $\tilde{X}_v$ is plotted as n

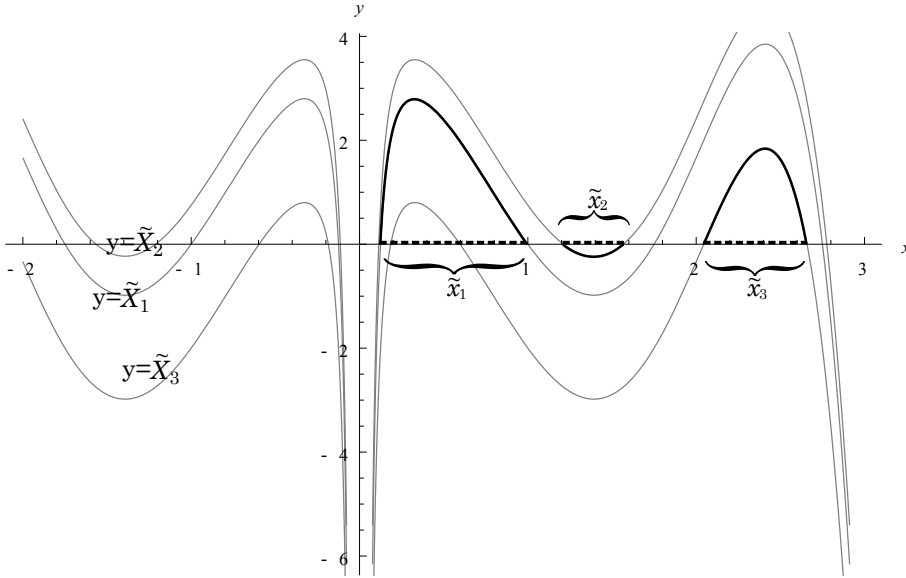


Figure 3.1 the range of Kerr-NUT-AdS solution with $n = 3$

gray lines respectively. All of them have $n+1$ nodes due to the cohomogeneous degree, and each of their difference arises from the NUT deformation b_v . Once the latitudinal directions $\{\tilde{x}_v\}$ are aligned somewhere, the singularity at $U_v = 0$ prohibits these alternation. Thus the range of $\tilde{x}_v$ must be separated each other. Simultaneously, that order fixes the sign of U_v at each the diagonal component, which must be canceled by $\tilde{X}_v$ from the requirement of the positive definiteness. Therefore, it restricts the efficient region of the metric functions as drawn by the black line, resulting in the occupation of each the $\tilde{x}_v$ range as the dashed line. In addition, the parameter tuning sometimes causes the enhancement of symmetry; the case when all the black lines are connected just means the round sphere.

After the limit (3.21), (3.22), the Sasaki–Einstein space inherits almost all the above properties. As shown in the figure 3.2, n occupations are preserved for new latitudinal

variables in accordance with the cohomogeneous degree; These are assured mainly by the transformation of $\{x_v, \alpha_k\}$. The transformation of Killing directions in Eq.(3.21) is nothing but the linear recombination with constant factors, keeping such a canonical expression. The benefit over the limit is surviving the $\{b_v\}$ with the EOM constraint up to scaling (3.21). If all of them cease at $\epsilon \rightarrow 0$, this operation has no meaning.

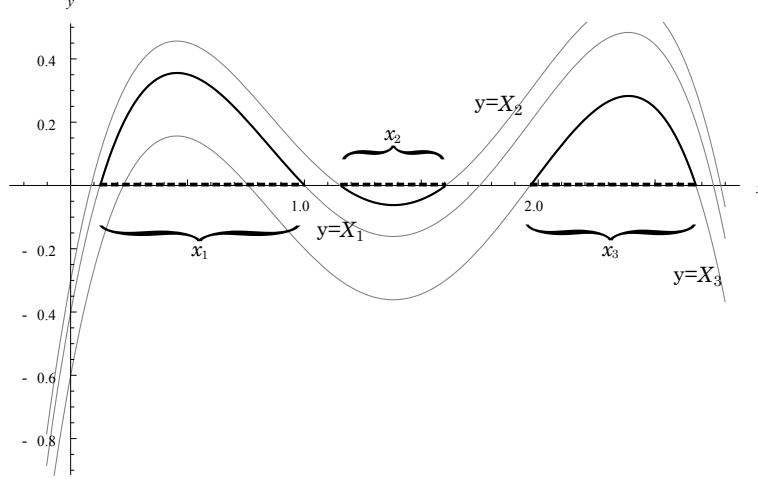


Figure 3.2 the range of toric Sasaki–Einstein with $n = 3$

On the other hand, a few lost properties can read from above graphs. One thing is mirror branches, presented at the Kerr-NUT-AdS case in the figure 3.1. This disappearance could be seen as the degeneration of horizons, reflecting the squared dependence on latitude variables. It seems compatible to emerge the Killing spinor in the Sasaki–Einstein space. Furthermore, the lack of mirror branches releases the whole translation along the horizontal axis at the figure 3.2. This induces one constraint among the n angular momenta from the gauge degree of the latitudinal shift. Therefore, taking this limit at the asymptotic AdS case, it is expected that the spacetime after the manipulation saturates the BPS bound; using the appropriate energy E and angular momenta J_i [64], it gives

$$E \geq J_1 + J_2 + \cdots + J_n. \quad (3.29)$$

The analysis toward such direction has been done in a restricted circumstance concretely [63], needless to say vanishing NUT deformation. In that work the shift gauge degree is tuned by the upper bound of the rotational parameter from the AdS radius, seen at the metric (3.1) as $a_i \rightarrow (-\lambda)^{-1/2}$.

In the low cohomogeneous case, such limit also connects them properly. Indeed, there exist concrete expressions of toric Sasaki–Einstein manifolds, which revive the degeneration of angular momenta or enhancing the isometry as in the table 3.2. In this variety, there is one thing different from the Kerr-NUT side: at the Sasakian side, the analytic continuation

is restricted severely. Consequently, this limit-relation is reliable only when the positive curvature radius in Euclidean regime or the negative curvature radius in Lorentzian regime.

					toric Sasaki
cohomogeneity	n	$n - 1$	$\cdots$	1	0
isometry	$U(1)^{n+1}$	$U(1)^{n-1} \times U(2)$	$\cdots$	$U(1) \times U(n)$	$SO(2n + 2)$
# Killing tensor	n	$n - 1$	$\cdots$	1	n
homogeneous case	—				S^{2n+1}

Table 3.2 the symmetric variation of toric Sasakian metrics

3.2 Off-shell generalization from hidden symmetry

When $\delta = 0$, the Kerr-NUT-AdS metric (3.14) can link with the Kähler base space of Eq. (3.23) by a slight modification of the limit formula. Comparing these local metrics, the relationship seems to connect the larger class of metrics over Einstein gravity. A natural extension is taking $\tilde{X}_v$ and X_v to be arbitrary functions of one variable respectively:

$$\tilde{X}_v \rightarrow {}^\forall \tilde{X}_v(\tilde{x}_v), \quad X_v \rightarrow {}^\forall X_v(x_v). \quad (3.30)$$

This extension breaks the Einstein condition, so that they are called off-shell metrics. While this off-shell generalization cannot provide the vision of the global behavior, it was confirmed that both side metrics maintain the rich local properties of tensorial symmetry.

In the both side, the common tensorial symmetry is the presence of the Killing tensor, which satisfies

$$\nabla_{(a} K_{bc)} = 0, \quad K_{(ab)} = K_{ab}. \quad (3.31)$$

The trivial example of this tensor is the metric itself. As we shall show, there exist non-trivial ones as many as the cohomogeneity in both off-shell metrics. Consequently, both of them have clarified the separability of several EOMs, such as, Hamilton–Jacobi and Klein–Gordon equations. Up to present, in neither sides the higher-form EOMs have obtained to separate yet, e.g. the Maxwell equation. By contrast, the Dirac equation has achieved the separation only in the Kerr-NUT-AdS side. This difference can be considered as the result from the origin of the Killing tensor; Although in the Kerr-NUT-AdS side the recent investigation has revealed a systematic construction of Killing tensors, the counterpart of the Sasakian side has left difficulty.

In the Kerr-NUT-AdS side, the systematic construction is called the tower of hidden symmetry. This is described by a specific sort of the anti-symmetric tensor fields, which is conformal Killing-Yano tensors. At the case of rank- p , a conformal Killing-Yano tensor k obeys

$$\nabla_X k = \frac{1}{p+1} X \lrcorner dk - \frac{1}{D-p+1} X^\flat \wedge *d*k. \quad (3.32)$$

If the second term vanishes everywhere, k becomes the Killing-Yano tensor, which becomes the Killing vector at the rank-1 case. Similarly, vanishing the second term, k is called the closed conformal Killing-Yano (CCKY) tensor. These tensors are transferred by the Hodge dual in each other. At the Kerr-NUT-AdS side, it was clarified that there must be a rank-2 CCKY tensor h , which provides a systematic understanding to the tensorial symmetry. In the case of the cohomogeneity- n with $(2n + \delta)$ dimensions, h can be written in the form

$$h = \sum_{v=1}^n \tilde{x}_v \tilde{e}^v \wedge \tilde{e}^{\hat{v}}, \quad (3.33)$$

with the orthonormal frame

$$\tilde{e}^v = \sqrt{\frac{\tilde{U}_v}{\tilde{X}_v}} dx_v, \quad \tilde{e}^{\hat{v}} = \sqrt{\frac{\tilde{X}_v}{\tilde{U}_v}} \sum_{k=0}^{n-1} \tilde{\sigma}_k^{\hat{v}} d\tilde{\psi}_k, \quad \tilde{e}^\delta = \sqrt{\frac{\lambda \prod_{k=1}^{n+1} \tilde{\alpha}_k^2}{\tilde{\sigma}_n}} \sum_{k=0}^n \tilde{\sigma}_k d\tilde{\psi}_k. \quad (3.34)$$

This gives n independent Killing tensors. Taking the Wedge product,

$$h^{(\ell)} \equiv h \underbrace{\wedge h \wedge \cdots \wedge h}_{\ell-1 \text{ times}}, \quad (3.35)$$

all the $h^{(\ell)}$ are also CCKY tensors with the rank- ℓ . ℓ is bounded by the spacetime dimension. Each the Hodge dual gives a Killing-Yano tensor $f^{(j)} = *h^{(j)}$. These tensors $f^{(j)}$ generate rank-2 Killing tensors $K^{(j)}$ by

$$K_{ab}^{(j)} = f^{(j)}_{ac_1 \cdots c_{D-2\ell-2}} f^{(j)}_b{}^{c_1 \cdots c_{D-2\ell-2}}. \quad (3.36)$$

In this case they are written explicitly as

$$K^{(j)} = \sum_{v=1}^n [\ell!^2 (n-2j-1)! \tilde{\sigma}_j^{\hat{v}} (\tilde{e}^v \otimes \tilde{e}^v + \tilde{e}^{\hat{v}} \otimes \tilde{e}^{\hat{v}})]. \quad (3.37)$$

They are all independent. This argument is almost available to apply for the low-cohomogenous case. Note that in such case the nontrivial Killing tensors cannot be obtained more than the cohomogeneous degree. Although the bi-linear of every Killing-Yano tensor must be a Killing tensor as in Eq.(3.36), it often occurs that other Killing-Yano tensors provide the same Killing tensor. Avoiding such redundancy, it is useful to focus on the eigenvalue of the CCKY 2-form h : As shown in Eq.(3.33), h admits a quasi-diagonal form, whose

eigenvalues is $\tilde{x}_v$ in this frame. When the cohomogeneity is highest, the eigenvalues are all the function of the latitudinal variables. As the cohomogeneity gets lower, the constant eigenvalues appear, providing the trivial Killing tensors. In consequence, the functional eigenvalues of h coincides with the cohomogeneous degree except for the homogeneous case, restricting the number of independent conserved quantities.

On the other side, Sasakian geometry has a relation to the Killing-Yano symmetry. It was clarified in [7, 8] that a Sasakian manifold of $(2n + 1)$ dimensions has rank- $(2p + 1)$ special Killing forms, a sort of Killing-Yano tensor. This is in the form

$$\underbrace{\eta \wedge d\eta \wedge \cdots \wedge d\eta}_{p-1 \text{ times}}, \quad 0 \leq p \leq n. \quad (3.38)$$

At first glance, this tensorial symmetry is suitable to the limit-relationship because of the tower construction with higher-rank tensors. Actually, there is a suggestion that the generalization of the limit should be regarded as their correspondence [20]. It is based on the limit formula modified as

$$\tilde{x}_v \rightarrow 1 + \frac{\epsilon}{2}x_v, \quad \text{then } h \rightarrow d\eta, \quad (3.39)$$

with suitable redefinition to Eqs. (3.21), (3.22). This limit works almost well, connecting the off-shell metric functions. However, the series of Eq. (3.38) never provide the non-trivial Killing tensors. The origin of the tensorial symmetry disappears in this interpretation. It seems worthful to find out the systematic construction of the Killing tensor in the toric Sasakian geometry.

3.3 Odd-dimensional priority over the relationship

As for the above generalizing scenario with off-shell metrics, even-dimensional correspondence seems to have nothing worse than the odd-dimensional case, except for the signature. In that case, the map of Kerr-NUT-AdS metrics corresponds to the base part of Eq. (3.23),

	toric Kähler				
cohomogeneity	n	$n - 1$	$\cdots$	1	0
isometry	$U(1)^n$	$U(1)^{n-2} \times U(2)$	$\cdots$	$U(n)$	$U(n + 1)$
# Killing tensor	n	$n - 1$	$\cdots$	1	n
homogeneous case	—				CP_n

Table 3.3 the symmetric variation of toric Kähler metrics

which possesses a Kähler structure locally. It has the variety by the cohomogeneity as in

the table 3.3. These Kähler metrics were well investigated in the field of mathematics, which established the symmetrical classification by the Hamiltonian 2-form [65, 66]. Nevertheless, focusing on the concrete solution reveals several troubles, especially the global behavior. The base part of Eq. (3.23) certainly satisfies Einstein condition, but its global behavior is known to be so wrong that it must be the transverse structure. Such metrics also have difficulty to rotate the signature: Its Lorentzian counter part has been still unknown with the interference of Kähler structure. Furthermore, in this even dimensions, the Killing spinor is absent, otherwise the Ricci flat case. More defects come out by using the limit formula (3.39). This accompanies with some redefinitions of the parameter mapping, which results in the replacement of the cosmological constant. It seems relevant to the disagreement of the homogeneous case: the Kerr-NUT-AdS side is the variety around S^{2n} , while the Kähler side around CP_n . Such difficulty is not present at the odd-dimensional case. Hence, we assess to pioneer odd dimensions preferentially, so that it will also assist understanding of even dimensions.

One more suspicious situation was suggested in the even-dimensional case by a previous study [20], caused in 4 dimensions. The 4-dimensional Kerr-NUT-AdS metrics contain a familiar physical sample, that is, the Kerr–Newman solution. In the off-shell setup with the matter-part as

$$S_{\text{Matter}} = \frac{1}{4} \int F \wedge *F, \quad (3.40)$$

where $F = dA$, the following 1-form becomes a gauge potential:

$$A = \sum_{v=1}^2 \left[\frac{\tilde{q}_v \tilde{x}_v}{U_v} \sum_{k=0}^1 \tilde{\sigma}_k^{\hat{v}} d\tilde{\psi}_k \right], \quad (3.41)$$

with $\tilde{q}_v$ constant parameters. Indeed, this potential satisfies the Maxwell equation by the arbitrary $\tilde{X}_v$. The gravitational EOM settles down to

$$\tilde{X}_v = \sum_{i=0}^2 \tilde{c}_i \tilde{x}_v^{2i} + b_v \tilde{x}_v + \frac{\tilde{q}_v^2}{4}. \quad (3.42)$$

This is the Euclidean expression of the Kerr–Newman solution in the present coordinate, adding a cosmological constant and a NUT charge. The limit (3.39) suggests that the matter flux should transfer as

$$A \longrightarrow \sum_{v=1}^2 \left[\frac{q_v}{U_v} \sum_{k=1}^n \sigma_{k-1}^{\hat{v}} d\psi_k \right]. \quad (3.43)$$

This flux certainly satisfies the Maxwell equation, but does not reply the gravitational back reaction. This is because the present flux is anti self-dual:

$$*F = -F, \quad \Rightarrow \quad T_{ab} = 0. \quad (3.44)$$

If accepting this scenario, the Kerr–Newman solution falls into the Kähler Einstein, equivalent to the loss of the Maxwell charge on the limit. This might give one explanation of the specialty about the Kerr–Newman solution from the conformal feature in 4 dimensions. However, it cannot explain the presence of a Maxwell-charged solution in the Kähler side, then we consider there should be a scope for improvement based on a correct mapping. For that reason it is important to study new samples satisfying some EOMs around the off-shell metrics. While both the off-shell metrics do not possess other samples, a few solutions were found out only close to the Kerr-NUT-AdS side. One sort of them is relevant to the solution-generating technique [3, 4, 5], yielded by the on-shell Kerr-NUT-AdS. Recent study has implied a candidate of the even-dimensional counterpart for the reduced limit, which may be reproduced from the Kähler–Einstein one by the solution-generating technique [67]. It possesses an interesting breaking of the Kähler condition. The even-dimensional difficulty unfortunately has disturbed its detailed analysis, such as the confirmation of limit mapping or the global regularity.

Chap. 4

Torsional solutions in Supergravities

In previous chapters, we have clarified the meaning to find out new exact samples with Sasakian structure, that is, the possibility as a clue for spinning black holes and of the global definiteness. Here let us show the investigation in SUGRAs. First, we present the new 5-dimensional gauged solution, which properly links to a known black hole with the reduced limit in Sec. 4.1. Next, in Sec. 4.2 we discuss a generalization of Sasakian structure suited to this solution, adopting the deformation by the presence of torsion. This argument ensures the rich tensorial symmetry in a concrete class of metrics. Then, using such class as metric ansatz, we investigate 5-dimensional and 11-dimensional SUGRAs in Sec. 4.3. In this section we also confirm that our solutions are certainly able to acquire the well global behavior.

4.1 Reduced limit in 5-dimensional SUGRA

Among a number of supergravity theories, the gauged minimal supergravity of 5 dimensions are one of the most active theory. Many of interesting researches have been done, e.g., success in embedding to the type IIB SUGRA [24, 25, 26]. This theory is described by the following Lagrangian:

$$\mathcal{L}_5 = *(\mathcal{R} - \Lambda) - \frac{1}{2}F \wedge *F + \frac{1}{3\sqrt{3}}F \wedge F \wedge A . \quad (4.1)$$

The word “gauged” is relevant to the presense of the cosmological constant, which balances with the gauge coupling constant. This condition decreases the parameters of the action, so that the theory possesses one free parameter. In the above action, this degree is shown as the cosmological constant by scaling out the $U(1)$ gauge coupling constant. This notation is useful for going to the un-gauged limit, which just coincides to $\Lambda = 0$ case.

On this theory, a rotational charged black hole solution was constructed at 2005 [27].

The solution can be written in the form,

$$ds^2 = \sum_{v=1}^2 \left\{ \frac{\tilde{U}_v}{\tilde{X}_v} d\tilde{x}_v^2 + \frac{\tilde{X}_v}{\tilde{U}_v} \left(\sum_{k=0}^1 \tilde{\sigma}_k^{\hat{v}} d\tilde{\psi}_k \right)^2 \right\} + \frac{\tilde{c}}{\tilde{\sigma}_2} \left(\sum_{k=0}^2 \tilde{\sigma}_k d\tilde{\psi}_k + \tilde{\sigma}_2 \sum_{v=1}^2 \left[\frac{\tilde{q}_v}{\tilde{x}_v^2 \tilde{U}_v} \sum_{k=0}^1 \tilde{\sigma}_k^{\hat{v}} d\tilde{\psi}_k \right] \right)^2, \\ \tilde{X}_v = \sum_{i=1}^2 \tilde{c}_i \tilde{x}_v^{2i} + \tilde{b}_v + \frac{\tilde{c}(1 - \tilde{q}_v)^2}{\tilde{x}_v^2}, \quad \tilde{c}_2 = -\frac{1}{12}\Lambda, \quad (4.2)$$

$$A = -2\sqrt{3}\tilde{c} \sum_{v=1}^2 \left[\frac{\tilde{q}_v}{\tilde{U}_v} \sum_{k=0}^1 \tilde{\sigma}_k^{\hat{v}} d\tilde{\psi}_k \right], \quad (4.3)$$

where $\tilde{q}_\mu$ denotes the electric and magnetic charges. Since the similar fashion of the Kerr-Newman-AdS solution (3.42), (3.41) in this canonical form, this solution is sometimes regarded as its natural uplift to 5 dimensions. The Charn-Simon term admirably closes the back reaction in such mixing EOMs. For this solution, the limit (3.21) suggests that the matter-flux transfers to

$$A \sim \sum_{v=1}^2 \left[\frac{q_v}{U_v} \sum_{k=1}^2 \sigma_{k-1}^{\hat{v}} d\psi_k \right], \quad (4.4)$$

This form is apparently same to the 4-dimensional case, nevertheless, replying the gravitational back reaction. Indeed in 5 dimensions the 2-form field strength cannot inherit the self-dual property, instead obeying the following condition:

$$*F = -F \wedge \eta, \quad \Rightarrow \quad T_{ab} \neq 0. \quad (4.5)$$

This condition has considerably clean feature, which makes the matter-EOM so simple to forecast a part of the back reaction to the gravity. In consequence, we find an exact solution as follows,

$$ds^2 = \sum_{v=1}^2 \left\{ \frac{X_v}{U_v} dx_v^2 + \frac{U_v}{X_v} \left(\sum_{k=1}^2 \sigma_{k-1}^{\hat{v}} d\psi_k \right)^2 \right\} - c_3 \left(d\psi_0 + \sum_{k=1}^2 \sigma_k d\psi_k + \frac{1}{\sqrt{3|c_3|}} A \right)^2, \\ X_v = \sum_{i=1}^3 c_i x_v^i + b_v + 2c_3 q_v x_v, \quad c_3 = -\frac{1}{3}\Lambda, \quad (4.6)$$

$$A = -2\sqrt{3|c_3|} \sum_{v=1}^2 \left[\frac{q_v}{U_v} \sum_{k=1}^2 \sigma_{k-1}^{\hat{v}} d\psi_k \right]. \quad (4.7)$$

Solving EOMs tells how to transfer Maxwell charges on the limit, which is definitely non-vanishing. Owing to the limit-correspondence, our solution maintains almost all the information of the charged black hole. Comparing these solutions, the one after the limit is clearly simpler than before, which exerts its power at analyzing the complicated system of mixed EOMs. Hence, we consider it is worthful to find the exact solutions in the extent like Eq. (4.6), under the anticipation of the limit-correspondence.

Here, two intricacies face us. One thing is that it is no longer regarded as a BPS limit: this solution is not supersymmetric because its Riemann tensor manifestly breaks the SUSY requirement in [68]. Rather the black hole solution (4.2) has been already confirmed to possess the parameter region remaining one SUSY [27], which manifestly differs from our solution. Adding the electric charge permits the black hole to saturate the BPS bound without the limit manipulation (3.21). Another thing is that Eq. (4.6) breaks a part of the traditional Sasakian conditions slightly. This indicates that Sasakian geometry could not supply some symmetrical guarantee for the extent where we want to search. At the same time, it might imply the existence of a generalization of Sasakian geometry, which contains the other limited solutions whose original black hole solution has not been found. Therefore, we discuss a generalization of Sasakian geometry, which is suited to the present solution.

4.2 Torsional generalization of Sasakian structure

4.2.1 Deformation of definitions

For the generalization of Sasakian geometry, we shall adopt the deformation by a totally skew-symmetric torsion. It seems compatible to introduce such torsion in SUGRAs: It can be identified with a 3-form or other-form flux inhabiting in the theories [69]. In particular, it was found that the previous black hole solution admits such a torsion relating to the 2-form flux from the viewpoint of a sort of the tensorial symmetry, called generalized hidden symmetry or GCCKY tensors. In the long Sasakian history, a different study has been done about the presence of torsion in Sasakian geometry: It has been previously considered by introducing a connection with totally skew-symmetric torsion, which preserves the Sasakian structure [70, 71]. Given a Sasakian structure, such a connection always exists uniquely. Then the torsion is written as $T = \eta \wedge d\eta$ where η is the contact 1-form as in the previous chapter. On the other hand, our framework expects that the existence of torsion no longer preserves the Sasakian geometry because of the effect of the energy-momentum tensor, which may change a crucial geometric property as well as Einstein's equation. Therefore, we discuss one possible deformation of the Sasakian structure in the presence of totally skew-symmetric torsion.

First of all, we make up the notation for a connection with totally skew-symmetric torsion for maintaining the minimal mathematical rigidity. Let (M, g) be a Riemannian manifold, T be a 3-form on M and $\{e_a\}$ be an orthonormal frame on TM . We define a connection with totally skew-symmetric torsion, ∇^T , by

$$g(\nabla_X^T Y, Z) = g(\nabla_X Y, Z) + \frac{1}{2}T(X, Y, Z), \quad (4.8)$$

where X and Y denote vector fields on M , and ∇ is the Levi-Civita connection of g . This connection satisfies a metricity condition, $\nabla^T g = 0$, and has the same geodesics as ∇ , $\nabla_{\dot{\gamma}}^T \dot{\gamma} = \nabla_{\dot{\gamma}} \dot{\gamma} = 0$ for a geodesic γ . Thus the commutators are related to the Lie brackets by

$$\nabla_X^T Y - \nabla_Y^T X = [X, Y] + T(X, Y) , \quad (4.9)$$

where $T(X, Y) = \sum_a T(X, Y, e_a) e_a$. For a p -form Π the torsional covariant derivative relates to the ordinary covariant derivative as

$$\nabla_X^T \Pi = \nabla_X \Pi - \frac{1}{2} \sum_a (X \lrcorner e_a \lrcorner T) \wedge (e_a \lrcorner \Pi) , \quad (4.10)$$

where $\lrcorner$ represents the inner product. Then, we have

$$\begin{aligned} d^T \Pi &= \sum_a e^a \wedge \nabla_{e_a}^T \Pi \\ &= d\Pi - \sum_a (e_a \lrcorner T) \wedge (e_a \lrcorner \Pi) , \end{aligned} \quad (4.11)$$

$$\begin{aligned} \delta^T \Pi &= - \sum_a e_a \lrcorner \nabla_{e_a}^T \Pi \\ &= * d * \Pi - \frac{1}{2} \sum_{a,b} (e_a \lrcorner e_b \lrcorner T) \wedge (e_a \lrcorner e_b \lrcorner \Pi) , \end{aligned} \quad (4.12)$$

with $\{e^a\}$ a dual 1-form frame of $\{e_a\}$, $e^a(e_b) = \delta^a_b$, d^T the exterior-derivative and δ^T the co-derivative with torsion.

Suppose $\mathcal{M}$ is a Hermitian manifold equipped with a complex structure J and a Hermitian metric g obeying $g(X, Y) = g(J(X), J(Y))$ for any vector field X and Y . Then it is known that there exists a unique Hermitian connection ∇^{Bis} with totally skew-symmetric torsion Bis , i.e., $\nabla^{Bis} g = 0$ and $\nabla^{Bis} J = 0$. This connection ∇^{Bis} is known as a *Bismut connection*, the corresponding totally skew-symmetric torsion Bis called a *Bismut torsion* [72]. Generally this can be written in the form

$$Bis(X, Y, Z) = d\Omega(J(X), J(Y), J(Z)) , \quad (4.13)$$

where Ω is the fundamental 2-form $\Omega(X, Y) \equiv g(J(X), Y)$. A Hermitian manifold $\mathcal{M}$ equipped with the Bismut torsion Bis is called a *Kähler with torsion manifold*.

As explained, a Riemannian manifold (M, g) becomes Sasaki manifold if its metric cone $(C(M), \mathring{g}) = (M \times R_+, \mathring{g} = dr^2 + r^2 g)$ is Kähler. Once defined in such manner, the Sasakian structure is derived from the Kähler cone structure. Following this manner, we suggest a generalization of Sasakian structure deformed by the presence of torsion:

Definition 4.2.1 *Let (M, g) be a Riemannian manifold and T be a 3-form on M . Then, we call (M, g, T) a Sasaki with torsion (ST) manifold if its metric cone $(C(M), \mathring{g})$ is a Kähler with torsion (KT) manifold whose Bismut torsion Bis is given by $Bis = r^2 T$.*

Let X and Y be vector fields on M , and $\overset{\circ}{\nabla}^{Bis}$ be the Bismut connection associated with the metric cone $C(M)$. Chosen appropriately, X and Y can be also regarded as vector fields on $C(M)$. Then the following formulae hold:

$$\begin{aligned}\overset{\circ}{\nabla}_{\partial_r}^{Bis} \partial_r &= 0, \quad \overset{\circ}{\nabla}_{\partial_r}^{Bis} X = \overset{\circ}{\nabla}_X^{Bis} \partial_r = \frac{1}{r} X, \\ \overset{\circ}{\nabla}_X^{Bis} Y &= \nabla_X^T Y - rg(X, Y) \partial_r,\end{aligned}\tag{4.14}$$

where $\nabla_X^T Y$ is the connection on M with totally skew-symmetric torsion T . Taking the complex structure J on the KT cone into account, we define a vector field ζ on $C(M)$ by

$$\zeta = J(r\partial_r),\tag{4.15}$$

whose length is given by $\dot{g}(\zeta, \zeta) = r^2$. Since $\overset{\circ}{\nabla}_X^{Bis} J = 0$, it satisfies

$$\dot{g}(\overset{\circ}{\nabla}_X^{Bis} \zeta, Y) = \dot{g}(J(\overset{\circ}{\nabla}_X^{Bis}(r\partial_r)), Y) = \dot{g}(J(X), Y).\tag{4.16}$$

which is anti-symmetric with respect to exchange of X and Y . Hence, this assures that ζ is a Killing vector field, i.e., $\dot{g}(\overset{\circ}{\nabla}_X^{Bis} \zeta, Y) + \dot{g}(\overset{\circ}{\nabla}_Y^{Bis} \zeta, X) = 0$. We identify M with $M \times \{1\} \subset C(M)$, finding that ζ is a Killing vector field of unit length on M . We also define a 1-form η and a $(1, 1)$ -tensor Φ respectively on M by

$$\eta(X) = g(\zeta, X), \quad \Phi(X) = \nabla_X^T \zeta.\tag{4.17}$$

Then we have

$$J(X) = \Phi(X) - r\eta(X)\partial_r.\tag{4.18}$$

The integrability condition of J impose a condition for the Bismut torsion, that is,

$$Bis(X, Y, Z) = Bis(X, J(Y), J(Z)) + Bis(J(X), Y, J(Z)) + Bis(J(X), J(Y), Z).\tag{4.19}$$

Making use of Eq. (4.18) and the relation $Bis = r^2 T$, the condition (4.19) leads to a constraint for the torsion T as

$$T(X, Y, Z) = T(X, \Phi(Y), \Phi(Z)) + T(\Phi(X), Y, \Phi(Z)) + T(\Phi(X), \Phi(Y), Z).\tag{4.20}$$

In the similar fashion to the Sasakian geometry [43, 44, 45], we thus have obtained a triple of (ζ, η, Φ) given by (4.15) and (4.17), and a totally skew-symmetric torsion T obeying (4.20). Since this can be interpreted as a generalization of the Sasakian structure, we call it a *Sasaki with torsion (ST) structure* on M . The following proposition provides four equivalent characterizations of the ST structure:

Proposition 4.2.2 *Let (M, g) be a Riemannian manifold and T be a 3-form on M obeying (4.20). Then the following conditions are equivalent:*

(1) (M, g, T) is an ST manifold.

(2) There exists a Killing vector field ζ of unit length on M so that the dual 1-form η satisfies

$$\nabla_X^T(d^T\eta) = -2X^\flat \wedge \eta, \quad (4.21)$$

for any vector field X , where $X^\flat = g(X, -)$.

(3) There exists a Killing vector field ζ of unit length on M so that the $(1,1)$ -tensor field Φ defined by $\Phi(X) = \nabla_X^T\zeta$ satisfies

$$(\nabla_X^T\Phi)(Y) = g(\zeta, Y)X - g(X, Y)\zeta, \quad (4.22)$$

for any pair of vector fields X, Y .

(4) There exists a Killing vector field ζ of unit length on M so that the curvature satisfies

$$R^T(X, Y)\zeta = g(\zeta, Y)X - g(\zeta, X)Y + \Phi(T(X, Y)), \quad (4.23)$$

for any pair of vector fields X, Y , where the curvature $R^T(X, Y)$ is defined by

$$R^T(X, Y)Z = \nabla_X^T\nabla_Y^TZ - \nabla_Y^T\nabla_X^TZ - \nabla_{[X, Y]}^TZ. \quad (4.24)$$

It is worth mentioning that a p -form ϕ is called a *special Killing p -form with torsion* if it satisfies the equations

$$\nabla_X^T\phi = \frac{1}{p+1}X \lrcorner d^T\phi, \quad \nabla_X^T(d^T\phi) = kX \wedge \phi, \quad (4.25)$$

with a constant k . For $\phi = \eta$, the first equation in (4.25) implies that its dual vector field ζ is a Killing vector field. Hence the 1-form η in Prop. 4.2.2 is a special Killing 1-form with torsion. Here, we find that the $(2\ell + 1)$ -forms

$$\eta^{(\ell)} = \eta \wedge (d^T\eta)^\ell, \quad (4.26)$$

for $\ell = 0, \dots, n$, are also special Killing forms with torsion on an ST manifold. For general properties of the special Killing forms with torsion, we refer to Appendix of [67]. In general, given a special Killing p -form with torsion ϕ on M ,

$$\hat{\phi} = r^p dr \wedge \phi + \frac{r^{p+1}}{p+1} d^T\phi, \quad (4.27)$$

is a parallel $(p+1)$ -form on $C(M)$, i.e., $\overset{\circ}{\nabla}^{Bis}\hat{\phi} = 0$, see [73]. In particular, for the 1-form η in Prop. 4.2.2 we have a parallel 2-form

$$\hat{\phi} = r dr \wedge \eta + \frac{r^2}{2} d^T\eta = \frac{1}{2} d^T(r^2\eta), \quad (4.28)$$

which is nothing but a fundamental 2-form $\Omega = \hat{\phi}$ on $C(M)$.

The following statement is an immediate consequence of Prop. 4.2.2.

Proposition 4.2.3 *Let (M, g, T) be an ST manifold and (ζ, η, Φ) be a triple of its ST structure on M , given in Prop. 4.2.2. Then we have*

$$\eta(\zeta) = 1, \quad (4.29)$$

$$\Phi(\Phi(X)) = -X + \eta(X)\zeta, \quad (4.30)$$

$$g(\Phi(X), \Phi(Y)) = g(X, Y) - \eta(X)\eta(Y), \quad (4.31)$$

$$\Phi(\zeta) = 0, \quad \eta(\Phi(X)) = 0, \quad (4.32)$$

$$N_\Phi(X, Y) + d\eta(X, Y)\zeta = 0, \quad (4.33)$$

$$d^T\eta = 2\varpi, \quad \zeta \lrcorner d\varpi = 0, \quad (4.34)$$

where the fundamental 2-form ϖ is defined by $\varpi(X, Y) = g(\Phi(X), Y)$, and N_Φ is the Nijenhuis tensor of type-(1, 2) with respect to Φ , given by

$$N_\Phi(X, Y) \equiv [\Phi(X), \Phi(Y)] + \Phi(\Phi([X, Y])) - \Phi([X, \Phi(Y)]) - \Phi([\Phi(X), Y]). \quad (4.35)$$

Equipped with a structure (ζ, η, Φ) satisfying Eqs. (4.29)–(4.31), a Riemannian manifold (M, g) is known as an *almost contact metric manifold*. Eq. (4.32) is derived from such a structure, especially Eqs. (4.29) and (4.30). Furthermore, an almost contact metric structure (g, ζ, η, Φ) is called *normal* if it satisfies Eq. (4.33), and a *contact metric structure* if it satisfies $d\eta = 2\varpi$, respectively (e.g., see [44, 43]). Sasakian manifold is also known to be included in normal contact metric manifold. In contrast, as seen in Eq. (4.34) our ST manifold possesses a contact metric structure deformed by the torsion T , while be an almost normal contact metric manifold.

Definition 4.2.4 *Let (M, g, T) be a Riemannian manifold with a 3-form T obeying (4.20). We call the almost contact metric structure satisfying $d^T\eta = 2\varpi$ and $\zeta \lrcorner d\varpi = 0$ a T -contact metric structure, and call $(M, g, T, \zeta, \eta, \Phi)$ a T -contact metric manifold. We further call the T -contact metric manifold a TK -contact metric manifold if ζ is a Killing vector field.*

The sub-bundle of codimension one, $\mathcal{D} = \ker \eta \subset TM$ has an almost complex structure defined by $J_{\mathcal{D}} = \Phi|_{\mathcal{D}}$. Hence, the sub-bundle $\mathcal{D}$ together with the endomorphism $J_{\mathcal{D}}$ provides M with an almost CR structure of codimension one. The normality condition yields that the almost CR structure is integrable, i.e., the Nijenhuis tensor with respect to $J_{\mathcal{D}}$ vanishes. Now we demonstrate the following proposition.

Proposition 4.2.5 *An ST manifold is a normal T -contact metric manifold whose torsion $T_{\mathcal{D}} = T|_{\mathcal{D}}$ is given by a Bismut torsion*

$$T_{\mathcal{D}}(X, Y, Z) = d\varpi(J_{\mathcal{D}}(X), J_{\mathcal{D}}(Y), J_{\mathcal{D}}(Z)), \quad (4.36)$$

for all $X, Y, Z \in \mathcal{D}$.

Proof. Let us start at $(M, g, T, \zeta, \eta, \Phi)$ being an almost contact metric manifold equipped with a 3-form T satisfying (4.20) and define a fundamental 2-form ϖ by $\varpi(X, Y) = g(\Phi(X), Y)$. Then a straightforward calculation leads us to

$$\begin{aligned} 2g((\nabla_X^T \Phi)Y, Z) &= -d\varpi(X, \Phi Y, \Phi Z) + d\varpi(X, Y, Z) + M(X, Y, Z) \\ &\quad + g(N^{(1)}(Y, Z), \Phi X) + \eta(X)N^{(2)}(Y, Z) \\ &\quad + d^T \eta(X, \Phi Z)\eta(Y) - d^T \eta(X, \Phi Y)\eta(Z) , \end{aligned} \quad (4.37)$$

where we have introduced tensor fields $N^{(i)}$ ($i = 1, 2$) defined in Sec. 6 of [44]:

$$N^{(1)}(X, Y) = N_\Phi(X, Y) + d\eta(X, Y)\zeta , \quad (4.38)$$

$$N^{(2)}(X, Y) = (\mathcal{L}_{\Phi X}\eta)(Y) - (\mathcal{L}_{\Phi Y}\eta)(X) , \quad (4.39)$$

The tensor M is defined by

$$M(X, Y, Z) = T(X, \Phi Y, Z) + T(X, Y, \Phi Z) - T(\zeta, X, \Phi Y)\eta(Z) + T(\zeta, X, \Phi Z)\eta(Y), \quad (4.40)$$

then it satisfies

$$M(\zeta, X, Y) = M(X, \zeta, Y) = M(X, Y, \zeta) = 0 . \quad (4.41)$$

In the case of a T-contact metric manifold, this formula reduces more. Indeed, we have

$$\begin{aligned} N^{(2)}(X, Y) &= d\eta(X, \Phi Y) + d\eta(\Phi X, Y) \\ &= d^T \eta(X, \Phi Y) + d\eta^T(\Phi X, Y) + T(\zeta, X, \Phi Y) + T(\zeta, \Phi X, Y) = 0 , \end{aligned} \quad (4.42)$$

where we have used (4.20), (4.32) and (4.34) at the last equality.

Furthermore, ζ holds

$$\zeta \lrcorner d^T \eta = \zeta \lrcorner d\eta = 0 , \quad (4.43)$$

which implies $\mathcal{L}_\zeta \eta = 0$ and $\mathcal{L}_\zeta d\eta = 0$. We also see

$$\mathcal{L}_\zeta d^T \eta = d\zeta \lrcorner d^T \eta + \zeta \lrcorner dd^T = 2\zeta \lrcorner d\varpi = 0 , \quad (4.44)$$

which leads to

$$2(\mathcal{L}_\zeta g)(X, Y) = d^T \eta(X, (\mathcal{L}_\zeta \Phi)(Y)) . \quad (4.45)$$

Thus ζ is a Killing vector field if and only if $\mathcal{L}_\zeta \Phi = 0$.

If (M, g, T) is an ST manifold, then Prop. 4.2.3 yields that M is a normal T-contact metric manifold. Hence, we have $N^{(i)} = 0$ ($i = 1, 2$), so that (4.37) reduces to

$$\begin{aligned} 2g((\nabla_X^T \Phi)Y, Z) &= -d\varpi(X, \Phi Y, \Phi Z) + d\varpi(X, Y, Z) + M(X, Y, Z) \\ &\quad + d^T \eta(X, \Phi Z)\eta(Y) - d^T \eta(X, \Phi Y)\eta(Z) . \end{aligned} \quad (4.46)$$

Noting that

$$\begin{aligned} d^T \eta(X, \Phi Z) \eta(Y) - d^T \eta(X, \Phi Y) \eta(Z) &= 2\varpi(X, \Phi Z) \eta(Y) - 2\varpi(X, \Phi Y) \eta(Z) \\ &= 2g(X, Z)g(\zeta, Y) - 2g(X, Y)g(\zeta, Z) \end{aligned} \quad (4.47)$$

in this equation and using (4.22) we have

$$d\varpi(X, \Phi Y, \Phi Z) - d\varpi(X, Y, Z) - M(X, Y, Z) = 0. \quad (4.48)$$

When $X = \zeta$ or $Y, Z = \zeta$, the equation above is automatically satisfied from (4.34) and (4.41). Now it is easy to verify (4.48) is equivalent to (4.36). Conversely, the normality condition $N^{(1)} = 0$ leads to $\mathcal{L}_\zeta \Phi = 0$, see [44], so that using (4.45) we have a Killing vector field ζ . It is easy to see that (3) in Prop. II.2 is satisfied. $\square$

Since an almost contact metric structure is normal if and only if the almost CR structure is integrable and $\mathcal{L}_\zeta \Phi = 0$, see [44], the following proposition immediately follows.

Proposition 4.2.6 *An ST manifold is a TK-contact metric manifold whose CR structure is integrable and torsion $T_{\mathcal{D}} = T|_{\mathcal{D}}$ is given by a Bismut torsion.*

A curvature property is determined albeit this abstract argument. It is confirmed that the Ricci tensor of a Sasakian manifold of $(2n + 1)$ dimensions is given by $\text{Ric}(X, \zeta) = 2n\eta(X)$. In the ST manifold case, the Ricci curvature follows from Eq. (4.23) that

$$\begin{aligned} \text{Ric}^T(X, \zeta) &= - \sum_a g(R^T(X, e_a)\zeta, e_a) \\ &= 2n\eta(X) - \sum_a T(X, e_a, \Phi(e_a)). \end{aligned} \quad (4.49)$$

4.2.2 Concrete class of local metrics

We shall present a concrete class of local metrics admitting the deformed Sasakian structure introduced in subsection 4.2.1, which we call Sasaki with torsion (ST) metrics.

The ST metric in $(2n + 1)$ dimensions is given in local coordinates $(x^\mu) = (x_v, \psi_k)$ with $v = 1, \dots, n$ and $k = 0, \dots, n$ by

$$g_{2n+1} = \sum_{v=1}^n \frac{dx_v^2}{Q_v} + \sum_{v=1}^n Q_v \left(\sum_{k=1}^n \sigma_{k-1}^{\hat{v}} d\psi_k \right)^2 + 4 \left(\sum_{k=0}^n \sigma_k d\psi_k + \mathcal{A}_{[ST]} \right)^2, \quad (4.50)$$

where

$$Q_v = \frac{X_v}{U_v}, \quad U_v = \prod_{\substack{u=1 \\ u \neq v}}^n (x_v - x_u), \quad X_v = X_v(x_v). \quad (4.51)$$

The functions $\sigma_k^{\hat{v}}$ and σ_k are the k -th elementary symmetric polynomials as in Eq. (3.17). We impose that the 1-form $\mathcal{A}_{[ST]}$ is restricted as

$$\mathcal{A}_{[ST]} = \sum_{v=1}^n \frac{N_v}{U_v} \sum_{k=1}^n \sigma_{k-1}^{\hat{v}} d\psi_k, \quad N_v = N_v(x_v). \quad (4.52)$$

We should note that the metric (4.50) is an “off-shell” metric, that is, it contains $2n$ unknown functions $X_v(x_v)$ and $N_v(x_v)$ depending only on one variable x_v . As we will see in Sec. 4.3, these metric functions are determined by considering EOMs of theories.

On this setup, N_v has the interesting gauge degree, which can be seen as the coincidence of the diffeomorphism and Wilson line. If and only if N_v are given by the polynomials in x_v of the form

$$N_v = \sum_{i=0}^{n-1} a_i x_v^i, \quad (4.53)$$

where a_i ($i = 0, \dots, n-1$) are arbitrary constants, the 1-form $\mathcal{A}_{[ST]}$ becomes

$$\mathcal{A}_{[ST]} = \sum_{k=1}^{n-1} (-1)^{k-1} a_{n-k} d\psi_k. \quad (4.54)$$

This is equivalent to taking $\mathcal{A}_{[ST]} = 0$ because a_i are eliminated by the gauge transformation of $d\psi_0$. These parameters sometimes happen to get the importance from the global viewpoint.

The present metric (4.50) expresses nothing but a limited part of Sasaki with torsion geometry: Besides the cohomogeneous argument, its $2n$ -dimensional base space can be extended from ones known as an orthotoric Kähler metric established in [66, 65] to KT metric at least locally (see Prop. 4.2.5 and 4.2.6).

Let us confirm that this concrete metric properly satisfies ST conditions presented in the previous subsection. Firstly we shall see the conditions in Prop. 4.2.3. To do this, it is convenient to introduce an orthonormal frame $\{e^a\} = \{e^v, e^{\hat{v}} = e^{n+v}, e^0 = e^{2n+1}\}$ and compute connection 1-forms from the first structure equation

$$de^a + \sum_b \omega^a_b \wedge e^b = 0, \quad (4.55)$$

with $\omega_{ab} = -\omega_{ba}$. For the metric (4.50), the following choice of the orthonormal frame is suitable:

$$e^v = \frac{dx_v}{\sqrt{Q_v}}, \quad e^{\hat{v}} = \sqrt{Q_v} \sum_{k=1}^n \sigma_{k-1}^{\hat{v}} d\psi_k, \quad e^0 = 2 \left(\sum_{k=0}^n \sigma_k d\psi_k + \mathcal{A}_{[ST]} \right). \quad (4.56)$$

Then the connection 1-forms ω^a_b are calculated as

$$\omega^v_u = -\frac{\sqrt{Q_u}}{2(x_v - x_u)} e^v - \frac{\sqrt{Q_v}}{2(x_v - x_u)} e^u, \quad v \neq u \quad (4.57)$$

$$\omega^v_{\hat{v}} = -\partial_v \sqrt{Q_v} e^{\hat{v}} + \sum_{u \neq v} \frac{\sqrt{Q_u}}{2(x_v - x_u)} e^{\hat{u}} - (1 + \partial_v H) e^0, \quad (4.58)$$

$$\omega^v_{\hat{u}} = \frac{\sqrt{Q_u}}{2(x_v - x_u)} e^{\hat{v}} - \frac{\sqrt{Q_v}}{2(x_v - x_u)} e^{\hat{u}}, \quad v \neq u \quad (4.59)$$

$$\omega^{\hat{v}}_{\hat{u}} = -\frac{\sqrt{Q_u}}{2(x_v - x_u)} e^v - \frac{\sqrt{Q_v}}{2(x_v - x_u)} e^u, \quad v \neq u \quad (4.60)$$

$$\omega^v_0 = - (1 + \partial_v H) e^{\hat{v}}, \quad (4.61)$$

$$\omega^{\hat{v}}_0 = (1 + \partial_v H) e^v, \quad (4.62)$$

where H is a function

$$H = \sum_{v=1}^n \frac{N_v}{U_v}. \quad (4.63)$$

This variable implies the deviation from Sasakian geometry.

We introduce a 1-form $\eta = e^0$, vector field $\zeta = e_0$ and $(1, 1)$ -tensor Φ in Prop. 4.2.3 as

$$\Phi(e_v) = e_{\hat{v}}, \quad \Phi(e_{\hat{v}}) = -e_v, \quad \Phi(e_0) = 0. \quad (4.64)$$

For the triple (ζ, η, Φ) together with the metric g , the conditions (4.29)–(4.31) hold. Namely, the set (g, ζ, η, Φ) is an almost contact metric structure on M . Moreover, by using relations $\nabla_{e_a} e^b(e_c) = -\omega^b_c(e_a)$ we can compute the covariant derivatives as Eqs. (B.1)–(B.13) in App. B and the commutation relations $[e_a, e_b] \equiv \nabla_{e_a} e_b - \nabla_{e_b} e_a$ are obtained. Eq. (4.33) is demonstrated from these commutation relations, implying that the almost contact metric structure is normal. However, we find that η is not in general a contact 1-form because of its concrete expression: that is,

$$d\eta = 2 \sum_{v=1}^n (1 + \partial_v H) e^v \wedge e^{\hat{v}}, \quad (4.65)$$

hence there is a possibility that $\eta \wedge (d\eta)^n = 0$ at some points on the manifold. If H is constant, we have $d\eta(X, Y) = 2g(\Phi(X), Y)$ so that η is a contact 1-form. We further find that this local class is a quasi-Sasakian metric [74], i.e., $d\varpi = 0$ where ϖ is the fundamental 2-form given by $\varpi(X, Y) = g(\Phi(X), Y)$. In fact, we have

$$\varpi = \sum_{v=1}^n e^v \wedge e^{\hat{v}} = d \left[\sum_{k=0}^n \sigma_k d\psi_k \right]. \quad (4.66)$$

Next, let us see the conditions in Prop. 4.2.2. We introduce the torsion T and compute the covariant derivatives with respect to the torsion connection ∇^T . Since the torsion T satisfying Eq. (4.34) is given by

$$T = 2 \sum_{v=1}^n \partial_v H e^v \wedge e^{\hat{v}} \wedge e^0 , \quad (4.67)$$

we can check that Eq. (4.20) holds. We emphasize again that the torsion (4.67) differs from the torsion preserving the Sasakian structure, $\eta \wedge d\eta$, discussed in [70]. Namely, $\nabla^T \zeta \neq 0$ and $\nabla^T \Phi \neq 0$. The covariant derivatives $\nabla_{e_a}^T e_b$ are calculated as (B.29)–(B.41) in App. B. Using these expressions, we find that

$$\nabla_X^T \zeta = \Phi(X) . \quad (4.68)$$

It is also shown that for any vector field X ,

$$\nabla_X^T \eta = \frac{1}{2} X \lrcorner d^T \eta , \quad \nabla_X^T (d^T \eta) = -2X \wedge \eta , \quad (4.69)$$

which proves Eq. (4.25) with $k = -2$ so that η is a special Killing 1-form with torsion.

The cone metrics

Let us look at the property of the uplift space: the Riemannian cone of the metric (4.50) given by

$$\dot{g} = dr^2 + r^2 g_{2n+1} . \quad (4.70)$$

For the cone metric $\dot{g}$, we introduce an orthonormal basis $\{\dot{e}^\alpha\}$ as

$$\dot{e}^r = dr , \quad \dot{e}^a = r e^a . \quad (4.71)$$

The connection 1-forms $\dot{\omega}^\alpha{}_\beta$ with respect to $\dot{g}$ are calculated as

$$\dot{\omega}^r{}_a = -\frac{1}{r} \dot{e}^a , \quad \dot{\omega}^a{}_b = \omega^a{}_b , \quad (4.72)$$

where $\omega^a{}_b$ is as in Eqs. (4.57)–(4.62). The commutation relations $[\dot{e}_\alpha, \dot{e}_\beta]$ are calculated in the parallel form to the previous section. An almost complex structure J is introduced by

$$J(\dot{e}_r) = \dot{e}_0 , \quad J(\dot{e}_0) = -\dot{e}_r , \quad J(\dot{e}_v) = \dot{e}_{\hat{v}} , \quad J(\dot{e}_{\hat{v}}) = -\dot{e}_v . \quad (4.73)$$

Then the integrability condition is directly checked for the almost complex structure J . The Nijenhuis tensor vanishes so that J is integrable, and the cone metric $\dot{g}$ becomes Hermitian,

$$\dot{g}(X, Y) = \dot{g}(J(X), J(Y)) . \quad (4.74)$$

The fundamental form $\Omega(X, Y) = \dot{g}(J(X), Y)$ can be written as

$$\Omega = \dot{e}^r \wedge \dot{e}^0 + \sum_{v=1}^n \dot{e}^v \wedge \dot{e}^{\hat{v}} = \frac{1}{2} dT (r^2 e^0) . \quad (4.75)$$

Since (M, g, J) is a Hermitian manifold, the existence of the Bismut connection is guaranteed: that is, a unique Hermitian connection $\overset{\circ}{\nabla}^{Bis}$ with totally skew-symmetric torsion Bis . From Eq. (4.13) the Bismut torsion is explicitly obtained as

$$Bis = \sum_{v=1}^n \frac{2}{r} \partial_v H \dot{e}^v \wedge \dot{e}^{\hat{v}} \wedge \dot{e}^0 = r^2 T , \quad (4.76)$$

where T is given by (4.67). Additionally, an important condition is manifest in this class. The straightforward calculation shows that the azimuthal Killing vector fields $\partial_k \equiv \partial/\partial\psi_k$ ($k = 0, 1, \dots, n$) preserve the KT structure on this cone,

$$\mathcal{L}_{\partial_k} \Omega = 0, \quad \mathcal{L}_{\partial_k} Bis = 0. \quad (4.77)$$

Hidden symmetry

Sasakian geometry was known to relate to a tensorial symmetry, called Killing–Yano symmetry. It was confirmed that a Sasakian manifold of $2n+1$ dimensions has rank- $(2p+1)$ special Killing forms in the form $\eta \wedge (d\eta)^p$ ($0 \leq p \leq n$) [7, 8]. Killing–Yano symmetry was originally defined as Killing–Yano (KY) tensors [75] and conformal Killing–Yano (CKY) tensors [76, 77, 73] from a purely mathematical view point. Later it has played an important role in the study of black hole physics. In recent study general metrics admitting a rank-2 closed CKY tensor were obtained from four [78, 79] to higher [80, 81, 82, 83] dimensions. Then such metrics was found to possess remarkable properties in mathematical physics: separation of variables for the Hamilton–Jacobi, Klein–Gordon and Dirac equations, e.g., see reviews [84, 85, 86]. Considering this successful achievement, it is expected that they possess similar integrability structures due to Killing–Yano symmetry,

By the presence of totally skew-symmetric torsion, Killing–Yano symmetry can be naturally extended. A *generalized conformal Killing–Yano (GCKY) tensor k with respect to torsion T* was introduced by [28] as a p -form satisfying for any vector field X and a 3-form T ,

$$\nabla_X^T k = \frac{1}{p+1} X \lrcorner d^T k - \frac{1}{D-p+1} X^\flat \wedge \delta^T k , \quad (4.78)$$

where $X^\flat$ is a dual 1-form of X . In particular, we call a GCKY tensor f obeying $\delta^T f = 0$ a *generalized Killing–Yano (GKY) tensor*, and a GCKY tensor h obeying $d^T h = 0$ a *generalized closed conformal Killing–Yano (GCCKY) tensor*. It was revealed that the

generalized Killing–Yano symmetry occurs in several rotating black hole solution, i.e., not only the present one Eq. (4.2) in 5-dimensional minimal supergravity [28], but abelian heterotic supergravity and its higher-dimensional generalization [87]. Fertile properties of such a generalized symmetry and general metrics admitting a rank-2 GCCKY tensor have been studied in [87, 88, 89].

As we have seen in previously, the metric (4.50) can be regarded as a natural generalization of Sasakian metrics in the presence of torsion. Thus it is natural to expect the existence of the generalized Killing–Yano symmetry. Actually, we also find that for the metric (4.50), the Hamilton-Jacobi equation of geodesics,

$$\partial_\tau S + g^{\mu\nu}(\partial_\mu S)(\partial_\nu S) = 0, \quad (4.79)$$

can be solved by separation of variables. This is attributed to the existence of enough Killing tensors as well as Killing vector fields $\partial/\partial\psi_k$. In the case of the presence of torsion, a general fact is that if a rank- p GKY tensor f is given, the rank-2 Killing tensor K is generated by

$$K_{ab} = f_{ac_1 \dots c_{p-1}} f_b^{c_1 \dots c_{p-1}}. \quad (4.80)$$

Therefore, it is worth asking whether the metric (4.50) admits the existence of GKY tensors with respect to a torsion.

Exploring GKY tensors gives rise to the problem what the torsion is. The natural torsion is the 3-form T related to the ST structure, given by Eq. (4.67). Since the first equation in (4.25) is same as the GKY equation, a special Killing p -form with torsion is alternatively said to be a *rank- p special GKY tensor*. As was already seen in Eq. (4.26), $\eta^{(\ell)} \equiv \eta \wedge (d^T \eta)^\ell$ for $\ell = 0, \dots, n$ are rank- $(2\ell + 1)$ special GKY tensors with respect to torsion T . Thus we have $n + 1$ GKY tensors. However, these GKY tensors $\eta^{(\ell)}$ do not give rise to nontrivial rank-2 Killing tensors. In fact, every GKY tensor generates the only metric essentially. By contrast, adopting another torsion, we find other GKY tensors $f^{(j)}$ which are not special and generating nontrivial Killing tensors.

Introduce a 2-form $\hat{h}$ and 3-form Gcc as

$$\hat{h} = \sum_{v=1} \sqrt{x_v} e^v \wedge e^{\hat{v}}, \quad (4.81)$$

$$Gcc = \sum_{v \neq u} \frac{1}{\sqrt{x_v} + \sqrt{x_u}} \sqrt{\frac{Q_u}{x_u}} e^v \wedge e^{\hat{v}} \wedge e^{\hat{u}} + \sum_{v=1}^n 2(1 + \partial_v H) e^v \wedge e^{\hat{v}} \wedge e^0. \quad (4.82)$$

Then it is demonstrated that for the metric (4.50), the $(2j + 1)$ -forms

$$h^{(\ell)} \equiv e^0 \wedge (\hat{h})^j = e^0 \wedge \hat{h} \wedge \dots \wedge \hat{h}, \quad (4.83)$$

for $j = 1, \dots, n$, are rank- $(2j + 1)$ GCKY tensors with respect to torsion G_{cc} , obeying for any vector field X

$$\nabla_X^{G_{cc}} h^{(j)} = -\frac{1}{D-2j} X^\flat \wedge *d^{G_{cc}} * h^{(j)} . \quad (4.84)$$

From general properties of GCKY tensors (e.g., see [87]), GCKY tensors $h^{(j)}$ generate GKY tensors $f^{(j)}$ by $f^{(j)} = *h^{(j)}$. These GKY tensors $f^{(j)}$ generate rank-2 Killing tensors $K^{(j)}$ by $K_{ab}^{(j)} = [\ell!^2(n-2j-1)!]^{-1} f^{(j)}_{ac_1 \dots c_{D-2\ell-2}} f^{(j)}_{b \ c_1 \dots c_{D-2\ell-2}}$. This explicit expression is successfully reappeared as

$$K^{(j)} = \sum_{v=1}^n \sigma_j^{\hat{v}} (e^v \otimes e^v + e^{\hat{v}} \otimes e^{\hat{v}}) . \quad (4.85)$$

4.3 Exploring Supergravities

4.3.1 Exact solutions with torsion

Adopting the metric ansatz in the form (4.50), we shall present the investigation of finding out the supergravity solutions. We proceed in Euclidean signature for convenience, but every solution is able to obtain the Lorentzian signature, preserving the matter-flux reality.

5-dimensional minimal (un-)gauged supergravity

We first consider the 5-dimensional minimal gauged supergravity, reproducing the limit-corresponding solution with the present ansatz. The action is given by Eq. (4.1), then the EOMs read

$$R_{ab} = -4g_{ab} + \frac{1}{2} \left(F_{ac} F_b^c - \frac{1}{6} g_{ab} F_{cf} F^{cf} \right) , \quad (4.86)$$

$$d * F - \frac{1}{\sqrt{3}} F \wedge F = 0 , \quad (4.87)$$

where the cosmological constant is normalized as $\Lambda = -12$, adjusted to the Sec. 4.2. Note that for the Euclidean solutions, we must consider the Euclidean action that is obtained by Wick rotation; We perform this as changing the sign of the whole right-hand side in Eq. (4.86), which accompanies with flipping the sign of the cosmological constant. The adequacy of this flipping is due to the discussion of the Einstein gravitational case in Chap. 3. Hence, we find the solutions of Einstein equation for the Euclidean signature,

$$R_{ab} = 4g_{ab} - \frac{1}{2} \left(F_{ac} F_b^c - \frac{1}{6} g_{ab} F_{cf} F^{cf} \right) . \quad (4.88)$$

As we will see later, our Wick rotation is achieved by making the fiber direction of solutions timelike, so as to satisfy the original Einstein equation (4.86). It will be also confirmed that this transformation does not break the reality of the matter flux.

We assume the form of the gauge potential A and the metric functions N_v so as to solve the Maxwell–Chern–Simon equation (4.87),

$$A = C_F \sum_{v=1}^2 \frac{q_v}{U_v} \sum_{k=1}^2 \sigma_{k-1}^{\hat{v}} d\psi_k , \quad (4.89)$$

$$N_v = a_1 x_v + q_v , \quad (4.90)$$

with C_F , a_1 and q_v constant parameters. Then the field strength is

$$F = C_F \left(\partial_1 H e^1 \wedge e^{\hat{1}} + \partial_2 H e^2 \wedge e^{\hat{2}} \right) , \quad (4.91)$$

where we have $\partial_1 H = -\partial_2 H$. Since F is closed and has the following properties

$$*F = -F \wedge \eta , \quad F \wedge \varpi = 0 , \quad (4.92)$$

with $\varpi(X, Y) = g(\Phi(X), Y)$, Eq. (4.87) can be solved easily

$$\begin{aligned} d * F &= -F \wedge d\eta \\ &= -\frac{2}{C_F} F \wedge F . \end{aligned} \quad (4.93)$$

Therefore the constant C_F is determined as $C_F = -2\sqrt{3}$. Eq. (4.88) requires that $X_v(x_v)$ takes the form

$$X_v = -4x_v^3 + \sum_{i=1}^2 c_i x_v^i + b_v - 8q_v x_v , \quad (4.94)$$

where c_i , b_v and q_v are constants.

This metric admits our torsion 3-form T , which preserves the KT structure of the cone. Note that the torsion can be constructed from the matter F . It is written as

$$T = \frac{1}{\sqrt{3}} * F . \quad (4.95)$$

In contrast, the torsion preserving the almost metric contact structure needs the additional term $\eta \wedge \varpi$. These solutions have a certain correspondence to the black hole solutions (4.2), (4.3). This black hole admits the similar torsion which can be written in the same form by their matter flux; It was founded that this torsion constructs a GCCKY 2-form on the black holes [28]. On the other hand, the hidden symmetry itself exists on our solutions in the different form; GCCKY tensors $h^{(j)}$ are odd rank tensors with the torsion G_{cc} , which cannot be written only by the matter flux.

This structure leads us to find the un-gauged minimal supergravity solution in the similar form. The solution is provided when Eqs. (4.50) and (4.89) takes the form

$$X_v = \sum_{i=1}^2 c_i x_v^i + b_v, \quad N_v = -x_v^2 + a_1 x_v + q_v. \quad (4.96)$$

The solutions possess Lorentzian counterparts. In the un-gauged case, the suitable Wick rotation with matter-reality only changes the whole metric form as

$$g_L = \sum_{v=1}^2 \frac{dx_v^2}{Q_v} + \sum_{v=1}^2 Q_v \left(\sum_{k=1}^2 \sigma_{k-1}^{\hat{v}} d\psi_k \right)^2 - 4 \left(\sum_{k=0}^2 \sigma_k d\psi_k + A \right)^2. \quad (4.97)$$

For the gauged minimal supergravity solution, it is necessary to rearrange signs in the metric function X_v as

$$X_v = 4x_v^3 + \sum_{i=1}^2 c_i x_v^i + b_v + 8q_v x_v. \quad (4.98)$$

This requirement arises from the inverting the cosmological constant to be negative. In both cases, the vector potential remains the form as Eq. (4.89).

11-dimensional supergravity

We next consider the 11-dimensional supergravity. The action is

$$\mathcal{L}_{11} = *\mathcal{R} - \frac{1}{2} F_{(4)} \wedge *F_{(4)} + \frac{1}{6} F_{(4)} \wedge F_{(4)} \wedge A_{(3)}, \quad (4.99)$$

where $F_{(4)} = dA_{(3)}$ is a 4-form flux of a 3-form gauge potential $A_{(3)}$. This produces the EOMs as

$$R_{ab} = \frac{1}{12} \left(F_{(4)acfe} F_{(4)b}{}^{cfe} - \frac{1}{12} g_{ab} F_{(4)cdef} F_{(4)}^{cdef} \right), \quad (4.100)$$

$$d * F_{(4)} - \frac{1}{2} F_{(4)} \wedge F_{(4)} = 0. \quad (4.101)$$

While this expression is Lorentzian, we examine the Euclidean solutions satisfying the Einstein equation which are the one changing the sign of the right-hand side in Eq. (4.100).

We assume that the field strength $F_{(4)}$ takes the form

$$F_{(4)} = \frac{1}{2} \sum_{v \neq u} F_{vu} e^v \wedge e^{\hat{v}} \wedge e^u \wedge e^{\hat{u}}, \quad (4.102)$$

where

$$F_{vu} = 2\ell_1 + \ell_2 (\partial_v H + \partial_u H), \quad (4.103)$$

and H is still given by Eq. (4.63) and ℓ_1, ℓ_2 are constant. Under this assumption, the field strength becomes closed, $dF_{(4)} = 0$, and the co-derivative is given by

$$\begin{aligned} *d*F_{(4)} = & - \sum_{v \neq u} \sqrt{Q_v} \left(\partial_v F_{vu} + \sum_{w \neq v, u} \frac{F_{vu} - F_{wu}}{x_v - x_w} \right) e^{\hat{v}} \wedge e^u \wedge e^{\hat{u}} \\ & + 2 \sum_{v \neq u} F_{vu} (1 + \partial_v H) e^u \wedge e^{\hat{u}} \wedge e^0 . \end{aligned} \quad (4.104)$$

Substituting the expressions (4.102) and (4.104) into Eq. (4.101), we obtain $\ell_1 = \ell_2 = -2$ and

$$N_v = -x_v^5 + \sum_{i=1}^4 a_i x_v^i + q_v . \quad (4.105)$$

Then we have

$$F_{(4)} = -2 \sum_{v \neq u} (1 + \partial_v H) e^v \wedge e^{\hat{v}} \wedge e^u \wedge e^{\hat{u}} . \quad (4.106)$$

The Einstein equation (4.100) reduces to

$$\partial_v^2 \mathcal{Q} - 4 \sum_{u \neq v} \mathcal{K}_{vu} = 0 , \quad (4.107)$$

where

$$\mathcal{K}_{vu} \equiv -\frac{1}{4} \frac{\partial_v \mathcal{Q}}{x_v - x_u} + \frac{1}{4} \frac{\partial_u \mathcal{Q}}{x_v - x_u} , \quad \mathcal{Q} \equiv \sum_{v=1}^n Q_v . \quad (4.108)$$

This equation can be solved by

$$X_v = \sum_{i=1}^5 c_i x_v^i + b_v , \quad (4.109)$$

with free parameters c_i and b_v .

Therefore, we have obtained the concrete ST samples in the extent of SUGRAs. The demonstration proceeds with the cohomogeneity- n , but the lower-cohomogenous extension is almost straightforward. The fact such variety certainly exists seems suited to the reduced-limit picture from the coincidence with the rotational parameters. The 11-dimensional solution possesses the possibility as a partner for the unknown rotating black hole, because there has never been the solution which is present only in the Sasakian side.

4.3.2 Global analysis

One of the crucial feature for the Euclidean metric is the global regularity or compactness. Such consistency seems to enlighten the physical application occasionally. We shall argue about it briefly for the present solutions.

Compactness of generalized $L^{a,b,c}$

Firstly, we analyze the global structure of the 5-dimensional minimal gauged supergravity solution. Here our aim is to construct regular metrics on compact manifolds. In this analysis the boundary of latitudinal variables acquires importance, which is determined by the node of the metric function. For clarifying this, let us rewrite the metric in the form

$$g_5 = \frac{x-y}{X}dx^2 + \frac{y-x}{Y}dy^2 + \frac{X}{x-y}(d\psi_1 + yd\psi_2)^2 + \frac{Y}{y-x}(d\psi_1 + xd\psi_2)^2 + 4\left(d\psi_0 + (x+y)d\psi_1 + xyd\psi_2 + \frac{q_1-q_2}{x-y}d\psi_1 + \frac{q_1y-q_2x}{x-y}d\psi_2\right)^2, \quad (4.110)$$

where

$$X = -4x(x - \alpha_1)(x - \alpha_2) + b_1 - 8q_1x, \quad Y = -4y(y - \alpha_1)(y - \alpha_2) + b_2 - 8q_2y, \quad (4.111)$$

and α_i ($i = 1, 2$), b_v and q_v ($v = 1, 2$) are free parameters. Here, not all the parameters are nontrivial: two degrees of the redundancy occurs from the scaling and shift transformations, as explained in Chap. 3.

In order to obtain the regular condition on a compact manifold, we must set regions of the latitudinal coordinates (x, y) , making an appropriate choice of the parameters. Let us assume that $\bar{x}_i$ and $\bar{y}_i$ ($i = 1, 2, 3$) are real roots of the equations $X(x) = 0$ and $Y(y) = 0$, which are satisfying the inequalities $\bar{x}_1 < \bar{x}_2 < \bar{x}_3$ and $\bar{y}_1 < \bar{y}_2 < \bar{y}_3$. If we choose the region of the coordinates as $\bar{x}_1 \leq x \leq \bar{x}_2 < \bar{y}_2 \leq y \leq \bar{y}_3$, then the metric is positive definite, except for the boundaries $x = \bar{x}_1$ and $\bar{x}_2$ as well as $y = \bar{y}_2$ and $\bar{y}_3$. From the relationship between the coefficients and solutions, we have

$$\alpha_1 + \alpha_2 = \bar{x}_1 + \bar{x}_2 + \bar{x}_3 = \bar{y}_1 + \bar{y}_2 + \bar{y}_3, \quad (4.112)$$

$$\alpha_1\alpha_2 + 2q_1 = \bar{x}_1\bar{x}_2 + \bar{x}_1\bar{x}_3 + \bar{x}_2\bar{x}_3, \quad \alpha_1\alpha_2 + 2q_2 = \bar{y}_1\bar{y}_2 + \bar{y}_1\bar{y}_3 + \bar{y}_2\bar{y}_3, \quad (4.113)$$

$$b_1 = 4\bar{x}_1\bar{x}_2\bar{x}_3, \quad b_2 = 4\bar{y}_1\bar{y}_2\bar{y}_3. \quad (4.114)$$

Applying for the argument of [15, 17], we obtain the constraint condition to extend the metric smoothly onto the boundaries: Since $\partial/\partial\psi_0$, $\partial/\partial\psi_1$ and $\partial/\partial\psi_2$ are linearly independent Killing vector fields, the generic Killing vector field is written as

$$\xi = \sum_{k=0}^2 \xi^{(k)} \frac{\partial}{\partial\psi_k}, \quad (4.115)$$

where $\xi^{(k)}$ are constants. The length of ξ reads

$$(\xi)^2 = \frac{X(x)}{x-y}(\xi^{(1)} + y\xi^{(2)})^2 + \frac{Y(y)}{y-x}(\xi^{(1)} + x\xi^{(2)})^2 + 4\left(\xi^{(0)} + (x+y)\xi^{(1)} + xy\xi^{(2)} + \frac{q_1-q_2}{x-y}\xi^{(1)} + \frac{q_1y-q_2x}{x-y}\xi^{(2)}\right)^2. \quad (4.116)$$

Using this expression, we impose its periodicity around each nodes. $\xi^{(k)}$ can be chosen to vanish this norm at each boundaries $x = \bar{x}_i$ and $y = \bar{y}_j$. Such Killing vector fields are

$$\begin{aligned}\xi_{[\bar{x}_i]} &= \frac{2}{X'(\bar{x}_i)} \left((q_1 + \bar{x}_i^2) \frac{\partial}{\partial \psi_0} - \bar{x}_i \frac{\partial}{\partial \psi_1} + \frac{\partial}{\partial \psi_2} \right) , \\ \xi_{[\bar{y}_j]} &= \frac{2}{Y'(\bar{y}_j)} \left((q_2 + \bar{y}_j^2) \frac{\partial}{\partial \psi_0} - \bar{y}_j \frac{\partial}{\partial \psi_1} + \frac{\partial}{\partial \psi_2} \right) .\end{aligned}\quad (4.117)$$

We normalize them in such a way that the surface gravity is equal to be unity,

$$\frac{g^{ab}(\partial_a \xi_{[\bar{x}_i]}^2)(\partial_b \xi_{[\bar{x}_i]}^2)}{4\xi_{[\bar{x}_i]}^2} \Big|_{x=\bar{x}_i} = \frac{g^{ab}(\partial_a \xi_{[\bar{y}_j]}^2)(\partial_b \xi_{[\bar{y}_j]}^2)}{4\xi_{[\bar{y}_j]}^2} \Big|_{y=\bar{y}_j} = 1 . \quad (4.118)$$

Thus if Killing vector fields $\xi_{[\bar{x}_i]}$ and $\xi_{[\bar{y}_j]}$ have period 2π , the metric extends smoothly onto the boundaries. Because we have four vector fields $\xi_{[\bar{x}_i]}$ and $\xi_{[\bar{y}_j]}$ with three independent Killing vectors, they must satisfy a linear relation which can be written as:

$$n_1 \xi_{[\bar{x}_1]} + n_2 \xi_{[\bar{x}_2]} + m_1 \xi_{[\bar{y}_2]} + m_2 \xi_{[\bar{y}_3]} = 0, \quad (4.119)$$

with integral coefficients (n_1, n_2, m_1, m_2) which are assumed to be coprime. Any three of the integers also must be coprime for avoiding conical singularities. Substituting Eq. (4.117) into Eq. (4.119), it can be solved as

$$\begin{aligned}\frac{n_1}{(\bar{x}_3 - \bar{x}_1)[q + (\bar{x}_2 - \bar{y}_2)(\bar{x}_2 - \bar{y}_3)]} &= \frac{n_2}{(\bar{x}_3 - \bar{x}_2)[q + (\bar{x}_1 - \bar{y}_2)(\bar{x}_1 - \bar{y}_3)]} \\ &= \frac{m_1}{(\bar{y}_2 - \bar{y}_1)[q - (\bar{x}_1 - \bar{y}_3)(\bar{x}_2 - \bar{y}_3)]} = \frac{m_2}{(\bar{y}_3 - \bar{y}_1)[q - (\bar{x}_1 - \bar{y}_2)(\bar{x}_2 - \bar{y}_2)]} ,\end{aligned}\quad (4.120)$$

where

$$q \equiv q_1 - q_2 = \frac{\bar{x}_1 \bar{x}_2 + \bar{x}_1 \bar{x}_3 + \bar{x}_2 \bar{x}_3 - \bar{y}_1 \bar{y}_2 - \bar{y}_1 \bar{y}_3 - \bar{y}_2 \bar{y}_3}{2} . \quad (4.121)$$

Without loss of generality the value of (4.120) can be set to 1 and b_2 to 0 since there are redundant degrees of freedom by the scaling and shift transformations. Then we have $\bar{y}_1 = 0$ by Eq. (4.114), and obtain by Eq. (4.120) as

$$n_1 = (\bar{x}_3 - \bar{x}_1)[q + (\bar{x}_2 - \bar{y}_2)(\bar{x}_2 - \bar{y}_3)] , \quad (4.122)$$

$$n_2 = (\bar{x}_3 - \bar{x}_2)[q + (\bar{x}_1 - \bar{y}_2)(\bar{x}_1 - \bar{y}_3)] , \quad (4.123)$$

$$m_1 = \bar{y}_2[q - (\bar{x}_1 - \bar{y}_3)(\bar{x}_2 - \bar{y}_3)] , \quad (4.124)$$

$$m_2 = \bar{y}_3[q - (\bar{x}_1 - \bar{y}_2)(\bar{x}_2 - \bar{y}_2)] , \quad (4.125)$$

where

$$q = \frac{\bar{x}_1 \bar{x}_2 + \bar{x}_1 \bar{x}_3 + \bar{x}_2 \bar{x}_3 - \bar{y}_2 \bar{y}_3}{2} . \quad (4.126)$$

Therefore, we have reduced the problem of metric regularity on compact manifolds to solving four coupled algebraic equations (4.122)–(4.125) for a set of coprime integers (n_1, n_2, m_1, m_2) under the constraints for the real roots $\bar{x}_i$ and $\bar{y}_i$: These are

$$\begin{aligned}\bar{x}_3 &= \bar{y}_2 + \bar{y}_3 - \bar{x}_1 - \bar{x}_2, \\ \bar{x}_1 &< \bar{x}_2 < \bar{x}_3, \quad 0 < \bar{y}_2 < \bar{y}_3, \quad \bar{x}_2 < \bar{y}_2.\end{aligned}\tag{4.127}$$

This system includes the toric Sasaki–Einstein case. When we take $q = 0$, Eqs. (4.122)–(4.125) give rise to the condition

$$n_1 + n_2 + m_1 + m_2 = 0, \tag{4.128}$$

which is surely the condition of the toric Sasaki–Einstein metrics L^{n_1, n_2, m_1} on $S^2 \times S^3$ discussed in [17, 15, 16]. When the parameter q is non-zero, we find Sasaki with torsion metrics L^{n_1, n_2, m_1, m_2} parameterized by independent four integers. Indeed, we confirm the existence of such integer sets: The table 4.1 shows numerical results of solving four coupled algebraic equations (4.122)–(4.125) for some sets of coprime integers (n_1, n_2, m_1, m_2) under the conditions (4.127). The global regularity is provided by all the integer sets at the left side columns.

Topological feature of generalized $Y^{p,q}$

Making use of the 5-dimensional minimal gauged supergravity solution (4.110), we have discussed the global feature on compact manifolds M_5 . Then it has been resulted that they can be regarded as a generalization of $L^{a,b,c}$. Now we can also argue about the cohomogeneity-1 case of our solution, which can be regarded as a generalization of $Y^{p,q}$ [13]. Owing to the higher symmetry, we show rather precise discussions for the global properties than the L^{n_1, n_2, m_1, m_2} case, i.e., the topological aspects.

In the cohomogeneity-1 case, our solution acquires the following form:

$$\begin{aligned}g &= (\check{\alpha} - x)(d\theta^2 + \sin^2 \theta d\phi^2) + \frac{dx^2}{Q(x)} + Q(x)(d\psi_1 + \cos \theta d\phi)^2 \\ &\quad + 4 \left(d\psi_0 + \left(x + \frac{q}{x - \check{\alpha}} \right) d\psi_1 + \left(x - \check{\alpha} + \frac{q}{x - \check{\alpha}} \right) \cos \theta d\phi \right)^2,\end{aligned}\tag{4.129}$$

where

$$Q(x) = \frac{4x^3 + (1 - 12\check{\alpha})x^2 + (8q - 2\check{\alpha} + 12\check{\alpha}^2)x + k}{\check{\alpha} - x}, \tag{4.130}$$

and q , $\check{\alpha}$ and k are free parameters. The metric is certainly a Sasaki with torsion metric, satisfying the EOMs of 5-dimensional minimal gauged supergravity with the following Maxwell potential:

$$A = -\frac{2\sqrt{3}q}{x - \check{\alpha}}(d\psi_1 + \cos \theta d\phi). \tag{4.131}$$

n_1	n_2	m_1	m_2	$\bar{x}_1$	$\bar{x}_2$	$\bar{x}_3$	$\bar{y}_1$	$\bar{y}_2$	$\bar{y}_3$	q
-4	-3	-1	-2	-1.32023	-1.25127	3.11486	0	0.167499	0.375858	-3.21042
-4	-2	-1	-3	-1.27727	-1.14007	3.17205	0	0.155888	0.59882	-3.15254
-4	-1	-2	-3	-1.25653	-1.04966	3.20068	0	0.329791	0.564696	-3.12434
-4	1	-2	-3	-1.18938	-0.78852	3.0232	0	0.372652	0.672647	-2.6462
-4	2	-1	-3	-1.11468	-0.543466	2.82041	0	0.188869	0.9734	-2.12735
-4	3	-1	-2	-1.11385	-0.202506	2.46249	0	0.263875	0.882254	-1.62438
-3	-2	-1	-4	-1.15876	-1.09101	3.24621	0	0.142997	0.853444	-3.08052
-3	-1	-2	-4	-1.14654	-1.00989	3.26472	0	0.302705	0.805588	-3.06305
-3	1	-2	-4	-1.05629	-0.741916	3.11358	0	0.3276	0.987783	-2.56939
-3	2	-1	-4	-0.899549	-0.461233	3.04032	0	0.146529	1.53301	-1.97347
-2	-1	-3	-4	-1.06256	-0.9939	3.29153	0	0.483516	0.751555	-3.0381
-2	1	-3	-4	-0.969754	-0.731764	3.13442	0	0.536657	0.896244	-2.55231
-1	2	-3	-4	-0.761789	-0.47932	2.987	0	0.578562	1.16732	-2.00871
-1	4	-3	-2	-0.358631	0.309102	2.83766	0	0.50795	2.28018	-0.704807
1	4	-3	-2	0.167966	1.36545	2.52543	0	1.90031	2.15854	0
2	1	-3	4	1.58023	2.19861	2.46249	0	2.66499	3.57634	1.62438
2	1	3	4	2.739	2.94736	3.11486	0	4.36613	4.43508	3.21042
2	3	-4	-1	0.36689	0.625118	2.52543	0	1.15997	2.35746	0
2	3	-4	1	0.557479	2.32971	2.83766	0	2.52855	3.19629	0.704807
3	1	-2	4	1.84701	2.63154	2.82041	0	3.36388	3.93509	2.12735
3	1	2	4	2.57323	3.01616	3.17205	0	4.31211	4.44931	3.15254
3	2	-1	4	2.35055	2.65055	3.0232	0	3.81172	4.21258	2.6462
3	2	1	4	2.63598	2.87089	3.20068	0	4.25033	4.45721	3.12434
4	1	-2	3	1.50731	2.8938	3.04032	0	3.50156	3.93987	1.97347
4	1	2	3	2.39276	3.10321	3.24621	0	4.33722	4.40497	3.08052
4	2	-1	3	2.1258	2.78598	3.11358	0	3.8555	4.16987	2.56939
4	2	1	3	2.45913	2.96201	3.26472	0	4.2746	4.41126	3.06305
4	3	-2	1	1.81967	2.40843	2.987	0	3.46631	3.74878	2.00871
4	3	-1	2	2.23817	2.59776	3.13442	0	3.86618	4.10417	2.55231
4	3	1	2	2.53997	2.80801	3.29153	0	4.28543	4.35409	3.0381

Table 4.1 numerical results for consistent integer sets

The torsion 3-form is also given by $T = *F/\sqrt{3}$.

Let us study its global properties in accordance with [13, 14]. For this analysis, we perform the following coordinate transformation

$$\psi_1 = -\psi + \Psi, \quad \psi_0 = \check{\alpha}\psi. \quad (4.132)$$

Then the metric is

$$\begin{aligned} g_5 = & (\check{\alpha} - x)(d\theta^2 + \sin^2 \theta d\phi^2) + \frac{dx^2}{Q(x)} + \frac{4\check{\alpha}^2 Q(x)}{F(x)}(d\psi - \cos \theta d\phi)^2 \\ & + F(x) \left(d\Psi - f(x)(d\psi - \cos \theta d\phi) \right)^2, \end{aligned} \quad (4.133)$$

where

$$F(x) = Q(x) + 4 \left(x + \frac{q}{x - \check{\alpha}} \right)^2, \quad (4.134)$$

$$f(x) = \frac{Q(x) + 4 \left(x + \frac{q}{x - \check{\alpha}} \right) \left(x - \check{\alpha} + \frac{q}{x - \check{\alpha}} \right)}{F(x)}. \quad (4.135)$$

Intuitively, introducing $\check{\alpha}$ means avoiding the illness of 4-dimensional Kähler transverse structure. In this expression, the Sasaki–Einstein metric $Y^{p,q}$ is located at $q = 0$ and

$\check{\alpha} = 1/6$. The homogeneous Sasaki–Einstein metric $T^{1,1}$ also arises if we take the coordinate transformation $x = C/6y$ and send $C \rightarrow 0$. Moreover, it can be seen that when $k = -4\check{\alpha}^3 + \check{\alpha}^2 - 8q\check{\alpha}$, the function $Q(x)$ degenerates to a polynomial of degree-2 and we have $Q = -4x^2 + (8\check{\alpha} - 1)x - 4\check{\alpha}^2 + \check{\alpha} - 8q$. Then the metric is the standard S^5 metric when $q = 0$. Otherwise, $Q(x)$ is a rational function and henceforth we will focus on the case.

The positive definiteness of the metric g_5 imposes the restriction to the metric function: There exist three distinct real roots $\bar{x}_1$, $\bar{x}_2$ and $\bar{x}_3$ of the equation $Q(x) = 0$, such as,

$$\bar{x}_1 < \bar{x}_2 < \bar{x}_3, \quad \bar{x}_2 < \check{\alpha}, \quad (4.136)$$

and the coordinate x takes the range $\bar{x}_1 \leq x \leq \bar{x}_2$.

Firstly, let us reveal the topological property of the 4-dimensional part in this expression, which does not have Kähler structure but global advantage. This part means

$$g_{\check{B}} = (\check{\alpha} - x)(d\theta^2 + \sin^2 \theta d\phi^2) + \frac{dx^2}{Q(x)} + \frac{4\check{\alpha}^2 Q(x)}{F(x)}(d\psi - \cos \theta d\phi)^2. \quad (4.137)$$

We show that $g_{\check{B}}$ can extend globally on S^2 -bundle over S^2 . Fixing the coordinates θ and ϕ in Eq. (4.137) and introducing a new coordinate $r = 2|x - \bar{x}_i|^{1/2}/|Q'(\bar{x}_i)|^{1/2}$, we can evaluate the behavior near $x = \bar{x}_i$ of the residual line element as

$$dr^2 + \left(\frac{\check{\alpha}(\bar{x}_i - \check{\alpha})Q'(\bar{x}_i)}{2(\bar{x}_i(\bar{x}_i - \check{\alpha}) + q)} \right)^2 r^2 d\psi^2. \quad (4.138)$$

Hence, avoiding conical singularities at $x = \bar{x}_i$ requires both of the condition

$$\frac{\check{\alpha}(\bar{x}_i - \check{\alpha})Q'(\bar{x}_i)}{\bar{x}_i(\bar{x}_i - \check{\alpha}) + q} = \pm n, \quad (4.139)$$

and the range of ψ given by $0 \leq \psi \leq 4\pi/n$ with a constant $n \neq 0$. Eq. (4.139) is explicitly written as

$$(12\check{\alpha} - n_i)\bar{x}_i^2 + \check{\alpha}(2 - 24\check{\alpha} + n_i)\bar{x}_i + 2\check{\alpha}(4q - \check{\alpha} + 6\check{\alpha}^2) - qn_i = 0, \quad (i = 1, 2), \quad (4.140)$$

where n_i take $\pm n$, respectively. Thus, two of three parameters q , k and $\check{\alpha}$ are fixed by the regular condition (4.140). Since the Chern number is calculated as

$$c_1(\check{B}) = \frac{n}{4\pi} \int_{S^2} d(-\cos \theta d\phi) = n, \quad (4.141)$$

the 4-dimensional space $\check{B}$ is a trivial bundle $S^2 \times S^2$ for even integer n and a twisted S^2 -bundle for odd integer n , respectively.

We now deal with a simple case, setting $n_1 = n_2 = n$. Eq. (4.140) possesses the trivial case when $q = 0$, $\check{\alpha} = 1/6$ and $n = 2$, reproducing the Sasaki–Einstein metric $Y^{p,q}$. We

obtain more general solutions for discretized parameters in the case $\check{\alpha} \neq n/12$ nor $n/16$. Since Eq. (4.140) becomes

$$q = \frac{(2-n)\check{\alpha}(-n+4\check{\alpha}+4n\check{\alpha})}{4(n-16\check{\alpha})(n-12\check{\alpha})}, \quad k = \frac{\check{\alpha}(-n+4\check{\alpha}+8n\check{\alpha}-48\check{\alpha}^2)L(\check{\alpha})}{4(n-16\check{\alpha})(n-12\check{\alpha})^3}, \quad (4.142)$$

with

$$\begin{aligned} L(\check{\alpha}) = & 2n^2 - n^3 + 4(n^3 - n^2 - 6n)\check{\alpha} \\ & + 16(n^2 + 16n + 4)\check{\alpha}^2 - 192(7n + 8)\check{\alpha}^3 + 9216\check{\alpha}^4, \end{aligned} \quad (4.143)$$

the roots of the function $Q(x)$, $\bar{x}_1$, $\bar{x}_2$ and $\bar{x}_3$ are given by

$$\bar{x}_{1,2} = \frac{2\check{\alpha} + n\check{\alpha} - 24\check{\alpha}^2 \pm \sqrt{\frac{(n-2)n\check{\alpha}(-n+5n\check{\alpha}10\check{\alpha}-48\check{\alpha}^2)}{n-16\check{\alpha}}}}{2(n-12\check{\alpha})}, \quad (4.144)$$

$$\bar{x}_3 = \frac{-n+4\check{\alpha}+8n\check{\alpha}-48\check{\alpha}^2}{4(n-12\check{\alpha})}, \quad (4.145)$$

where the choice of the sign in (4.144) depends on the sign of $n-12\check{\alpha}$. $\bar{x}_2$ must satisfy all the constraint, i.e., the reality condition and the inequalities (4.136). This requires the following ranges of $\check{\alpha}$ for each integer n :

$$(a) \quad n \geq 4, \quad \check{\alpha}_1 < \check{\alpha} < \frac{n}{4(n+1)}, \quad (4.146)$$

$$(b) \quad n = 1, \quad \frac{15 - \sqrt{33}}{96} < \check{\alpha} < \frac{1}{8}, \quad (4.147)$$

$$(c) \quad n \leq -1, \quad \frac{n}{8} < \check{\alpha} < \check{\alpha}_1 \quad \text{or} \quad \check{\alpha}_2 < \check{\alpha} < \check{\alpha}_3, \quad (4.148)$$

where the quantities $\check{\alpha}_1$, $\check{\alpha}_2$ and $\check{\alpha}_3$ are defined by

$$\check{\alpha}_1 = \frac{1}{96}(10 + 5n - \sqrt{100 - 92n + 25n^2}), \quad (4.149)$$

$$\check{\alpha}_2 = \frac{1}{96}(10 + 5n + \sqrt{100 - 92n + 25n^2}), \quad (4.150)$$

$$\check{\alpha}_3 = \frac{1}{48}(5 + n + \sqrt{25 - 14n + n^2}). \quad (4.151)$$

The regular condition for the 5-dimensional metric g_5 gives rise to a further constraint from the period of the fiber direction Ψ in Eq. (4.133). Namely, tuning this period should realize the 5-dimensional description as a principal S^1 -bundle over $\check{B}$. Since the connection 1-form is given by

$$\mathcal{A}_{[\check{B}]} = f(x)(d\psi - \cos\theta d\phi), \quad (4.152)$$

the periods P_i ($i = 1, 2$) are calculated as [14],

$$\begin{aligned} P_1 &= \frac{1}{2\pi} \int_{C_1} d\mathcal{A}_{[\check{B}]} = \frac{2}{n}(f(\bar{x}_2) - f(\bar{x}_1)), \\ P_2 &= \frac{1}{2\pi} \int_{C_2} d\mathcal{A}_{[\check{B}]} = 2f(\bar{x}_2), \end{aligned} \quad (4.153)$$

where n is the Chern number* given by Eq. (4.141), and $\mathcal{C}_1$ and $\mathcal{C}_2$ represent the basis for $H_2(\check{B}, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$. Now we require

$$\frac{f(\bar{x}_1)}{f(\bar{x}_2)} = \frac{\ell}{m}, \quad (4.154)$$

where $\ell, m \in \mathbb{Z}$. Then, the period of $\kappa^{-1}d\mathcal{A}_{[\check{B}]} / 2\pi$ becomes an integer if we set $\kappa = 2hf(\bar{x}_2)/(mn)$ with $h = \gcd(\ell - m, nm)$. Thus we take the range $0 \leq \Psi \leq 2\pi\kappa$. In this setup, we can confirm the appropriate discretization for free parameters numerically: A calculation shows that our solution indeed admits the parameter $\check{\alpha}$ satisfying the condition (4.154), which means it is topologically an S^1 -bundle over $\check{B}$, parameterized by three integers ℓ, m and n . Hence our solution seems a generalization of $Y^{p,q}$, labeled one more quantum numbers with the suitable topology.

In addition, it is straightforward to verify that the following four Killing vectors

$$\begin{aligned} \xi_{[0]} &= \frac{\partial}{\partial\phi} + \frac{\partial}{\partial\psi}, \quad \xi_{[\pi]} = -\frac{\partial}{\partial\phi} + \frac{\partial}{\partial\psi}, \\ \xi_{[\bar{x}_1]} &= \frac{2}{n} \left(\frac{\partial}{\partial\psi} + f(\bar{x}_1) \frac{\partial}{\partial\Psi} \right), \quad \xi_{[\bar{x}_2]} = \frac{2}{n} \left(\frac{\partial}{\partial\psi} + f(\bar{x}_2) \frac{\partial}{\partial\Psi} \right), \end{aligned} \quad (4.155)$$

vanish respectively at $\theta = 0, \theta = \pi, x = \bar{x}_1$ and $x = \bar{x}_2$ with the surface gravity 1, and connect each other to a linear relation,

$$(\check{N}_1 - \check{N}_2)(\xi_{[0]} + \xi_{[\pi]}) + n\check{N}_2\xi_{[\bar{x}_1]} - n\check{N}_1\xi_{[\bar{x}_2]} = 0, \quad (4.156)$$

with $\check{N}_1 = n\ell/h \in \mathbb{Z}$, $\check{N}_2 = nm/h \in \mathbb{Z}$ (cf. Eq. (4.119)). Moreover, the volume formulation is obtained as

$$\text{Vol}(M_5) = \pi^3 \left| \frac{32\check{\alpha}\kappa(\bar{x}_2 - \bar{x}_1)(2\check{\alpha} - \bar{x}_1 - \bar{x}_2)}{n} \right|. \quad (4.157)$$

Since $\check{B}$ is a simply-connected manifold, it follows that M_5 is also simply-connected. It is notable that M_5 should be a spin manifold [14]. Smale's theorem states that any simply-connected compact five-manifold which is spin and has no torsion in the second homology group is diffeomorphic to $S^5 \sharp k(S^2 \times S^3)$ for some non-negative integer k . Therefore, together with the analysis in App. A of [13], we can confirm that our gravitational solution properly has the case of topologically $S^2 \times S^3$.

Non-compact manifolds in 11 dimensions

Next, we turn to discussing the global structure of the 11-dimensional supergravity solution. The solution possesses the metric function (4.105) that is identical to the chargeless

*The cycle $\mathcal{C}_1$ is the S^2 -fiber of $\check{B}$ at some fixed point (θ, ϕ) on the base space. While, $\mathcal{C}_2$ is the sub-manifold S^2 of $\check{B}$ at $x = \bar{x}_2$, where the length of $\partial/\partial\psi$ vanishes.

case, and the nontrivial fiber component (4.109) from the matter-flux effect. Considering the charge effect, we need to focus on the global behavior of the Killing vector field. We assume that the functions $X_v(x_v)$ take the form

$$\begin{aligned} X(x) &\equiv X_1(x_1) = c(x - a)P(x) , \\ Y_k(y_k) &\equiv X_{k+1}(x_{k+1}) = c \prod_{i=1}^5 (y_k - \beta_i) , \quad k = 1, \dots, 4 , \end{aligned} \quad (4.158)$$

where $P(x)$ is a positive definite polynomial of degree-4 constrained by EOMs, and a, c, β_i are real constants satisfying

$$c > 0, \quad \beta_1 < \beta_2 < \dots < \beta_5 < a. \quad (4.159)$$

Then we choose the region of the coordinates x, y_k as

$$\beta_1 \leq y_1 \leq \beta_2 \leq \dots \leq y_4 \leq \beta_5 < a \leq x < \infty. \quad (4.160)$$

The fact that the region of x is infinite corresponds to non-compactness of manifold. This is relevant to the absence of the cosmological constant in the 11-dimensional supergravity. Then the metric is positive definite except for the boundaries $y_k = \beta_k, y_k = \beta_{k+1}$ and $x = a$. From App. B we see that the curvature is finite at the points $y_k = y_{k+1} = \beta_k$. Analogous to the 5-dimensional case, calculations yield that the following vector fields are Killing vector fields vanishing at the boundaries $x = a, y_k = \beta_k$ and $y_k = \beta_{k+1}$, ($k = 1, 2, 3, 4$), respectively,

$$\begin{aligned} \xi_{[x=a]} &= \frac{2}{X'(a)} \left((N_1(a) + a^5) \frac{\partial}{\partial \psi_0} + \sum_{\ell=1}^5 (-1)^k \ell a^{5-\ell} \frac{\partial}{\partial \psi_\ell} \right) , \\ \xi_{[y_k=\beta_k]} &= \frac{2}{Y'_k(\beta_k)} \left((N_{k+1}(\beta_k) + \beta_k^5) \frac{\partial}{\partial \psi_0} + \sum_{\ell=1}^5 (-1)^\ell \beta_k^{5-\ell} \frac{\partial}{\partial \psi_\ell} \right) , \\ \xi_{[y_k=\beta_{k+1}]} &= \frac{2}{Y'_k(\beta_{k+1})} \left((N_{k+1}(\beta_{k+1}) + \beta_{k+1}^5) \frac{\partial}{\partial \psi_0} + \sum_{\ell=1}^5 (-1)^\ell \beta_{k+1}^{5-\ell} \frac{\partial}{\partial \psi_\ell} \right) . \end{aligned} \quad (4.161)$$

These Killing vector fields have a unit surface gravity. If we impose the condition $q_2 = q_3 = q_4 = q_5$, then we have $N_2 = N_3 = N_4 = N_5$, which implies the relation $\xi_{[y_k=\beta_k]} = \xi_{[y_{k-1}=\beta_k]}$, ($k = 2, 3, 4$). Hence we can use them as the new Killing coordinates ϕ_α with period 2π representing the canonical coordinate of torus T^6 ,

$$\frac{\partial}{\partial \phi_0} = \xi_{[x=a]} , \quad \frac{\partial}{\partial \phi_1} = \xi_{[y_1=\beta_1]} , \quad (4.162)$$

$$\frac{\partial}{\partial \phi_k} = \xi_{[y_k=\beta_k]} = \xi_{[y_{k-1}=\beta_k]} \quad (k = 2, 3, 4) , \quad \frac{\partial}{\partial \phi_5} = \xi_{[y_4=\beta_5]} . \quad (4.163)$$

Chap. 5

Spinning solutions in Einstein–Gauss–Bonnet gravity

In this chapter, we present the investigation in one of the higher derivative gravity, Einstein–Gauss–Bonnet (EGB) gravity. We obtain new exact samples of Sasakian geometry, at the same time overcoming the rotational-spacetime problem in this theory [29]. In Sec. 5.1, we explain the already-known solutions and their problem in EGB gravity. It has been awaited so far that the rotational black hole is constructed in exact form. In Sec. 5.2, owing to Sasakian structure, we obtain new exact solutions as unexpectedly large class. It is also revealed that they have rich properties and interesting location. In Sec. 5.3, we evaluate the conserved charges by the Abbott–Deser (AD) formulation. We elucidate the reasons why the present setup needs a care to the current, then calculating out the finite angular momentum properly.

5.1 Theoretical backgrounds of EGB gravity

Due to modern studies of quantum gravity, Einstein–Gauss–Bonnet gravity has attracted much attention recently. Among the active arguments about the necessity of incorporating the quantum effect to general relativity, the potential of EGB gravity has been indicated as the quantum correction of the low-energy effective theory originating from superstring theory. It was discussed that in the α' -expansion the Gauss–Bonnet term appears as the first curvature correction to Einstein gravity [33, 34, 35]. Past research has unveiled some advantages of this theory, such as the ghost-free renormalizability [30, 31] or the quasi-linear property [32].

In EGB gravity one of the most fascinating solutions describes the stationary black hole spacetime, as in general relativity. The static spherically symmetric black hole was discovered in that corrected gravitational theory a long time ago [90, 91], and has been

studied in many stimulating investigations up to the present. The research on this black hole solution disclosed its theoretical relationship to the one in Einstein gravity, beyond the similar appearance of the Schwarzschild–Tangherlini solution: As the Gauss–Bonnet coupling goes weaker, one of two branches in that solution gets closer to the spacetime expected from general relativity. The extension to add the electric charge was also constructed in that gravitational theory with Maxwell electrodynamics [92], resembling the Reissner–Nordström solution in the previous theory. On the other hand, studies in the black hole have revealed many peculiar features of EGB gravity absent from its counterpart in Einstein gravity. For instance, the other branch of that solution goes away from general relativistic expectation in the weak coupling limit. Massless or negative mass black holes exist on that branch without the negative cosmological constant.

A much more troublesome situation confronts us in pioneering new exact solutions. Less exact solutions have been established than general relativity with the interference of the higher power term, while it has been confirmed that somewhat various solutions happen to appear [37, 38] only when the Gauss–Bonnet coupling is chosen to the special value [39]. This trouble is particularly noticeable when finding rotating black hole solutions. These solutions could be obtained as the generalization of the Myers–Perry solution [1] for some particular meaning, but such exact solutions have not been obtained yet. On the other hand, recent numerical analysis has indicated the existence of the rotating black hole in EGB theory, constructing the numerical solution in the 5-dimensional spacetime [40, 41]. Hence, investigating exact stationary solutions beyond static is an important theme in EGB gravity nowadays. In this context, it is still unclear how far insight accumulated in Einstein gravity is available for the present theory. It was still proven that the Kerr–Schild ansatz goes wrong in EGB gravity, besides one specific exception [36]. The exceptional case is just when the Gauss–Bonnet coupling is chosen specially only in the 5-dimensional spacetime, and thus any extension to either higher dimensions or matter-field couplings has not been found yet. Recent progress also indicates some defects in global behavior such as the absence of circularity or horizons [93, 94].

5.1.1 General feature

In the D -dimensional spacetime, EGB gravity with the cosmological constant Λ is described by the following action:

$$S_{\text{EGB}} = \frac{1}{16\pi G} \int \sqrt{|g|} d^D x (R + \alpha \mathcal{L}_{\text{GB}} - 2\Lambda), \quad (5.1)$$

$$\mathcal{L}_{\text{GB}} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}, \quad (5.2)$$

with α the coupling constant of the Gauss–Bonnet term and G the gravitational coupling constant. This square order term can be written in a more systematic form:

$$\mathcal{L}_{\text{GB}} = 6\delta_{[\mu_1}^{\nu_1}\delta_{\mu_2}^{\nu_2}\delta_{\mu_3}^{\nu_3}\delta_{\mu_4}^{\nu_4]}R^{\mu_1\mu_2}_{\nu_1\nu_2}R^{\mu_3\mu_4}_{\nu_3\nu_4}, \quad (5.3)$$

where $[\cdots]$ denotes the anti-symmetrization for indexes. This acts as

$$M_{[\mu_1\mu_2]} = \frac{1}{2}(M_{\mu_1\mu_2} - M_{\mu_2\mu_1}), \quad (5.4)$$

$$M_{[\mu_1\mu_2\cdots\mu_n]} = \frac{1}{n!}(M_{\mu_1\mu_2\cdots\mu_n} - M_{\mu_2\mu_1\cdots\mu_n} + (\text{all the other permutations})). \quad (5.5)$$

The systematic expression (5.3) shows some features of EGB gravity. The anti-symmetrization for four indexes indicates that this term cannot construct if the spacetime dimensions are lower than 4. Since the volume form ε has an identity in the D -dimensional spacetime,

$$\delta_{[\mu_1}^{\nu_1}\delta_{\mu_2}^{\nu_2}\cdots\delta_{\mu_{D-1}}^{\nu_{D-1}}\delta_{\mu_D}^{\nu_D]} = \frac{1}{D!}\varepsilon_{\mu_1\mu_2\cdots\mu_{D-1}\mu_D}\varepsilon^{\nu_1\nu_2\cdots\nu_{D-1}\nu_D}, \quad (5.6)$$

the Gauss–Bonnet term falls into a characteristic class with the Hodge dual in the 4-dimensional spacetime:

$$\mathcal{L}_{\text{GB}} \sim \varepsilon_{\mu_1\mu_2\mu_3\mu_4}R^{\mu_1\mu_2}_{\nu_1\nu_2}R^{\mu_3\mu_4}_{\nu_3\nu_4}dx^{\nu_1}\wedge dx^{\nu_2}\wedge dx^{\nu_3}\wedge dx^{\nu_4}. \quad (5.7)$$

This characteristic class is called Euler class. Because this term is the surface term in this dimensions, Einstein–Gauss–Bonnet gravity is nontrivial in the dimensions higher than 4. On the other hand, the Ricci curvature is also able to be written in the similar form,

$$R = \delta_{[\mu_1}^{\nu_1}\delta_{\mu_2}^{\nu_2]}R^{\mu_1\mu_2}_{\nu_1\nu_2}. \quad (5.8)$$

This results from the fact that in the 2-dimensional spacetime the Ricci curvature coincides with Euler class. Therefore, Einstein–Gauss–Bonnet gravity can be said one of the natural generalization of general relativity into the higher-dimensional spacetime. This feature may be regarded as assuring the good property of this theory, e.g., the ghost-free renormalizability or the quasi-linear property [30, 32].

Including this theory, a generalization has been investigated intensively: It is so-called Lovelock gravity [95]. The gravitational action is given by

$$S_{\text{Lovelock}} = \sum_{p=0}^{[\frac{D-1}{2}]} \alpha_p \mathcal{L}_p, \quad (5.9)$$

$$\mathcal{L}_p = \delta_{[\mu_1}^{\nu_1}\delta_{\mu_2}^{\nu_2}\cdots\delta_{\mu_{2p-1}}^{\nu_{2p-1}}\delta_{\mu_{2p}}^{\nu_{2p}]} \underbrace{R^{\mu_1\mu_2}_{\nu_1\nu_2}\cdots R^{\mu_{2p-1}\mu_{2p}}_{\nu_{2p-1}\nu_{2p}}}_{p \text{ times}}, \quad (5.10)$$

The $p = 1$ case is the Ricci term, the $p = 2$ case is the Gauss–Bonnet term and larger p cases are similar objects, that is, Hodge duals of the Euler class in the higher-dimensional

spacetimes. The upper bound for p in each spacetime dimensions is a result of the above construction. The $p = 0$ case is identified to the cosmological constant, setting $\mathcal{L}_0$ to -2 . The clearer study in EGB gravity has the expectation to explore this family more comfortably.

Adding some matter field, equations of motion for EGB gravity read

$$S = S_{\text{EGB}} + S_{\text{Matter}}, \quad (5.11)$$

$$G_{\mu\nu} + \alpha H_{\mu\nu} + \Lambda g_{\mu\nu} = (16\pi G) T_{\mu\nu}^{(\text{Matter})}, \quad (5.12)$$

where

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R, \quad (5.13)$$

$$H_{\mu\nu} = 2(RR_{\mu\nu} - 2R_{\mu\rho}R_{\nu}^{\rho} - 2R^{\rho\sigma}R_{\mu\rho\nu\sigma} + R_{\mu\rho\sigma\lambda}R_{\nu}^{\rho\sigma\lambda}) \\ - \frac{1}{2}g_{\mu\nu}(R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}). \quad (5.14)$$

This somewhat intricate expression also can be rearranged into

$$G_{\mu}^{\nu} = -\frac{3}{2}\delta_{[\mu}^{\nu}\delta_{\mu_1}^{\nu_1}\delta_{\mu_2}^{\nu_2]}R^{\mu_1\mu_2}_{\nu_1\nu_2}, \quad (5.15)$$

$$H_{\mu}^{\nu} = -15\delta_{[\mu}^{\nu}\delta_{\mu_1}^{\nu_1}\delta_{\mu_2}^{\nu_2}\delta_{\mu_3}^{\nu_3}\delta_{\mu_4}^{\nu_4]}R^{\mu_1\mu_2}_{\nu_1\nu_2}R^{\mu_3\mu_4}_{\nu_3\nu_4}. \quad (5.16)$$

This simplification is based on the following decomposition:

$$\delta_{[\mu}^{\nu}\delta_{\mu_1}^{\nu_1}\delta_{\mu_2}^{\nu_2}\cdots\delta_{\mu_{n-1}}^{\nu_{n-1}}\delta_{\mu_n}^{\nu_n}] = \frac{1}{n+1}\delta_{\mu}^{\nu}\delta_{[\mu_1}^{\nu_1}\delta_{\mu_2}^{\nu_2}\cdots\delta_{\mu_{n-1}}^{\nu_{n-1}}\delta_{\mu_n}^{\nu_n}] - \frac{n}{n+1}\delta_{\mu}^{[\nu_1}\delta_{[\mu_1}^{|\nu|}\delta_{\mu_2}^{\nu_2}\cdots\delta_{\mu_{n-1}}^{\nu_{n-1}}\delta_{\mu_n}^{\nu_n]}], \quad (5.17)$$

where $|\nu|$ means the exclusion of the index ν from the anti-symmetrization.

Static black hole

In this gravitational theory, the static solution has prevailed well, finding out the static black hole solution. This black hole can be seen as the counterpart of the Schwarzschild–Tangherlini solution in some meaning, then the exploration has proceeded to constructing the Reissner–Nordström counterpart [90, 92].

In order to describe the electric charge for the gravitational solution, Maxwell field $F = dA$ is introduced as the matter content,

$$S_{\text{Matter}} = -\frac{1}{4\pi g_c^2} \int \sqrt{|g|} d^D x F_{\mu\nu} F^{\mu\nu}, \quad (5.18)$$

with the gauge coupling constant g_c . This matter field obeys

$$\nabla_{\mu} F^{\mu\nu} = 0, \quad (5.19)$$

$$T_{\mu\nu}^{(\text{Matter})} = \frac{1}{2\pi g_c^2} (F_{\mu\lambda} F_{\nu}^{\lambda} - \frac{1}{4} g_{\mu\nu} F_{\lambda\rho} F^{\lambda\rho}). \quad (5.20)$$

Hereafter, we set the gauge coupling constant to $g_c = 4\sqrt{G}$ without loss of generality, absorbing it into the scale of the matter flux F .

The static black hole solution is written in the form,

$$ds^2 = -R(r)dt^2 + \frac{dr^2}{R(r)} + r^2 d\Omega_{D-2}^2, \quad (5.21)$$

$$R(r) = 1 + \frac{r^2}{2\tilde{\alpha}} \left\{ 1 \mp \sqrt{1 + 4\tilde{\alpha} \left(\frac{M}{r^{D-1}} + \frac{2\Lambda}{(D-1)(D-2)} - \frac{Q^2}{8(D-3)(D-2)r^{2D-4}} \right)} \right\}, \quad (5.22)$$

$$A_\mu dx^\mu = \frac{Q}{r^{D-3}} dt, \quad (5.23)$$

with $d\Omega_{D-2}^2$ the unit $(D-2)$ -dimensional sphere metric, $\tilde{\alpha} = (D-3)(D-4)\alpha$, M the mass parameter, and Q the electric charge parameter. This spacetime is static, spherically symmetric, and asymptotically (A)dS or flat. As shown above (5.22), this solution has two branches. The minus-sign case is called the Einstein branch because of its relation to general relativity; Taking the weak coupling expansion,

$$R(r) = \frac{(1 \mp 1)}{2\tilde{\alpha}} r^2 + 1 \mp \left(\frac{2\Lambda}{(D-1)(D-2)} r^2 + \frac{M}{r^{D-3}} - \frac{Q^2}{8(D-3)(D-2)r^{2D-6}} \right) + \mathcal{O}(\alpha), \quad (5.24)$$

the minus branch goes to the general relativistic one properly while the other goes bankrupt.

Among active studies of this solution, its exact extension can be said as one of the most important problem. While still unclear how extent of the insight in Einstein gravity is available for the present theory, one of the generalization has been achieved under the static condition: This is what is called the topological static black hole solutions [96]. The solutions are obtained by replacing the internal sphere $d\Omega_{D-2}^2$ of (5.21) to the other $(D-2)$ -dimensional space ds_{D-2}^2 . Then these variation has been constructed in exact form, but most of them possess some artificial boundary structure. In the general relativistic case, the residual Einstein equation is imposed to the $(D-2)$ -dimensional line element, restricting its Ricci tensor. In EGB gravity, usually these constraints are quite harder than in Einstein gravity, reached to the $(D-2)$ -dimensional Weyl tensor. Thus such variation in EGB gravity is usually less than in Einstein gravity, except for a special case.

On the other hand, pioneering the rotating solution has fallen in a more troublesome situation. Generalizing the Myers–Perry solution [1], the general relativistic rotating black hole solution, faces much uncertainty so that such exact solutions have not been obtained yet. Hence, investigating exact stationary solutions beyond static becomes important theme in EGB gravity. Recently, it was proven that the Kerr–Schild ansatz, one of the succeeded approach in Einstein gravity, goes wrong in EGB gravity besides one specific exception [36]. However, many of numerical analysis indicated the existence of the rotating solution, then the discovery of them has been awaited so far. This exception requires a specific

condition, which is the peculiar feature of this higher derivative theory. This condition is described by the value of coupling constant, causing the enhancement of the topological black hole solutions simultaneously. We shall explain this condition next.

5.1.2 Special coupling choice

As in the massless and chargeless case of (5.21), EGB gravity acquires two maximally symmetric vacua generically. It is worth mentioned that if these vacua degenerate, something special occurs in the gravitational solution space. This occurs when you choose the coupling constant as the following special value:

$$\alpha = -\frac{(D-1)(D-2)}{8(D-3)(D-4)\Lambda}. \quad (5.25)$$

It leads the metric function $R(r)$ with vanishing electric charge into a fairly simpler form

$$R(r) = \frac{r^2}{l^2} + 1 \mp \left(\frac{2M}{l^2 r^{D-5}} \right)^{\frac{1}{2}}, \quad (5.26)$$

$$l^2 = -\frac{(D-1)(D-2)}{4\Lambda}. \quad (5.27)$$

This specialty is also read from EOMs. Choosing the coupling constant as Eq. (5.25), Eq. (5.12) is factorized into

$$-\frac{15l^2}{2(D-3)(D-4)}\delta_{[\mu}^{[\nu}\delta_{\mu_1}^{\nu_1}\delta_{\mu_2}^{\nu_2}\delta_{\mu_3}^{\nu_3}\delta_{\mu_4}^{\nu_4]}\widehat{R}^{\mu_1\mu_2}_{\nu_1\nu_2}\widehat{R}^{\mu_3\mu_4}_{\nu_3\nu_4} = (16\pi G) T_{\mu}^{\nu}, \quad (5.28)$$

$$\widehat{R}^{\mu\rho}_{\nu\sigma} = R^{\mu\rho}_{\nu\sigma} + \frac{2}{l^2}\delta_{[\nu}^{\mu}\delta_{\sigma]}^{\rho}. \quad (5.29)$$

This expression just coincides to the Gauss–Bonnet EOM tensor H_{μ}^{ν} with replacing $R^{\mu\rho}_{\nu\sigma} \rightarrow \widehat{R}^{\mu\rho}_{\nu\sigma}$. The special choice probably induces the enhancement on the gravitational solution space, thus some noteworthy solutions happen to appear only at this choice of parameters [36, 37, 38].

The enhancement of the gravitational solution is particularly prominent in the 5-dimensional spacetime. In this dimensions more assorted solutions have been found than in the other dimensions. The 5-dimensional curiousness of the coupling choice can be seen at the black hole metric function (5.27). On this choice the mass parameter can absorb the contribution of the internal 3-dimensional space. The action also inherits the specialty; In this choice it acquires the peculiar name, the Chern–Simon case or the dimensional reduced gravity in Lovelock gravity [39]. The reason why it is called the Chern–Simon is interpreted by embedding to the 6-dimensional spacetime. The 6-dimensional Euler class is written by

the exterior derivative of a 5-form:

$$\begin{aligned}
& \varepsilon \mathcal{R} \wedge \mathcal{R} \wedge \mathcal{R} \\
&= \varepsilon_{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5 \mu_6} R^{\mu_1 \mu_2}_{\nu_1 \nu_2} R^{\mu_3 \mu_4}_{\nu_3 \nu_4} R^{\mu_5 \mu_6}_{\nu_5 \nu_6} dx^{\nu_1} \wedge dx^{\nu_2} \wedge dx^{\nu_3} \wedge dx^{\nu_4} \wedge dx^{\nu_5} \wedge dx^{\nu_6} \\
&= d\mathcal{L}_{CS}.
\end{aligned} \tag{5.30}$$

Assuming the extra-dimensional space as the diagonal product to the original 5-dimensional spacetime, you can confirm that the 5-form is just the Hodge dual of the Lagrangian for that gravitational theory on the original spacetime. This situation just corresponds to the case of a $U(1)$ field strength 2-form because the 5-dimensional Chern–Simon term relates to the 6-dimensional surface term,

$$F \wedge F \wedge F = d(A \wedge F \wedge F). \tag{5.31}$$

Enhanced solutions

One of the enhanced solutions in 5 dimensions has been discovered by using the Kerr–Schild ansatz [36]. This solution has been known so far as the only one exact solution that is rotating in EGB gravity. It possesses two rotational parameters and no electric charge with any matter fields. It is notable that the solution has a peculiar feature: When these rotational parameters coincide, the line element can be taken to the nontrivial static chart. Let us explain this solution and its feature.

In the Kerr–Schild form, the line element is given by

$$ds^2 = f(x)(h_\mu dx^\mu)^2 + d\bar{s}^2, \tag{5.32}$$

where $d\bar{s}^2$ is the seed spacetime, h_μ is a null geodesic vector field on the seed line element, and $f(x)$ is a function. The Kerr–Schild ansatz is a method to find the new solution in the form of the deformation from the seed spacetime. This method is attributed to the property of h_μ : It must be the null geodesic vector field on the new spacetime generally. This results in the powerful reduction of the gravitational EOMs for Einstein gravity, so that it becomes the rank-2 partial differential equation with the linear order of h_μ exactly. Thus in the case of Einstein gravity, this method has greatly succeeded in catching new analytic solutions, e.g. Kerr black hole spacetime and some of its generalization [1, 2].

Demonstrating this solution and its specialty, here we set the seed of the line element as the AdS spacetime in this coordinate:

$$\begin{aligned}
d\bar{s}^2 = & -\frac{(1 + \frac{r^2}{\ell^2})\Delta(\dot{\theta})}{\Xi_a \Xi_b} dt^2 + \frac{r^2 U dr^2}{(1 + \frac{r^2}{\ell^2})(r^2 + a^2)(r^2 + b^2)} + \frac{U d\dot{\theta}^2}{\Delta(\dot{\theta})} \\
& + \frac{(r^2 + a^2)}{\Xi_a} \sin^2 \dot{\theta} d\dot{\phi}_1^2 + \frac{(r^2 + b^2)}{\Xi_b} \cos^2 \dot{\theta} d\dot{\phi}_2^2,
\end{aligned} \tag{5.33}$$

where $\mathcal{L}$ is the seed AdS radius, a and b are constant parameters with

$$\Xi_a = 1 - \frac{a^2}{\mathcal{L}^2}, \quad \Xi_b = 1 - \frac{b^2}{\mathcal{L}^2}, \quad (5.34)$$

and U and $\Delta(\dot{\theta})$ are functions such that

$$U = r^2 + a^2 \cos^2 \dot{\theta} + b^2 \sin^2 \dot{\theta}, \quad \Delta(\dot{\theta}) = \Xi_a \cos^2 \dot{\theta} + \Xi_b \sin^2 \dot{\theta}. \quad (5.35)$$

This coordinate is suited to search the rotating solution because of concealing the spherical isometry. The 2-rotational solution in EGB gravity [36] is constituted by

$$h_\mu dx^\mu = \frac{\Delta(\dot{\theta})dt}{\Xi_a \Xi_b} + \frac{r^2 U dr}{(1 + \frac{r^2}{\mathcal{L}^2})(r^2 + a^2)(r^2 + b^2)} + \frac{a}{\Xi_a} \sin^2 \dot{\theta} d\dot{\phi}_1 + \frac{b}{\Xi_b} \cos^2 \dot{\theta} d\dot{\phi}_2, \quad (5.36)$$

$$f(r, \dot{\theta}) = (\frac{1}{\mathcal{L}^2} - \frac{1}{l^2})(r^2 + a^2 \cos^2 \dot{\theta} + b^2 \sin^2 \dot{\theta}). \quad (5.37)$$

In this expression, it is remarkable that only the form of $f(r, \dot{\theta})$ differs from the 5-dimensional Kerr-AdS black hole solution in Einstein gravity. This analogy assists the intuition at Einstein gravity; Similar to the 5-dimensional Kerr-AdS black hole possessing two independent angular momentum, this solution is considered to have two rotational parameters as a and b . On the other hand, the difference from usual one allows a wonder coordinate transformation in the case that the rotational parameters are adjusted to equal. Tuning $a = b$ and changing the coordinates as

$$\begin{cases} \theta = 2\dot{\theta}, \\ \phi = \dot{\phi}_1 - \dot{\phi}_2, \\ \psi = \dot{\phi}_1 + \dot{\phi}_2, \end{cases} \quad (5.38)$$

the solution is written in

$$ds^2 = f(r)(h_\mu dx^\mu)^2 + d\bar{s}^2, \quad (5.39)$$

$$h_\mu dx^\mu = \frac{1}{\Xi_a} (dt + \frac{a}{2}(d\psi - \cos \theta d\phi)) + \frac{r^2 dr}{(\frac{r^2}{\mathcal{L}^2} + 1)(r^2 + a^2)}, \quad (5.40)$$

$$f(r) = (\frac{1}{\mathcal{L}^2} - \frac{1}{l^2})(r^2 + a^2), \quad (5.41)$$

with the seed line element,

$$d\bar{s}^2 = -\frac{(\frac{r^2}{\mathcal{L}^2} + 1)}{\Xi_a} dt^2 + \frac{r^2 dr^2}{(\frac{r^2}{\mathcal{L}^2} + 1)(r^2 + a^2)} + \frac{(r^2 + a^2)}{4\Xi_a} (d\theta^2 + \sin^2 \theta d\phi^2 + (d\psi - \cos \theta d\phi)^2), \quad (5.42)$$

On this coordinate there are two physical parameters which mean respectively that $\mathcal{L}$ is the seed AdS radius independent of the special coupling parameter l , and a is the oblateness

parameter into which two rotational parameters degenerate. Here the following coordinate transformation reveals the staticity:

$$\begin{cases} \tilde{t} = \sqrt{\frac{\Xi_a^3}{\Xi_l}} t + \left(\frac{1}{\mathcal{L}^2} - \frac{1}{l^2}\right) \int \frac{r^2 dr}{(\frac{r^2}{l^2} + 1)(\frac{r^2}{\mathcal{L}^2} + 1)}, \\ \tilde{\psi} = \psi + \frac{2a}{\Xi_l} \left(\frac{1}{\mathcal{L}^2} - \frac{1}{l^2}\right) t + \frac{2a\Xi_a}{\Xi_l} \left(\frac{1}{l^2} - \frac{1}{\mathcal{L}^2}\right) \int \frac{dr}{(r^2 + a^2)(\frac{1}{\mathcal{L}^2} + \frac{1}{r^2})}, \\ \tilde{r} = \sqrt{r^2 + a^2}. \end{cases} \quad (5.43)$$

It makes the line element transfer to

$$ds^2 = -\left(\frac{\tilde{r}^2}{l^2} - \frac{a^2}{l^2} + 1\right) d\tilde{t}^2 + \frac{d\tilde{r}^2}{\left(\frac{\tilde{r}^2}{l^2} - \frac{a^2}{l^2} + 1\right)} + \frac{\tilde{r}^2}{4\Xi_a} (d\theta^2 + \sin^2 \theta d\phi^2 + \frac{\Xi_l}{\Xi_a} (d\tilde{\psi} - \cos \theta d\phi)^2), \quad (5.44)$$

$$\Xi_l = 1 - \frac{a^2}{l^2}. \quad (5.45)$$

On this static chart, the 3-dimensional space with $\tilde{t}$ and $\tilde{r}$ fixed is not the round sphere but the squashed one over the $U(1)$ direction, i.e. the squashed S^3 . Therefore, the adjusted solution (5.39) is not identical to the familiar static black hole (5.21) but a member of topological black holes [96]. Seeing it as the static black hole, parameters a and $\mathcal{L}$ correspond to the mass and squashing rate after some recombination. Its asymptotic AdS radius is described by l rather than $\mathcal{L}$.

In the vacuum case of EGB gravity, static topological black holes were constructed in arbitrary coupling values and general spacetime dimensions [37, 38]. Whereas that class is usually smaller than the general relativistic one, it has been found that static topological black hole solutions can extend to much larger class only in the case of the special choice. This is because the choice considerably relaxes the EOM requirement [37]. In particular, the relaxation in 5 dimensions is so prominent that any Euclidean line element ds_3^2 is permitted for the internal 3-dimensional space. Therefore, as drawn in the figure 5.1, the solution (5.39) belongs to two families of gravitational solutions: one of them is the 2-rotating spacetime and the other is the enhancement of static topological black holes. While the other class of static solutions has been confirmed to enhance in the special coupling choice, the intersection does not occur besides here. It does not seem so unnatural that the solution has more families other than them, which has been unknown so far.

Whereas the higher-dimensional static topological black holes were researched [38, 97], each of these two families still has the problem how far it can generalize to more other directions. The research for topological black holes shows that the prominent relaxation to any internal $(D - 2)$ -dimensional space happens at the Chern–Simon case of Lovelock gravity. On the other hand, the higher-dimensional rotating solution has not been found at all. Moreover, both of them have not been deformed with coupling any matter fields such as Maxwell field in the exact manner.

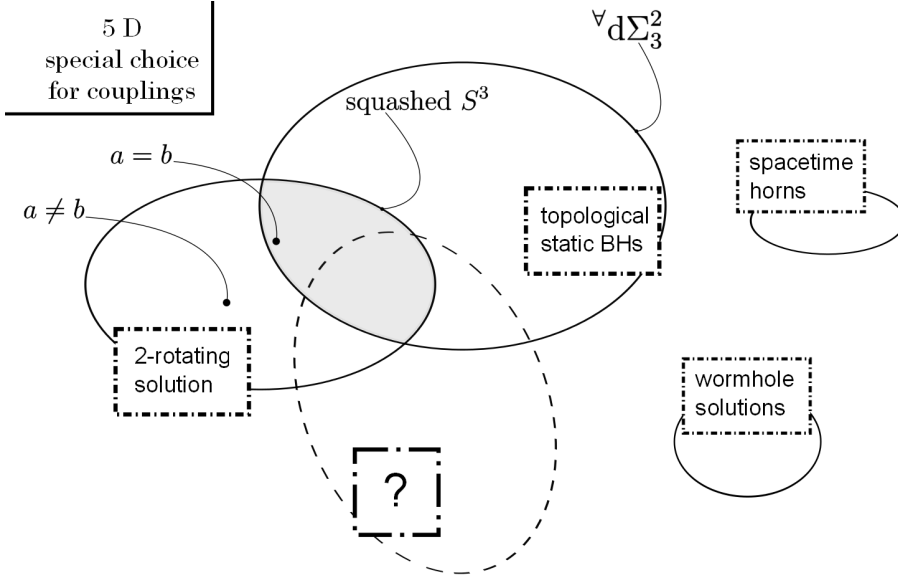


Figure 5.1 the location of 5-dimensional solutions in EGB gravity

5.2 Spinning exact solutions with Sasakian structure

In the previous section, we have seen the unbalance of explored solution in this theory: Whereas the special coupling choice happens to the enhancement in the exact-solution space, rotational solutions have been rarely found, only the 5-dimensional one. Concerning this difficulty, we expect that if there exists Sasakian solution in EGB gravity, such spacetime should rotate because of the reduced-limit picture. This prospect can read from the off-shell Sasakian metric, apart from Einstein gravity. In this section, we shall obtain the new solution, clarifying its feature and the location from others.

5.2.1 Charged spinning solutions in odd dimensions

In order to investigate new solutions in Gauss–Bonnet–Maxwell theory on arbitrary $(2n + 1)$ -dimensional spacetime, we adopt the ansatz restricting the metric to

$$ds^2 = - (d\tau + \lambda y \mathcal{A})^2 + \frac{dy^2}{Y(y)} + Y(y) \mathcal{A}^2 + y d\check{s}^2, \quad (5.46)$$

$$\mathcal{A} = d\varphi + \check{A}, \quad (5.47)$$

with $Y(y)$ an arbitrary function of y , λ a nonzero constant parameter, and $d\check{s}^2$ the Fubini–Study metric of the internal CP_{n-1} space*. We normalize this $2(n - 1)$ -dimensional space

*As is the case of topological black holes, there are some variations of the internal space besides CP_{n-1} . A part of them is expressed by γ : $\gamma = 0$ is flat space, and $\gamma^2 < 0$ is the negative curvature homogeneous space analogous to CP_{n-1} .

as

$$Ric_{CP_{n-1}} = \frac{n\gamma^2}{2} d\check{s}^2, \quad d\check{A} = \check{J}, \quad (5.48)$$

where $\check{A}$ is the vector potential of the Kähler form $\check{J}$ on the internal $2(n-1)$ -dimensional space $d\check{s}^2$. This ansatz is precisely respected with Sasakian structure, whose cohomogeneity is 1 in the Lorentzian signature.

Owing to Sasakian structure, this ansatz has some good properties. We shall show them in order. First, we present some technical benefits. On this ansatz, there is an assured Maxwell solution. The following vector potential satisfies the Maxwell equation regardless of the concrete form of $Y(y)$,

$$A_\mu dx^\mu = \frac{Q}{y^{n-1}} \mathcal{A}. \quad (5.49)$$

Then switching on the matter flux, we assign this electromagnetic charge.

To solve the gravitational equation of motion (5.12), the technical chart (5.46) exerts its power. Using the following orthonormal frame $\{e_\mu^A \mid A = (0, a, 2n-1, 2n)\}$,

$$\begin{aligned} e^0 &= d\tau + \lambda y \mathcal{A}, & e^a &= \sqrt{y} \check{e}^a, & \check{e}_p^a \check{e}_q^a dx^p dx^q &= d\check{s}^2, \\ e^{2n-1} &= \frac{dy}{\sqrt{Y(y)}}, & e^{2n} &= \sqrt{Y(y)} \mathcal{A}, \end{aligned} \quad (5.50)$$

with $a = 1, \dots, 2n-2$ and $\{x^p\}$ the coordinate of CP_{n-1} , the Riemann tensor acquires a considerably simple expression,

$$R^{0C}{}_{AB} = -\frac{\lambda^2}{2} \delta_{[A}^0 \delta_{B]}^C, \quad (5.51)$$

$$R^{2n-12n}{}_{AB} = \left(\frac{3}{2}\lambda^2 - Y''(y)\right) \delta_{[A}^{2n-1} \delta_{B]}^{2n} + \frac{1}{2}(\lambda^2 + \mathcal{N}_2) \check{J}_{ab} \delta_{[A}^a \delta_{B]}^b, \quad (5.52)$$

$$R^{c2n-1}{}_{AB} = \frac{\mathcal{N}_2}{2} \delta_{[A}^c \delta_{B]}^{2n-1} + \frac{1}{2}(\lambda^2 + \mathcal{N}_2) \check{J}^c{}_b \delta_{[A}^b \delta_{B]}^{2n}, \quad (5.53)$$

$$R^{c2n}{}_{AB} = \frac{\mathcal{N}_2}{2} \delta_{[A}^c \delta_{B]}^{2n} - \frac{1}{2}(\lambda^2 + \mathcal{N}_2) \check{J}^c{}_b \delta_{[A}^b \delta_{B]}^{2n-1}, \quad (5.54)$$

$$R^{cd}{}_{AB} = \frac{\mathcal{N}_1}{2} \delta_{[A}^c \delta_{B]}^d + \frac{1}{2}(\lambda^2 + \mathcal{N}_1) (\check{J}^c{}_{[a} \check{J}^d{}_{b]} + \check{J}^{cd} \check{J}_{ab}) \delta_{[A}^a \delta_{B]}^b + (\lambda^2 + \mathcal{N}_2) \check{J}^{cd} \delta_{[A}^{2n-1} \delta_{B]}^{2n}, \quad (5.55)$$

$$\mathcal{N}_1 = \frac{\gamma^2}{y} - \frac{1}{y^2} Y(y), \quad (5.56)$$

$$\mathcal{N}_2 = \frac{1}{y^2} Y(y) - \frac{1}{y} Y'(y), \quad (5.57)$$

As above, the 0-th direction of the Riemann tensor is coincident with the maximal symmetric case, and the $(2n-1)$ -th or $2n$ -th direction is interchangeable up to the sign. Every component is described in systematical forms, only by δ and $\check{J}$. These simplicity results in diagonalizing EOMs (5.12) as well as the Ricci tensor. In addition the diagonalization

accompanies specific degeneracies, that is, at most three different eigenvalues are allowed respectively on the $(0,0)$, $(2n-1, 2n-1)=(2n, 2n)$ and (a, a) component. All these components are ordinary differential equations of y , mutually correlated by the conservation law. In the case of the special coupling choice, these equations are further reduced to two equations with the choice $\lambda = \frac{2}{l}$. The one equation comes from its trace part,

$$\begin{aligned}
& G^A_A + \frac{l^2}{4(n-1)(2n-3)} H^A_A - \frac{n(2n-1)(2n+1)}{2l^2} - (16\pi G) T^A_A \\
&= n \left(1 + \frac{l^2}{4y^2} Y(y) - \frac{l^2}{4y} Y'(y) \right) \left(-\frac{4(2n-3)}{l^2} - \frac{2(n-2)\gamma^2}{y} + \frac{(2n-5)}{y^2} Y(y) + \frac{1}{y} Y'(y) \right) \\
&+ n \left(\frac{\gamma^2 l^2}{4y} + 1 - \frac{l^2}{4y^2} Y(y) \right) \left((n-1) \left(\frac{n}{2} - 1 \right) \left(-\frac{\gamma^2}{y} - \frac{4}{l^2} + \frac{1}{y^2} Y(y) \right) - \frac{8}{l^2} + Y''(y) \right) \\
&+ \frac{n(n-1)(2n-3)Q^2}{4y^{2n}} \\
&= 0,
\end{aligned} \tag{5.58}$$

and the other from the $(2n-1, 2n-1)$ component of EOMs,

$$\begin{aligned}
& G_{2n-1, 2n-1} + \frac{l^2}{4(n-1)(2n-3)} H_{2n-1, 2n-1} - \frac{n(2n-1)}{2l^2} - (16\pi G) T_{2n-1, 2n-1} \\
&= \frac{n(n-2)}{(2n-3)} \left(\frac{\gamma^2 l^2}{4y} + 1 - \frac{l^2}{4y^2} Y(y) \right) \left(\frac{(1-n)\gamma^2}{2y} - \frac{2(n+1)}{l^2} + \frac{(n-3)}{2y^2} Y(y) + \frac{1}{y} Y'(y) \right) \\
&- \frac{(n-1)(n-2)Q^2}{4y^{2n}} \\
&= 0.
\end{aligned} \tag{5.59}$$

All the other components can be written down linear combinations of (5.58) and (5.59). Here, we find that the following metric function satisfies the trace equation (5.58),

$$Y(y) = \frac{4}{l^2} y^2 + \gamma^2 y \pm \sqrt{\mu_1 y^{3-n} + \mu_2 y^{4-n} + \frac{2(2n-3)}{nl^2} Q^2 y^{4-2n}}, \tag{5.60}$$

with μ_1, μ_2 constant parameters and l the special coupling parameter as (5.27). In the case of $n = 2$, this is the new solution itself because the residual equation (5.59) automatically vanishes by the overall factor. Note that it is just the exposure of the 5-dimensional curiousness. In $n \geq 3$ the residual constraint, being the rank-1 ordinary differential equation, inhibits one integration constant. As a result, the higher-dimensional solution is (5.60) with $\mu_2 = 0$.

Non-staticity

We shall explain the affinity of Lorentz-Sasakian structure and its non-static property: The static condition cannot be fulfilled by any tune of the time-like Killing vector field, besides the specific form of the metric function $Y(y)$.

Whether the spacetime is static or not is due to the existence of the time-like Killing vector field ξ that satisfies the following condition everywhere in that spacetime[†],

$$\xi^b \wedge d\xi^b = 0. \quad (5.61)$$

According to our metric ansatz (5.46), Killing vector fields are included in $SU(n) \times U(1) \times \mathbf{R}$. Considering the property of the Reeb vector, the everywhere time-like Killing direction, any ξ can be written in

$$\xi = \partial_\tau + \Omega_v \partial_\varphi + \xi_{SU(n)}, \quad (5.62)$$

where Ω_v is an arbitrary constant and $\xi_{SU(n)}$ is any combination of $SU(n)$ Killing vector fields originated from the internal CP_{n-1} . The exterior derivative of its dual 1-form reads

$$d\xi^b = \left\{ \Omega_v \left(\frac{1}{y} Y(y) - \lambda^2 y \right) - \lambda \right\} J + \Omega_v d \left(\frac{1}{y} Y(y) - \lambda^2 y \right) \wedge (y\mathcal{A}) + d\xi_{SU(n)}^b, \quad (5.63)$$

with J the Kähler form on the $2n$ -dimensional base space, satisfying $J = d(y\mathcal{A}) = \varpi$. At this point, the exceptional static case can be read from this formula. When the metric function fixes $Y(y) = \lambda^2 y^2 + C_v y$ with any constant C_v , the second term vanishes automatically. The first term is also able to cancel out with choosing $\Omega_v = \frac{\lambda}{C_v}$. Hence the static condition is satisfied without $\xi_{SU(n)}$. In any other cases, however, first two terms of (5.63) must remain on some components of $\xi^b \wedge d\xi^b$. This contains the term $\mathcal{A} \wedge J$, which is the $SU(n)$ isometry singlet. Therefore even if you turn on $\xi_{SU(n)}$, the vector representation of $SU(n)$, it does not cancel that term but raises rather some terrible components.

The exceptional static case above can be checked by the concrete form. After changing the coordinate as follows,

$$\tau = \sqrt{\frac{\lambda}{\Omega_v}} t, \quad \varphi = \phi + \sqrt{\lambda \Omega_v} t, \quad (5.64)$$

the metric with $Y(y) = \lambda^2 y^2 + C_v y$ is given by

$$ds^2 = -(\lambda^2 y + C_v) dt^2 + \frac{dy^2}{\lambda^2 y^2 + C_v y} + y (d\check{s}^2 + \frac{\lambda}{\Omega_v} (d\phi + \check{A})^2), \quad (5.65)$$

If the similar transformation is done in any other metric function, the line element becomes like the Boyer–Lindquist coordinate [98]. It was analyzed that the existence of the Boyer–Lindquist coordinate relates the circularity of the metric, which our solution possesses but does not the previous rotating solution [93]. It is remarkable that the nontrivial static case above cannot occur in the present solution (5.60) unless the 5-dimensional spacetime.

5.2.2 The location on 5 dimensions

As explained the previous section, several solutions have been established in 5 dimensions. Then we shall show how to differ our solution from other solutions, and relate

[†]_b denotes the dual vector, lowering the index by the metric.


$$ds^2 = -(\frac{1}{l^2}r^2 + 1)dt^2 + \frac{V(r)}{16}(\frac{2}{l}dt + d\phi + \cos\theta d\psi)^2 + \frac{dr^2}{\frac{1}{l^2}r^2 + 1 + \frac{V(r)}{4r^2}} + \frac{r^2}{4}(d\theta^2 + \sin^2\theta d\psi^2 + (d\phi + \cos\theta d\psi)^2), \quad (5.66)$$

$$A_\mu dx^\mu = \frac{Q}{4r^2}(\frac{2}{l}dt + d\phi + \cos\theta d\psi). \quad (5.68)$$

The 5-dimensional solution is also different from the previous rotating solution [36] even if the electromagnetic charge Q vanishes, except for the static case. The previous solution

has the above large isometry only when its two rotational parameters coincide, as is the case of the Myers–Perry solution or its generalization with the cosmological constant [1, 2] in general relativity. In the Kerr–Schild form, their solution consists of the same null geodesic vector as the 5-dimensional Kerr–AdS solution [36, 2]. Thus, once the coincidence breaks into two angular momentum, they decrease the isometry to $U(1) \times U(1) \times \mathbf{R}$. Moreover, there is no Killing vector field possessing the constant norm in the previous solution besides the static case. This means the absence of Sasakian structure on their solution, then it concludes the difference from our solution. What this variation around non-static solutions means physically in EGB gravity is not so obvious, but much interesting subject. EGB gravity might allow somewhat larger class of rotating solutions than expected from the knowledge accumulated in general relativity.

Considering the global structure, the coordinate r almost corresponds to the radial direction. As $r \rightarrow +\infty$, the Riemann tensor of our solution gets closer to the one on the AdS spacetime, that is, $R^{AB}_{CD} \rightarrow -\frac{2}{l^2} \delta^A_C \delta^B_D$. For this reason, our spacetime is regarded as asymptotically AdS at least locally [36]. Tuning parameters could raise the Killing horizon on the positive finite radius of that spacetime. The Killing vector field ∂_t has the norm $(\frac{1}{4l^2}V(r) - \frac{1}{l^2}r^2 - 1)$, which allows some nodes. Even when the Maxwell charge Q vanishes, two residual parameters can cause the extremal case. Furthermore, this solution has the curvature singularity at $r = 0$. Some curvature invariants diverge there, for instance the Kretschmann invariant $Kret = R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}$:

$$Kret = \frac{40}{l^4} + \frac{8V(r)^2}{r^8} - \frac{4V(r)V'(r)}{r^7} + \frac{3V'(r)}{l^2r^3} + \frac{17V'(r)^2}{16r^6} + \frac{V''(r)}{l^2r^2} - \frac{V'(r)V''(r)}{8r^5} + \frac{V''(r)^2}{16r^4}. \quad (5.69)$$

There is one more possibility of singularity at the node of $V(r)$, attributed to its derivative term with the square root form. However, the similar situation occurs in the non-Einstein branch of the static black hole solution (5.22), avoided in the Einstein branch owing to some physical conditions. Thus it is expected that our solution also does on the precisely global analysis. The present coordinate might be improved after the global inspection more cautiously and the adjustment from some physical requests. All these structures except for μ_2 are preserved to the higher-dimensional generalization.

5.3 Physical quantities from AD formulation

Finally let us evaluate the physical conserved values of this spacetime around the AdS background, confirming this solution properly has the rotation. In order to face this higher-curvature gravity theory, we adopt the AD formulation which has been applied to EGB gravity by S. Deser *et al.* [42, 99].

5.3.1 Abbott–Deser formulation

R. Arnowitt, S. Deser and C. Misner established one of the most reliable method to evaluate the conserved charge of the spacetime itself, what is called, ADM charge [100]. Since this construction requires its boundary to be asymptotically flat, the formalism was extended to permit asymptotically dS or AdS case, then achieved to AD formalism [42].

Extracting the energy-momentum of the spacetime in this formalism, first of all we decompose the tested metric $g_{\mu\nu}$ of some excited spacetime into the background metric $\bar{g}_{\mu\nu}$ and its deviation:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \quad (5.70)$$

where $\bar{g}_{\mu\nu}$ has the information of the background curvature radius L and the deviation field $h_{\mu\nu}$ has the information of the present solution. Next, the spacetime energy-momentum tensor $T_{\mu\nu}^{(h)}$ reads from the variation of the gravitational action,

$$\frac{\delta S_{\text{Gravity}}}{\delta g^{\mu\nu}} = \sqrt{|g|} T_{\mu\nu}^{(h)}. \quad (5.71)$$

Using this tensor and a Killing vector field ξ , a conserved current can be constructed as follows:

$$\mathcal{J}^\mu(\xi; h) \equiv T_{(h)}^{\mu\nu} \xi_\nu. \quad (5.72)$$

Owing to the diffeomorphic symmetry and isometry, the constituents respectively satisfy the differential conditions as

$$\bar{\nabla}_\mu T_{(h)}^{\mu\nu} = 0, \quad \bar{\nabla}_\mu \xi_\nu + \bar{\nabla}_\nu \xi_\mu = 0, \quad (5.73)$$

where the covariant derivative $\bar{\nabla}$ denotes the background AdS connection. These symmetry results in a current conservation:

$$\bar{\nabla}_\mu \mathcal{J}^\mu(\xi; h) = 0. \quad (5.74)$$

Since this conservation law can be regarded as a co-closed condition, it indicates the existence of a 2-form $\mathcal{B}$ from the co-exactness,

$$\mathcal{J}^\mu(\xi; h) = \bar{\nabla}_\lambda \mathcal{B}^{\lambda\mu}(\xi; h), \quad (5.75)$$

As a result, the conserved charge $Q^t(\xi; g)$ is written by the surface integral of $\mathcal{B}(\xi; h)$ as

$$Q^t(\xi | g) = \frac{1}{\mathcal{V}_{D-2}} \int_{\mathcal{M}_{D-1}} \mathcal{J}^t(\xi; h) dV \quad (5.76)$$

$$= \frac{1}{\mathcal{V}_{D-2}} \int_{\partial\mathcal{M}} \mathcal{B}^{rt}(\xi; h) d\Sigma_r. \quad (5.77)$$

where $\mathcal{M}_{D-1}$ is a spatial $(D-1)$ -dimensional hypersurface, $\partial\mathcal{M}$ is its $(D-2)$ -dimensional boundary, and $\mathcal{V}_{D-2}$ is its volume. The following equation is useful to regard this as the generalization of Gauss law: converting the space integral into the surface integral on a curved background,

$$\bar{\nabla}_\lambda \mathcal{B}^{\lambda\mu} = \frac{1}{\sqrt{|\bar{g}|}} \partial_\lambda (\sqrt{|\bar{g}|} \mathcal{B}^{\lambda\mu}). \quad (5.78)$$

Schwarzschild–Tangherlini case

Here we test this procedure on the Schwarzschild–Tangherlini black hole solution in cosmological Einstein gravity of any D -dimensional spacetime. The gravitational action is given by

$$S_{\text{Ein.}} = \frac{1}{16\pi G} \int \sqrt{|g|} d^D x (R - 2\Lambda). \quad (5.79)$$

The static spherical symmetric black hole metric is

$$g_{\mu\nu} dx^\mu dx^\nu = -R(r) dt^2 + \frac{dr^2}{R(r)} + r^2 d\Omega_{D-2}^2, \quad (5.80)$$

$$R(r) = \frac{r^2}{L^2} + 1 - \frac{M}{r^{D-3}}, \quad (5.81)$$

$$L^2 = -\frac{(D-1)(D-2)}{2\Lambda}. \quad (5.82)$$

We set the back ground metric g as the AdS spacetime, $M = 0$ case of (5.81). Thus the deviation tensor field h reads

$$h_{\mu\nu} dx^\mu dx^\nu = \frac{M}{r^{D-3}} dt^2 + \frac{\frac{M}{r^{D-3}} dr^2}{\left(\frac{r^2}{L^2} + 1\right)\left(\frac{r^2}{L^2} + 1 - \frac{M}{r^{D-3}}\right)}. \quad (5.83)$$

The variation of $S_{\text{Ein.}}$ gives the energy-momentum tensor of the spacetime,

$$\begin{aligned} \frac{1}{\sqrt{|g|}} \frac{\delta S_{\text{Ein.}}}{\delta g^{\mu\nu}} &= \frac{1}{16\pi G} (R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \Lambda g_{\mu\nu}) \\ &= \frac{1}{16\pi G} [\bar{R}_{\mu\nu} - \frac{1}{2} \bar{R} g_{\mu\nu} + \frac{(D-1)(D-2)}{2L^2} \bar{g}_{\mu\nu} \\ &\quad - \frac{1}{2} (\bar{\nabla}^\sigma \bar{\nabla}_\sigma h_{\mu\nu} + \bar{\nabla}_\mu \bar{\nabla}_\nu h_\sigma^\sigma - \bar{\nabla}_\sigma \bar{\nabla}_\nu h_\mu^\sigma - \bar{\nabla}_\sigma \bar{\nabla}_\mu h_\nu^\sigma) \\ &\quad - \frac{1}{2} (\bar{R} h_{\mu\nu} + \frac{(D-1)(D-2)}{L^2} h_{\mu\nu}) \\ &\quad + \frac{1}{2} \bar{g}_{\mu\nu} (\bar{\nabla}^\sigma \bar{\nabla}_\sigma h_\rho^\rho - \bar{\nabla}^\sigma \bar{\nabla}_\rho h_\sigma^\rho + \bar{R}_\sigma^\rho h_\rho^\sigma)] + \mathcal{O}(h^2), \end{aligned} \quad (5.84)$$

On the h -expansion the 0-th order term automatically vanishes by the background equations of motion. Thus the energy-momentum tensor is evaluated by the first order of h . The existence of the 2-form co-potential $\mathcal{B}$ in this gravitational theory is well known:

$$\mathcal{B}^{\mu\nu}(\xi; \text{Ein}) = -9(h_\rho^{[\mu} \bar{\nabla}^{\nu]} \xi^{\rho]} + 2\xi^{[\mu} \bar{\nabla}^{\nu]} h_\rho^{\rho]}). \quad (5.85)$$

Confirming that the divergence of this 2-form actually gives the conserved current, the useful fact is that the first order terms of Eq. (5.84) is systematically factorized into

$$T_{\nu(h)}^\mu = -9\delta_{[\nu}^{[\mu} \left(\frac{1}{L^2} h_{\nu_1}^{\nu_1} \delta_{\nu_2]}^{\nu_2]} - \bar{\nabla}_{\nu_1} \bar{\nabla}^{\nu_1} h_{\nu_2]}^{\nu_2]} \right). \quad (5.86)$$

The above fact is due to the Killing vector condition, which is

$$\bar{\nabla}_\mu \bar{\nabla}_\rho \xi^\nu = \frac{1}{L^2} (\delta_\sigma^\nu \bar{g}_{\rho\mu} - \delta_\mu^\nu \bar{g}_{\rho\sigma}) \xi^\sigma. \quad (5.87)$$

The AD energy is identified as the conserved charge of the Killing vector field $\xi = -\partial_t$,

$$m = Q^t(-\partial_t \mid \text{Schwarzschild}) \quad (5.88)$$

$$= \frac{1}{\Omega_{D-2}} \int_{S_{D-2}} \mathcal{B}^{rt}(-\partial_t; \text{Ein}) d\Sigma_r, \quad (5.89)$$

with Ω_{D-2} the unit sphere volume. Because this surface integral is evaluated at $r \rightarrow +\infty$, the leading term which has the largest power of r determines this value. Considering the distribution of the $(D-2)$ -dimensional measure, the non-vanishing finite value is obtained only when the leading term of $\mathcal{B}$ is the $(2-D)$ -th power of r : The higher power term arises the divergence, and the lower power term vanishes asymptotically. In this case, the dominant distribution comes in the from

$$-\xi^t \bar{\nabla}^m h_m^r \sim \frac{(D-2)f}{r} h_r^r \sim \frac{(D-2)M}{r^{D-2}}. \quad (5.90)$$

This correctly gives the finite energy,

$$\begin{aligned} m &= Q^t(-\partial_t \mid \text{Schwarzschild}) \\ &= 3(D-2)M. \end{aligned} \quad (5.91)$$

It is worth mentioned that the value of energy is same as in the Reissner–Nordström solution. The distribution of the electric charge in Eq. (5.85) has enough the low power so that vanishing at the asymptotic region.

5.3.2 Conserved current on the special coupling choice

The above formalism can be adopted in EGB gravity. In particular, when the Gauss–Bonnet coupling constant stays in the weak region, the above current gives a good approximation for the Einstein branch of the static black hole (5.22). However, in the case of the special coupling choice, this approximation breaks down. This evidence is clearly seen at estimating the static black hole energy.

Substituting Eq. (5.26) for the tested metric g and AdS of the curvature radius l for the background g , we have the deviation tensor field h as

$$h_{\mu\nu}dx^\mu dx^\nu = \pm \sqrt{\frac{2M}{l^2 r^{D-5}}} dt^2 \pm \frac{\sqrt{\frac{2M}{l^2 r^{D-5}}} dr^2}{\left(\frac{r^2}{L^2} + 1\right)\left(\frac{r^2}{L^2} + 1 \mp \sqrt{\frac{2M}{l^2 r^{D-5}}}\right)}, \quad (5.92)$$

In this case, if Eq. (5.85) is adopted for the current formula, its higher power of r causes a divergence:

$$\mathcal{B}^{rt}(-\partial_t; \text{Ein}) \sim -3\xi^t \bar{\nabla}^m h_m^r \sim \frac{3(D-2)\sqrt{2M}}{lr^{\frac{D-3}{2}}}. \quad (5.93)$$

This divergence cannot cancel even if replacing the background metric. Hence, we must reconstruct the appropriate current with the AD formalism.

As explained previously, the special coupling tuning makes miracle at the action or EOMs. That results in

$$\begin{aligned} & \frac{16\pi G}{\sqrt{|g|}} \frac{\delta S_{\text{EGB}}}{\delta g^{\mu\nu}} \Big|_{\tilde{\alpha}=\frac{l^2}{2}} \\ &= -\frac{15l^2}{2(D-3)(D-4)} g_{\nu\mu} \delta_{[\mu}^{[\nu_1} \delta_{\mu_1}^{\nu_2} \delta_{\mu_2}^{\nu_3} \delta_{\mu_3}^{\nu_4} \delta_{\mu_4}^{\nu_5]} (R^{\mu_1\mu_2}_{\nu_1\nu_2} + \frac{2}{l^2} \delta_{\nu_1}^{\mu_1} \delta_{\nu_2}^{\mu_2}) (R^{\mu_3\mu_4}_{\nu_3\nu_4} + \frac{2}{l^2} \delta_{\nu_3}^{\mu_3} \delta_{\nu_4}^{\mu_4}). \end{aligned} \quad (5.94)$$

Here, we need to get the correct h -expansion of this expression. For this first step, we expand the Riemann tensor with this index-position, that is,

$$R^{\mu\rho}_{\nu\sigma} = -\frac{2}{l^2} \delta_{[\nu}^{[\mu} \delta_{\sigma]}^{\rho]} + \frac{2}{l^2} h_{[\nu}^{[\mu} \delta_{\sigma]}^{\rho]} - 2\bar{\nabla}_{[\nu} \bar{\nabla}^{[\mu} h_{\sigma]}^{\rho]} + \mathcal{O}(h^2). \quad (5.95)$$

The substitution of (5.95) for (5.94) unveils one more specialty of the present couplings: The h -expansion starts from the h -squared term. Thus we evaluate $T_{\mu\nu}^{(h)}$ at the lowest order,

$$T_{\mu\nu}^{(h)} = -\frac{15l^2}{8(D-3)(D-4)\pi G} g_{\mu\nu} \delta_{[\nu}^{[\nu_1} \left(\frac{1}{l^2} h_{\nu_1}^{\nu_2} \delta_{\nu_2}^{\nu_3} - \bar{\nabla}_{\nu_1} \bar{\nabla}^{\nu_2} h_{\nu_2}^{\nu_3} \right) \left(\frac{1}{l^2} h_{\nu_3}^{\nu_4} \delta_{\nu_4}^{\nu_5} - \bar{\nabla}_{\nu_3} \bar{\nabla}^{\nu_4} h_{\nu_4}^{\nu_5} \right)]. \quad (5.96)$$

We find out the co-potential 2-form of this theory:

$$\mathcal{B}^{\mu\nu}(\xi; \text{sp.EGB}) = -\frac{15l^2}{16(D-3)(D-4)\pi G} (h_{[\nu_1}^{[\mu} \bar{\nabla}^{\nu} \xi^{\nu_1]} + 2\xi^{[\mu} \bar{\nabla}^{\nu} h_{\nu_1}^{\nu_1]} \left(\frac{1}{l^2} h_{\nu_2}^{\nu_2} \delta_{\nu_3}^{\nu_3} - \bar{\nabla}_{\nu_2} \bar{\nabla}^{\nu_2} h_{\nu_3}^{\nu_3} \right)]. \quad (5.97)$$

In contrast to the Einstein gravity case, proving the consistency of this 2-form should need some bothered calculations. However, we obtain the systematic manner to prove it. We shall show this outline.

The divergence of this 2-form reads

$$\begin{aligned} & \bar{\nabla}_\lambda \mathcal{B}^{\lambda\mu}(\xi; \text{sp.EGB}) \\ &= -\frac{15l^2}{16(D-3)(D-4)\pi G} \left\{ \left(h_{[\nu_1}^{[\lambda} \bar{\nabla}_\lambda \bar{\nabla}^{\mu} \xi^{\nu_1]} + 2\xi^{[\lambda} \bar{\nabla}_{[\lambda} \bar{\nabla}^{\mu} h_{\nu_1}^{\nu_1]} \left(\frac{1}{l^2} h_{\nu_2}^{\nu_2} \delta_{\nu_3}^{\nu_3} - \bar{\nabla}_{\nu_2} \bar{\nabla}^{\nu_2} h_{\nu_3}^{\nu_3} \right] \right. \right. \\ & \quad + \left(h_{[\nu_1}^{[\lambda} \bar{\nabla}^{\mu} \xi^{\nu_1]} + 2\xi^{[\lambda} \bar{\nabla}^{\mu} h_{\nu_1}^{\nu_1]} \right) \bar{\nabla}_\lambda \left(\frac{1}{l^2} h_{\nu_2}^{\nu_2} \delta_{\nu_3}^{\nu_3} - \bar{\nabla}_{\nu_2} \bar{\nabla}^{\nu_2} h_{\nu_3}^{\nu_3} \right) \\ & \quad \left. \left. + \left(\bar{\nabla}_{[\lambda} h_{\nu_1}^{\lambda} \bar{\nabla}^{\mu} \xi^{\nu_1]} + 2\bar{\nabla}_{[\lambda} \xi^{[\lambda} \bar{\nabla}^{\mu} h_{\nu_1}^{\nu_1]} \left(\frac{1}{l^2} h_{\nu_2}^{\nu_2} \delta_{\nu_3}^{\nu_3} - \bar{\nabla}_{\nu_2} \bar{\nabla}^{\nu_2} h_{\nu_3}^{\nu_3} \right] \right) \right\}. \end{aligned} \quad (5.98)$$

The first line just corresponds to the current because the current is written in

$$\begin{aligned}
& \mathcal{J}^\mu(\xi; \text{sp.EGB}) \\
&= -\frac{15l^2}{8(D-3)(D-4)\pi G} \xi^{[\mu} \left(\frac{1}{l^2} h_{\nu_1}^{\nu_2} \delta_{\nu_2}^{\nu_3} - \bar{\nabla}_{\nu_1} \bar{\nabla}^{\nu_1} h_{\nu_2}^{\nu_3} \right) \left(\frac{1}{l^2} h_{\nu_3}^{\nu_4} \delta_{\nu_4}^{\nu_5} - \bar{\nabla}_{\nu_3} \bar{\nabla}^{\nu_3} h_{\nu_4}^{\nu_5} \right) \\
&= \frac{15l^2}{16(D-3)(D-4)\pi G} \left(h_{[\nu_2}^{[\mu} \bar{\nabla}_{\nu_1} \bar{\nabla}^{\nu_1} \xi^{\nu_2]} + 2\xi^{[\mu} \bar{\nabla}_{\nu_1} \bar{\nabla}^{\nu_1} h_{\nu_2}^{\nu_3]} \left(\frac{1}{l^2} h_{\nu_3}^{\nu_4} \delta_{\nu_4}^{\nu_5} - \bar{\nabla}_{\nu_3} \bar{\nabla}^{\nu_3} h_{\nu_4}^{\nu_5} \right) \right), \quad (5.99)
\end{aligned}$$

where the last equality uses Eq. (5.87). For showing all the other terms of Eq. (5.98) vanish exactly, the essential point is the Riemann tensorial character. Because of the h -expansion at Eq. (5.95), the following form inherits the tensorial symmetry of the Riemann tensor:

$$R_{(1) \quad \nu_1 \nu_2}^{\mu_1 \mu_2} = \frac{2}{l^2} h_{[\nu_1}^{[\mu_1} \delta_{\nu_2]}^{\mu_2]} - 2\bar{\nabla}_{[\nu_1} \bar{\nabla}^{[\mu_1} h_{\nu_2]}^{\mu_2]}. \quad (5.100)$$

Hereafter, we use this expression to emphasize this character. The trick for vanishing the second line of Eq. (5.98) is the second Bianchi identity of this tensor field:

$$\bar{\nabla}_{[\nu_1} R_{(1) \quad \nu_2 \nu_3]}^{\mu_1 \mu_2} = 0. \quad (5.101)$$

The third line of Eq. (5.98) can be rearranged to

$$\begin{aligned}
& (\bar{\nabla}_{[\nu_4} h_{\nu_1}^{[\nu_4} \bar{\nabla}^{\mu} \xi^{\nu_1]} + 2\bar{\nabla}_{[\nu_4} \xi^{[\nu_4} \bar{\nabla}^{\mu} h_{\nu_1}^{\nu_1]} R_{(1) \quad \nu_2 \nu_3]}^{\nu_2 \nu_3]} \\
&= -\frac{2}{5} \bar{\nabla}^\mu h_{[\nu_3}^{[\nu_3} \bar{\nabla}_{\nu_1} \xi^{\nu_2]} R_{(1) \quad \nu_2 \nu_4]}^{\nu_1 \nu_4]} \\
&\quad -\frac{2}{5} (\bar{\nabla}^\mu \xi^{[\nu_2} \bar{\nabla}_{[\nu_1} h_{\nu_4}^{\nu_3]} + \bar{\nabla}_{[\nu_1} \xi^\mu \bar{\nabla}^{[\nu_2} h_{\nu_4}^{\nu_3]} R_{(1) \quad \nu_2 \nu_3]}^{\nu_1 \nu_4]} \\
&\quad -\frac{1}{5} (\bar{\nabla}_{[\nu_1} h_{\nu_4}^{[\mu} \bar{\nabla}^{\nu_2} \xi^{\nu_3]} + 2\bar{\nabla}_{[\nu_1} h_{\nu_4}^{[\mu} \bar{\nabla}_{\nu_1} \xi^{\nu_2]} R_{(1) \quad \nu_2 \nu_3]}^{\nu_1 \nu_4]} \\
&\quad -\frac{2}{5} (\bar{\nabla}_{[\nu_1} h_{\nu_4}^{[\nu_1} \bar{\nabla}^{\nu_2} \xi^{\nu_3]} + 2\bar{\nabla}_{[\nu_1} h_{\nu_4}^{[\nu_1} \bar{\nabla}_{\nu_1} \xi^{\nu_2]} R_{(1) \quad \nu_2 \nu_3]}^{\nu_4] \mu}), \quad (5.102)
\end{aligned}$$

In the above rearrangement, the first term vanishes by the replacing symmetry of tensorial indexes:

$$R_{\mu_1 \mu_2 \nu_1 \nu_2}^{(1)} = R_{\nu_1 \nu_2 \mu_1 \mu_2}^{(1)}, \quad (5.103)$$

and the second terms cancel with the Killing vectorial condition. All the other terms contain a 5-rank tensor field, which is

$$\mathcal{K}_{\rho_1 \rho_2 \rho_3 \rho_4}^\rho = \bar{\nabla}_{[\rho_1} h_{\rho_2]}^\rho \bar{\nabla}_{[\rho_3} \xi_{\rho_4]} + 2\bar{\nabla}_{[\rho_3} h_{[\rho_1}^\rho \bar{\nabla}_{\rho_2]} \xi_{\rho_4]}, \quad (5.104)$$

This composite tensor field consists of the more symmetric tensor form which possesses the antisymmetry of 3-indexes' permutation:

$$\mathcal{K}_{\rho_1 \rho_2 \rho_3 \rho_4}^\rho = \frac{3}{2} \kappa_{\rho_1 [\rho_2 \rho_3 \rho_4]}^\rho - \frac{3}{2} \kappa_{\rho_2 [\rho_1 \rho_3 \rho_4]}^\rho, \quad (5.105)$$

where

$$\kappa_{\rho_1[\rho_2\rho_3\rho_4]}^\rho = \bar{\nabla}_{[\rho_2} h_{|\rho_1|}^\rho \bar{\nabla}_{\rho_4} \xi_{\rho_3]}. \quad (5.106)$$

Thus all the rest terms are deleted owing to the Riemann tensorial feature, especially the first Bianchi identity,

$$R_{(1)[\nu_1\nu_2\nu_3]}^\mu = 0. \quad (5.107)$$

We expect this systematic method should be effective for the wide variation of modified gravitational theories, e.g. Lovelock theory, because it is straightforward to extend to the higher power terms by factorizing $R_{(1)}^{\mu_1\mu_2}{}_{\nu_1\nu_2}$.

Adopting the current formula (5.97) seems reliable since this integral actually gives the correct mass value in the static black hole case (5.27). Seeing this, you can detect the leading term of the co-potential easily, attributed to the large symmetry as the spherical isometry,

$$\begin{aligned} \mathcal{B}^{rt}(-\partial_t; \text{sp.EGB}) &\sim - \frac{3l^2}{128(D-3)(D-4)\pi G} \xi^t \bar{\nabla}^{[n_1} h_{[n_1}^{[r]} R_{(1)}^{n_2 n_3]}{}_{n_2 n_3]} \\ &\sim + \frac{D(D-1)(D-2)M}{64(D-3)(D-4)\pi G r^{D-2}}. \end{aligned} \quad (5.108)$$

All the components can be read from the list in App. C. This results in the finite mass value.

$$\begin{aligned} m &= Q^t(-\partial_t \mid \text{EGB static BH}) \\ &= \frac{D(D-1)(D-2)M}{64(D-3)(D-4)\pi G}. \end{aligned} \quad (5.109)$$

5.3.3 Angular momentum and Energy of Sasakian spacetime

Applying this current for the new Sasakian solution, we obtain conserved charges of that spacetime. However, the evaluation is not so ease as the spherical case because of the lower isometry, or equivalently, the cross component of the deviation. To manage these terms, the ortho-normal frame is suitable. Then, taking the background AdS metric $\bar{g}$ as

$$\begin{aligned} \bar{g}_{\mu\nu} dx^\mu dx^\nu &= -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{D-2}^2, \\ &= -\left(\frac{r^2}{l^2} + \frac{\gamma^2}{4}\right)dt^2 + \frac{dr^2}{\left(\frac{r^2}{l^2} + \frac{\gamma^2}{4}\right)} + r^2(\check{g}_{ij}dx^i dx^j + \gamma^2(d\phi + \check{A}_i dx^i)^2), \end{aligned} \quad (5.110)$$

the vielbein is given by

$$\bar{g}_{\mu\nu}dx^\mu dx^\nu \equiv \eta_{AB}e^A e^B, \quad \eta_{AB} = \text{diag}(-, +, \dots, +), \quad (5.111)$$

$$A = (0, 1, 2, a), \quad (5.112)$$

$$\bar{e}^0 = \sqrt{f(r)}dt, \quad \bar{e}^1 = \frac{dr}{\sqrt{f(r)}}, \quad \bar{e}^2 = \gamma r(d\phi + \check{A}_i dx^i), \quad (5.113)$$

$$\bar{e}^a = r\check{e}^a, \quad \delta_{ab}\check{e}_i^a \check{e}_j^b dx^i dx^j = d\check{s}^2, \quad (5.114)$$

where $d\check{s}^2$ is the Fubini-Study metric of CP_{n-1} , and γ is its radius. Fixing $\gamma = 2$ can be done without loss of generality by the scaling of r . The deviation h is written in the ortho-normal frame:

$$\begin{aligned} h_{\mu\nu}dx^\mu dx^\nu &= h_{00}(r)e^0 e^0 + h_{11}(r)e^1 e^1 + 2h_{02}(r)e^0 e^2 + h_{22}(r)e^2 e^2 \\ &= \frac{V(r)}{4l^2 f(r)}e^0 e^0 - \frac{V(r)}{4r^2 f(r) + V(r)}e^1 e^1 + \frac{V(r)}{2lr\sqrt{f(r)}}e^0 e^2 + \frac{V(r)}{4r^2}e^2 e^2, \end{aligned} \quad (5.115)$$

$$V(r) = \pm \sqrt{\mu_1 r^{6-2n} + \mu_2 \delta_5^D r^{8-2n} + \frac{2(2n-3)}{nl^2} Q^2 r^{8-4n}}. \quad (5.116)$$

For calculating the co-potential, it is useful to factorize its constituent as

$$\mathcal{C}_{B_1 B_2 B_3}^A \equiv -\frac{15l^2}{64(n-1)(2n-3)\pi G}(h_{[B_1}^A \bar{\nabla}_{B_2} \xi_{B_3]} + 2\xi_{[B_1} \bar{\nabla}_{B_2} h_{B_3]}^A). \quad (5.117)$$

Using this on the present frame, the integrand of the current surface integral is decomposed into

$$\mathcal{B}^{rt} = \mathcal{B}_{tr} = \mathcal{B}_{01}e_t^0 e_r^1 = \mathcal{B}_{01}, \quad (5.118)$$

$$\begin{aligned} \mathcal{B}_{01} &= \mathcal{C}_{[01A_1}^{[A_1} R_{(1) A_2 A_3]}^{A_2 A_3]} \\ &= \mathcal{C}_{[01a_1}^{[a_1} R_{(1) a_2 a_3]}^{a_2 a_3]} + 3\mathcal{C}_{[012}^{[2} R_{(1) a_1 a_2]}^{a_1 a_2]} \\ &\equiv \mathcal{B}_{01}^{[1]} + \mathcal{B}_{01}^{[2]}, \end{aligned} \quad (5.119)$$

where

$$\mathcal{B}_{01}^{[1]} = \frac{1}{10}(3\mathcal{C}_{01[a_1}^{[a_1} R_{(1) a_2 a_3]}^{a_2 a_3]} + 3\mathcal{C}_{1[a_1 a_2}^{[a_1} R_{(1) a_3]0}^{a_2 a_3]} + \mathcal{C}_{[a_1 a_2 a_3]}^{[a_1} R_{(1) 01}^{a_2 a_3]} - 3\mathcal{C}_{0[a_1 a_2}^{[a_1} R_{(1) a_3]1}^{a_2 a_3]}), \quad (5.120)$$

$$\begin{aligned} \mathcal{B}_{01}^{[2]} &= \frac{1}{10}(\mathcal{C}_{012}^2 R_{(1) a_1 a_2}^{a_1 a_2} + 2\mathcal{C}_{012}^{[a_1} R_{(1) a_1 a_2}^{a_2]2} + 2\mathcal{C}_{12[a_1}^2 R_{(1) a_2]0}^{a_1 a_2} + 4\mathcal{C}_{12[a_1}^{[a_1} R_{(1) a_2]0}^{a_2]2} \\ &\quad + \mathcal{C}_{2[a_1 a_2]}^2 R_{(1) 01}^{a_1 a_2} + 2\mathcal{C}_{2[a_1 a_2]}^{[a_1} R_{(1) 01}^{a_2]2} + \mathcal{C}_{0[a_1 a_2]}^2 R_{(1) 12}^{a_1 a_2} + 2\mathcal{C}_{0[a_1 a_2]}^{[a_1} R_{(1) 12}^{a_2]2} \\ &\quad + 2\mathcal{C}_{01[a_1}^2 R_{(1) a_2]2}^{a_1 a_2} + 4\mathcal{C}_{01[a_1}^{[a_1} R_{(1) a_2]2}^{a_2]2} + 2\mathcal{C}_{20[a_1}^2 R_{(1) a_2]1}^{a_1 a_2} + 4\mathcal{C}_{20[a_1}^{[a_1} R_{(1) a_2]1}^{a_2]2} \\ &\quad + \mathcal{C}_{1[a_1 a_2]}^2 R_{(1) 20}^{a_1 a_2} + 2\mathcal{C}_{1[a_1 a_2]}^{[a_1} R_{(1) 20}^{a_2]2}). \end{aligned} \quad (5.121)$$

We identify the angular momentum as the charge of the axial Killing vector ∂_ϕ . To evaluate the charge, it should be clarified which components of $\mathcal{C}$ are effective. The $\mathcal{C}$ component in the current integrand must not have the index-0 or index-1 at the superscript because of the anti-symmetrization. All the non-zero component in this requirement is as follows:

$$\mathcal{C}_{012}^2(\xi : \partial_\phi) = \frac{5lV(r)}{128(n-1)(2n-3)\pi Gr} \left(\frac{rV(r)'}{V(r)} - 2 \right), \quad (5.122)$$

$$\mathcal{C}_{02b}^a(\xi : \partial_\phi) = \frac{5lV(r)}{128(n-1)(2n-3)\pi Gr\sqrt{f(r)}} \check{J}_b^a, \quad (5.123)$$

$$\mathcal{C}_{12b}^a(\xi : \partial_\phi) = \frac{5l^2V(r)}{128(n-1)(2n-3)\pi Gr^2\sqrt{f(r)}} \left(\frac{f(r)}{f(r) + \frac{V(r)}{4r^2}} \right) \delta_b^a, \quad (5.124)$$

$$\mathcal{C}_{0bc}^2(\xi : \partial_\phi) = - \frac{5lV(r)}{128(n-1)(2n-3)\pi Gr\sqrt{f(r)}} \check{J}_{bc}, \quad (5.125)$$

$$\mathcal{C}_{2bc}^2(\xi : \partial_\phi) = - \frac{15l^2V(r)}{128(n-1)(2n-3)\pi Gr^2} \check{J}_{bc}. \quad (5.126)$$

Every component has the index-2 somewhere, so that the distribution of $\mathcal{B}_{01}^{[1]}$ ceases in the axial Killing current. The $\mathcal{B}_{01}^{[2]}$ is calculated to

$$\begin{aligned} \mathcal{B}_{01}^{[2]} &= \frac{1}{10} (\mathcal{C}_{012}^2 R_{(1) a_1 a_2}^{a_1 a_2} + 4\mathcal{C}_{12[a_1}^{[a_1} R_{(1) a_2]0}^{a_2]2} + \mathcal{C}_{2[a_1 a_2]}^2 R_{(1) 01}^{a_1 a_2} + \mathcal{C}_{0[a_1 a_2]}^2 R_{(1) 12}^{a_1 a_2} + 4\mathcal{C}_{20[a_1}^{[a_1} R_{(1) a_2]1}^{a_2]2}) \\ &= \frac{lV(r)^2}{256(n-1)(2n-3)\pi Gr^5} \left(2 - \frac{rV(r)'}{V(r)} \right) \left\{ \left(\frac{f(r)}{f(r) + \frac{V(r)}{4r^2}} \right) \delta_{[a_1}^{[a_1} \delta_{a_2]}^{a_2]} + \check{J}^{a_1 a_2} \check{J}_{a_1 a_2} + \check{J}_{[a_1}^{[a_1} \check{J}_{a_2]}^{a_2]} \right\}. \end{aligned} \quad (5.127)$$

The leading distribution comes from all the above $\mathcal{C}$ component, then the current integrand goes to

$$\mathcal{B}^{rt}(\partial_\phi; \text{EGB Sasakian}) \rightarrow \frac{nlV(r)^2}{128(2n-3)\pi Gr^5} \left(2 - \frac{rV(r)'}{V(r)} \right). \quad (5.128)$$

This formula properly replies the finite angular momentum:

$$\begin{aligned} j &= Q^t(\partial_\phi \mid \text{EGB Sasakian}) \\ &= \frac{n(n-1)l\mu_1}{128(2n-3)\pi G}. \end{aligned} \quad (5.129)$$

This value is valid in all the dimensions, even in the 5-dimensional case. The reason why the effect of the μ_2 parameter vanishes reads from the following current structure:

$$\frac{V(r)^2}{r^5} \left(2 - \frac{rV(r)'}{V(r)} \right) = -\frac{1}{2} \partial_r \left[\frac{V(r)^2}{r^4} \right], \quad (5.130)$$

The derivation cancels the fourth power term of r in the square root of $V(r)$. The deleted power just coincides with the squashing parameter term in the 5-dimensional solution.

Next evaluated charge is the AD energy. It seems natural to identify the AD energy as the Killing charge of $-\partial_t$, as same as the static black hole case. The $\mathcal{C}$ component for the energy is calculated as

$$\mathcal{C}_{012}^2(\xi : -\partial_t) = \frac{5l^2 f(r)V(r)}{256(n-1)(2n-3)\pi G r^3} \left(2 - \frac{rV(r)'}{V(r)} - \frac{1}{f(r)} - \frac{1}{f(r) + \frac{V(r)}{4r^2}} \right), \quad (5.131)$$

$$\mathcal{C}_{01b}^a(\xi : -\partial_t) = - \frac{5l^2 V(r)}{256(n-1)(2n-3)\pi G r^3} \left(\frac{f(r)}{f(r) + \frac{V(r)}{4r^2}} \right) \delta_b^a, \quad (5.132)$$

$$\mathcal{C}_{02b}^a(\xi : -\partial_t) = - \frac{5l^2 \sqrt{f(r)}V(r)}{256(n-1)(2n-3)\pi G r^3} \check{J}_b^a, \quad (5.133)$$

$$\mathcal{C}_{0bc}^2(\xi : -\partial_t) = - \frac{5l^2 \sqrt{f(r)}V(r)}{128(n-1)(2n-3)\pi G r^3} \check{J}_{bc}. \quad (5.134)$$

$\mathcal{C}_{01b}^a$ gives the distribution for $\mathcal{B}_{01}^{[1]}$ unless the spacetime dimensions are 5. The co-potential is calculated into

$$\mathcal{B}_{01}^{[1]} = \frac{n(n-2)f(r)l^2 V(r)^2}{512(2n-3)\pi G (f(r) + \frac{V(r)}{4r^2})r^7} \quad (5.135)$$

$$\rightarrow \frac{n(n-2)l^2 V(r)^2}{512(2n-3)\pi G r^7}, \quad (5.136)$$

$$\begin{aligned} \mathcal{B}_{01}^{[2]} &= \frac{l^2 f(r)V(r)^2}{512(2n-3)\pi G r^7} \left\{ \left(\frac{rV(r)'}{V(r)} - 2 \right) \left(\frac{(2n-3)f(r)}{f(r) + \frac{V(r)}{4r^2}} + 3 \right) \right. \\ &\quad \left. + \frac{1}{2f(r)} \left(\frac{(2n-3)f(r)}{f(r) + \frac{V(r)}{4r^2}} + 3 \right) \left(1 + \frac{f(r)}{f(r) + \frac{V(r)}{4r^2}} \right) - \frac{(2n-3)V(r)}{4(f(r) + \frac{V(r)}{4r^2})^2 r^2} \right\} \end{aligned} \quad (5.137)$$

$$\rightarrow \frac{nV(r)^2}{256(2n-3)\pi G r^5} \left(\frac{rV(r)'}{V(r)} - 2 \right) + \frac{nl^2 V(r)^2}{256r^7} + \frac{nl^2 V(r)^2}{256(2n-3)\pi G r^7} \left(\frac{rV(r)'}{V(r)} - 2 \right). \quad (5.138)$$

When the spacetime dimensions are higher than 5, the leading distribution comes from the first term of $\mathcal{B}_{01}^{[2]}$ on the asymptotic region. The r power of this term just gives the finite energy value for Eq. (5.116). All the other terms are at least two lower power of r than the leading order, so that these do not work besides $n=2$ case. In the 5-dimensional case, only the second term of $\mathcal{B}_{01}^{[2]}$ does work. The μ_2 term in $V(r)^2$ just fits for compensating the low order of this next leading term. Thus the energy evaluates to

$$\begin{aligned} m &= Q^t(-\partial_t \mid \text{EGB Sasakian}) \\ &= \frac{n}{(2n-3)2^8 \pi G} (\delta_5^D l^2 \mu_2 - (n-1)\mu_1), \end{aligned} \quad (5.139)$$

We shall finally consider the case when the Maxwell-field flux is present. It is worth mentioned that the above values are correct even this case. As is the Reissner–Nordström case, the charge effect does not affect at the asymptotic region. On the other hand, the

Maxwell charge q itself can be obtained with the ordinary Gaussian integral, which results in the appropriate value:

$$q = \frac{1}{\Omega_{D-2}} \int_{S_{D-2}} *F = \frac{Q}{l}. \quad (5.140)$$

Chap. 6

Conclusions

In this thesis, we have studied the gravitational solution with Sasakian structure and its novel relation to higher-dimensional black hole physics. Our investigation has proceeded in several superstring inspired gravities because the higher-dimensional spacetime is a necessity framework for this theory of everything. Then, we have successfully obtained new solutions in exact form, using Sasakian geometry as the metric ansatz. They have much theoretical interest, especially the possibility as the reduced limit whose original black hole has not been found yet.

In supergravity, we have succeeded in the investigation of two theories, i.e., the 5-dimensional (un-)gauged minimal SUGRA and the 11-dimensional SUGRA. First, we have found the new reduced solution of the 5-dimensional gauged minimal SUGRA, and have clarified its correspondence from a known charged rotating black hole. Although this solution violates the Sasakian condition slightly, we regard this as means of extending the Sasakian condition in any odd-dimensional spacetime; it may be used as a guiding principle for finding other reduced solutions whose original black hole has not been found yet. Therefore, we have discussed a generalization of Sasakian geometry suited to the present solution. In this argument, we have adopted the deformation of Sasakian structure by the presense of torsion, which is expected from the tensorial symmetry of the charged black hole before the reduced limit [28]. This argument has ensured that a concrete class of any odd-dimensional metrics possesses rich tensorial symmetry, so that we have find exact solutions newly, i.e., the 5-dimensional un-gauged extension and the 11-dimensional one.

In Einstein–Gauss–Bonnet gravity, using the highly isometric ansatz that admits Sasakian structure, we have also found the new spacetimes in exact form of every odd dimensions higher than 3. These solutions are novel in the sense that (i) they are rotating (their angular momentum is finite), (ii) they are asymptotically AdS at least locally, (iii) they are charged (they couple to the Maxwell-field). Despite these physical importance, such analytic solutions had not been known previously in EGB gravity. Particularly in regard to (i), we note that this is the very first case of successful evaluation of conserved charges.

It has been made possible by our detail study of the Abbott-Deser formulation, estimating the correct higher-order distributions in the case of tuning the Gauss-Bonnet coupling constant specially. Furthermore, we have distinguished our solutions from other known solutions especially in the 5-dimensional chargeless case: some of the 2-rotating solution with zero angular momentum can be reproduced as a special case of our solution, and some of the topological static black holes are also obtained similarly. The higher-dimensional or charged case of our solution is all the more different from other known spacetimes obviously.

On the whole, we have succeeded in collecting interesting samples from the viewpoint of the reduced limit or Sasakian geometry. Each of them has the characteristic variety and defects, as it were, dislocating the conventional prospect by the off-shell metric scenario: SUGRA solutions preserve the variety of the cohomogeneity but require to introduce the torsion. EGB solutions have the large family of any odd dimensions and Maxwell coupling, but do not the cohomogenetic variety in the naive form. Further exploration will unveil the proper meaning of the reduced limit, the novel relationship accomplished in Einstein gravity.

Our method can be regarded as one alternative to the Kerr-Schild ansatz because the naive Kerr-Schild form cannot catch present solutions. We are convinced that more solutions can be caught in other circumstance: such as Lovelock or even-dimensional counterpart. Our investigation also yields the contribution to Sasakian geometry itself, as a concrete example. Recently, the mathematical argument of Sasakian structure has been revived intensively, but concrete solutions in physical theories are so few that only in Einstein gravity.

Much progress is expected at exploring more solutions and analyzing their global properties. Gathering more explicit samples will reveal the true meaning of the reduced limit from rotating black holes. Our result implies concrete candidates to explore them: In SUGRA, it is expected that ST geometry includes some solutions yielded by the solution-generating technique. We consider that it is worthful to confirm how far such solutions are included in ST geometry. Since several rotating black holes in gauged SUGRA were constructed by the guess from the generated ones, the confirmation will be important to explore them. On the other hand, the EGB solution should uplift to the Lovelock solutions, which must be the considerably rare solutions possessing the finite angular momentum in such higher-derivative gravity. We expect that their matter-coupling cannot uplift straightforwardly, requiring higher-form fluxes. Furthermore, local metrics of even-dimensional analogues will contribute to reveal the intrinsic meaning of the reduced limit, while they may not be able to define globally. If this exploration can be interpreted to a sort of physical universality, our study will provide the new method to find analytic solutions having certain priority to the conventional generating technique, e.g., the variety of applicable theories, or the presence of the cosmological constant. Proceeding with the global analysis also has the great

importance. In Euclidean regime, our technical development will assist to design the more useful compact space in superstring theory, e.g. $\mathcal{M}_5$ of $\text{AdS}_5/\text{CFT}_4$. In Lorentzian regime, physical conditions will be improved. Using the enough large isometry, we want to analyze the instability in the systematic manner [101, 102, 103, 104]. Clarifying the mysterious relationship between Sasakian manifolds and Kerr-NUT-AdS spacetimes [18, 60], we will contribute to the emergence of both fields.

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Appendix A

Vandermonde identities

In this appendix the useful and important identities are presented for investigating the n -cohomogeneous Kerr-NUT-AdS, toric Sasaki or Kähler metrics in this thesis; for instance, obtaining their curvatures [105]. See also Ref. [65] for commentary of identities.

All of them are consequently based on the following identities with assuming $\{x_v | v = 1, \dots, n\}$,

$$\prod_{u=1}^n (C + x_u) = \sum_{k=0}^n C^{n-k} \sigma_k, \quad (\text{A.1})$$

where C is an arbitrary parameter and σ_k is as Eq. (3.17). If one index is prohibited, similar one holds

$$\prod_{\substack{u=1 \\ u \neq v}}^n (C + x_u) = \sum_{k=0}^{n-1} C^{n-k-1} \sigma_k^{\hat{v}}. \quad (\text{A.2})$$

Choosing C adequately, the Vandermonde identity obeys

$$\sum_{k=0}^{n-1} \frac{(-1)^k x_v^{n-k-1} \sigma_k^{\hat{v}}}{U_u} = \delta_{vu}, \quad (\text{A.3})$$

with $1 \leq v, u \leq n$, and $0 \leq i, k \leq n-1$. This has the inverse identity:

$$\sum_{v=1}^n \frac{(-1)^i x_v^{n-i-1} \sigma_k^{\hat{v}}}{U_v} = \delta_{ik}. \quad (\text{A.4})$$

It is well confirmed that this identity has several extensions. They are

$$\sum_{v=1}^n \frac{x_v^n \sigma_k^{\hat{v}}}{U_v} = \sigma_{k+1}, \quad (\text{A.5})$$

$$\sum_{v=1}^n \frac{\sigma_k^{\hat{v}}}{x_v U_v} = (-1)^{n-1} \frac{\sigma_k}{\sigma_n}, \quad (\text{A.6})$$

$$\sum_{v=1}^n \frac{x_v^{n-1+p}}{U_v} = h_p, \quad p \in \mathbb{Z}^+, \quad (\text{A.7})$$

where h_p is the p -th complete symmetric polynomial of $\{x_v\}$.

Appendix B

Curvature properties with torsion

In this appendix we collect some technical results of Chap. 4, which is useful for understanding our concrete expressions of ST geometry.

B.1 Levi-Civita connection

For the orthonormal frame (4.56) of the metric (4.50), we obtain the connection 1-forms (4.57)–(4.62). By using relation $\nabla_{e_a} e^b(e_c) = -\omega^b_c(e_a)$, we can compute the covariant derivatives as follows:

$$\nabla_{e_v} e_v = \sum_{w \neq v} \frac{\sqrt{Q_w}}{2(x_v - x_w)} e_w, \quad (\text{B.1})$$

$$\nabla_{e_v} e_u = -\frac{\sqrt{Q_u}}{2(x_v - x_u)} e_v, \quad v \neq u \quad (\text{B.2})$$

$$\nabla_{e_v} e_{\hat{v}} = \sum_{w \neq v} \frac{\sqrt{Q_w}}{2(x_v - x_w)} e_{\hat{w}} - (1 + \partial_v H) e_0, \quad (\text{B.3})$$

$$\nabla_{e_v} e_{\hat{u}} = -\frac{\sqrt{Q_u}}{2(x_v - x_u)} e_{\hat{v}}, \quad v \neq u \quad (\text{B.4})$$

$$\nabla_{e_{\hat{v}}} e_v = \partial_v \sqrt{Q_v} e_{\hat{v}} - \sum_{w \neq v} \frac{\sqrt{Q_w}}{2(x_v - x_w)} e_{\hat{w}} + (1 + \partial_v H) e_0, \quad (\text{B.5})$$

$$\nabla_{e_{\hat{v}}} e_u = -\frac{\sqrt{Q_u}}{2(x_v - x_u)} e_{\hat{v}} + \frac{\sqrt{Q_v}}{2(x_v - x_u)} e_{\hat{u}}, \quad v \neq u \quad (\text{B.6})$$

$$\nabla_{e_{\hat{v}}} e_{\hat{v}} = -\partial_v \sqrt{Q_v} e_v + \sum_{w \neq v} \frac{\sqrt{Q_w}}{2(x_v - x_w)} e_w, \quad (\text{B.7})$$

$$\nabla_{e_{\hat{v}}} e_{\hat{u}} = \frac{\sqrt{Q_u}}{2(x_v - x_u)} e_v - \frac{\sqrt{Q_v}}{2(x_v - x_u)} e_u, \quad v \neq u \quad (\text{B.8})$$

$$\nabla_{e_v} e_0 = (1 + \partial_v H) e_{\hat{v}}, \quad (\text{B.9})$$

$$\nabla_{e_{\hat{v}}} e_0 = -(1 + \partial_v H) e_v, \quad (\text{B.10})$$

$$\nabla_{e_0} e_v = (1 + \partial_v H) e_{\hat{v}} , \quad (\text{B.11})$$

$$\nabla_{e_0} e_{\hat{v}} = -(1 + \partial_v H) e_v , \quad (\text{B.12})$$

$$\nabla_{e_0} e_0 = 0 , \quad (\text{B.13})$$

where the function H is given by (4.63).

Since the curvature 2-form $\mathcal{R}^a{}_b$ is defined by the second structure equation

$$\mathcal{R}^a{}_b = d\omega^a{}_b + \sum_c \omega^a{}_c \wedge \omega^c{}_b , \quad (\text{B.14})$$

we obtain

$$\begin{aligned} \mathcal{R}^v{}_u = & K_{vu} e^v \wedge e^u + \left(K_{vu} - (1 + \partial_v H)(1 + \partial_u H) \right) e^{\hat{v}} \wedge e^{\hat{u}} \\ & - \frac{\partial_v H - \partial_u H}{2(x_v - x_u)} \sqrt{Q_u} e^{\hat{v}} \wedge e^0 + \frac{\partial_v H - \partial_u H}{2(x_v - x_u)} \sqrt{Q_v} e^{\hat{u}} \wedge e^0, \quad (v \neq u) \end{aligned} \quad (\text{B.15})$$

$$\begin{aligned} \mathcal{R}^v{}_{\hat{v}} = & -\frac{1}{2} \left(\partial_v^2 \mathcal{Q} + 6(1 + \partial_v H)^2 \right) e^v \wedge e^{\hat{v}} + 2 \sum_{u \neq v} \left(K_{vu} - (1 + \partial_v H)(1 + \partial_u H) \right) e^u \wedge e^{\hat{u}} \\ & - \sqrt{Q_v} \partial_v^2 H e^v \wedge e^0 - \sum_{u \neq v} \frac{\partial_v H - \partial_u H}{x_v - x_u} \sqrt{Q_u} e^u \wedge e^0 \end{aligned} \quad (\text{B.16})$$

$$\begin{aligned} \mathcal{R}^v{}_{\hat{u}} = & K_{vu} e^v \wedge e^{\hat{u}} + \left(K_{vu} - (1 + \partial_v H)(1 + \partial_u H) \right) e^u \wedge e^{\hat{v}} \\ & - \frac{\partial_v H - \partial_u H}{2(x_v - x_u)} \sqrt{Q_u} e^v \wedge e^0 - \frac{\partial_v H - \partial_u H}{2(x_v - x_u)} \sqrt{Q_v} e^u \wedge e^0, \quad (v \neq u) \end{aligned} \quad (\text{B.17})$$

$$\begin{aligned} \mathcal{R}^{\hat{v}}{}_{\hat{u}} = & K_{vu} e^{\hat{v}} \wedge e^{\hat{u}} + \left(K_{vu} - (1 + \partial_v H)(1 + \partial_u H) \right) e^v \wedge e^u \\ & - \frac{\partial_v H - \partial_u H}{2(x_v - x_u)} \sqrt{Q_u} e^{\hat{v}} \wedge e^0 + \frac{\partial_v H - \partial_u H}{2(x_v - x_u)} \sqrt{Q_v} e^{\hat{u}} \wedge e^0, \quad (v \neq u) \end{aligned} \quad (\text{B.18})$$

$$\begin{aligned} \mathcal{R}^v{}_0 = & -\sqrt{Q_v} \partial_v^2 H e^v \wedge e^{\hat{v}} - \sum_{u \neq v} \frac{\partial_v H - \partial_u H}{2(x_v - x_u)} \sqrt{Q_u} e^u \wedge e^{\hat{v}} \\ & - \sum_{u \neq v} \frac{\partial_v H - \partial_u H}{2(x_v - x_u)} \sqrt{Q_u} e^v \wedge e^{\hat{u}} - \sum_{u \neq v} \frac{\partial_v H - \partial_u H}{x_v - x_u} \sqrt{Q_v} e^u \wedge e^{\hat{u}} \\ & + (1 + \partial_v H)^2 e^v \wedge e^0 \end{aligned} \quad (\text{B.19})$$

$$\begin{aligned} \mathcal{R}^{\hat{v}}{}_0 = & -\sum_{u \neq v} \frac{\partial_v H - \partial_u H}{2(x_v - x_u)} \sqrt{Q_u} e^v \wedge e^u - \sum_{u \neq v} \frac{\partial_v H - \partial_u H}{2(x_v - x_u)} \sqrt{Q_v} e^{\hat{v}} \wedge e^{\hat{u}} \\ & + (1 + \partial_v H)^2 e^{\hat{v}} \wedge e^0, \end{aligned} \quad (\text{B.20})$$

where

$$K_{vu} \equiv \frac{-\partial_v \mathcal{Q} + \partial_u \mathcal{Q}}{4(x_v - x_u)}, \quad \mathcal{Q} \equiv \sum_{v=1}^n Q_v. \quad (\text{B.21})$$

The Ricci curvature is defined by

$$Ric(e_a, e_b) = \sum_c \mathcal{R}^c{}_a(e_c, e_b). \quad (\text{B.22})$$

Thus nonzero components of the Ricci curvature are

$$Ric(e_v, e_v) = -\frac{1}{2}\partial_v^2 \mathcal{Q} + 2 \sum_{u \neq v} K_{vu} - 2(1 + \partial_v H)^2 , \quad (\text{B.23})$$

$$Ric(e_{\hat{v}}, e_{\hat{v}}) = -\frac{1}{2}\partial_v^2 \mathcal{Q} + 2 \sum_{u \neq v} K_{vu} - 2(1 + \partial_v H)^2 , \quad (\text{B.24})$$

$$Ric(e_0, e_0) = 2 \sum_{v=1}^n (1 + \partial_v H)^2 . \quad (\text{B.25})$$

Note that the Ricci curvature is diagonalized. The scalar curvature is defined by

$$Scal = \sum_a Ric(e_a, e_a) . \quad (\text{B.26})$$

Thus we obtain

$$Scal = - \sum_{v=1}^n \partial_v^2 \mathcal{Q} + 4 \sum_{v \neq u} K_{vu} - 2 \sum_{v=1}^n (1 + \partial_v H)^2 . \quad (\text{B.27})$$

B.2 Connection with totally skew-symmetric torsion

Next, we compute the curvature quantities with torsion with respect to the orthonormal frame (4.56) of the metric (4.50). Since we have obtained the covariant derivatives with respect to the Levi-Civita connection ∇ , (B.1)–(B.13), hence we can compute from Eq. (4.8) the covariant derivatives with respect to the torsion connection ∇^T as

$$\nabla_{e_a}^T e_b = \nabla_{e_a} e_b + \frac{1}{2} T(e_a, e_b) . \quad (\text{B.28})$$

Thus we obtain

$$\nabla_{e_v}^T e_v = \sum_{w \neq v} \frac{\sqrt{Q_w}}{2(x_v - x_w)} e_w , \quad (\text{B.29})$$

$$\nabla_{e_v}^T e_u = -\frac{\sqrt{Q_u}}{2(x_v - x_u)} e_v , \quad (\text{B.30})$$

$$\nabla_{e_v}^T e_{\hat{v}} = \sum_{w \neq v} \frac{\sqrt{Q_w}}{2(x_v - x_w)} e_{\hat{w}} - e_0 , \quad (\text{B.31})$$

$$\nabla_{e_v}^T e_{\hat{u}} = -\frac{\sqrt{Q_u}}{2(x_v - x_u)} e_{\hat{v}} , \quad (\text{B.32})$$

$$\nabla_{e_{\hat{v}}}^T e_v = \partial_v \sqrt{Q_v} e_{\hat{v}} - \sum_{w \neq v} \frac{\sqrt{Q_w}}{2(x_v - x_w)} e_{\hat{w}} + e_0 , \quad (\text{B.33})$$

$$\nabla_{e_{\hat{v}}}^T e_u = -\frac{\sqrt{Q_u}}{2(x_v - x_u)} e_{\hat{v}} + \frac{\sqrt{Q_v}}{2(x_v - x_u)} e_{\hat{u}} , \quad (\text{B.34})$$

$$\nabla_{e_{\hat{v}}}^T e_{\hat{v}} = -\partial_v \sqrt{Q_v} e_v + \sum_{w \neq v} \frac{\sqrt{Q_w}}{2(x_v - x_w)} e_w , \quad (\text{B.35})$$

$$\nabla_{e_{\hat{v}}}^T e_{\hat{u}} = \frac{\sqrt{Q_u}}{2(x_v - x_u)} e_v - \frac{\sqrt{Q_v}}{2(x_v - x_u)} e_u , \quad (\text{B.36})$$

$$\nabla_{e_v}^T e_0 = e_{\hat{v}} , \quad (\text{B.37})$$

$$\nabla_{e_{\hat{v}}}^T e_0 = -e_v , \quad (\text{B.38})$$

$$\nabla_{e_0}^T e_v = (1 + 2\partial_v H) e_{\hat{v}} , \quad (\text{B.39})$$

$$\nabla_{e_0}^T e_{\hat{v}} = -(1 + 2\partial_v H) e_v , \quad (\text{B.40})$$

$$\nabla_{e_0}^T e_0 = 0 , \quad (\text{B.41})$$

where the function H is again given by (4.63).

B.3 Calabi-Yau with torsion metric on a cone

We begin with the metric (4.70) and choose the same orthonormal frame as (4.71), then the connection 1-forms are calculated as (4.72). For the Hermitian connection $\overset{\circ}{\nabla}^{Bis}$ with respect to the Bismut torsion (4.76), the connection 1-form with torsion $\overset{\circ}{\omega}^{Bis\alpha}_{\beta}$ are calculated as

$$\overset{\circ}{\omega}^{Bis\alpha}_{\beta} = \overset{\circ}{\omega}^{\alpha}_{\beta} - \frac{1}{2} \sum_{\gamma} Bis^{\alpha}_{\beta\gamma} \overset{\circ}{e}^{\gamma} . \quad (\text{B.42})$$

That is, we have

$$\overset{\circ}{\omega}^{Bisr}_a = -\frac{\overset{\circ}{e}^a}{r} , \quad (\text{B.43})$$

$$\overset{\circ}{\omega}^{Bisv}_u = -\frac{\sqrt{Q_u}}{2(x_v - x_u)} \frac{\overset{\circ}{e}^v}{r} - \frac{\sqrt{Q_v}}{2(x_v - x_u)} \frac{\overset{\circ}{e}^u}{r} , \quad v \neq u \quad (\text{B.44})$$

$$\overset{\circ}{\omega}^{Bisv}_{\hat{v}} = -\partial_v \sqrt{Q_v} \frac{\overset{\circ}{e}^{\hat{v}}}{r} + \sum_{u \neq v} \frac{\sqrt{Q_u}}{2(x_v - x_u)} \frac{\overset{\circ}{e}^{\hat{u}}}{r} - (1 + 2\partial_v H) \frac{\overset{\circ}{e}^0}{r} , \quad (\text{B.45})$$

$$\overset{\circ}{\omega}^{Bisv}_{\hat{u}} = \frac{\sqrt{Q_u}}{2(x_v - x_u)} \frac{\overset{\circ}{e}^{\hat{v}}}{r} - \frac{\sqrt{Q_v}}{2(x_v - x_u)} \frac{\overset{\circ}{e}^{\hat{u}}}{r} , \quad v \neq u \quad (\text{B.46})$$

$$\overset{\circ}{\omega}^{Bis\hat{v}}_{\hat{u}} = -\frac{\sqrt{Q_u}}{2(x_v - x_u)} \frac{\overset{\circ}{e}^v}{r} - \frac{\sqrt{Q_v}}{2(x_v - x_u)} \frac{\overset{\circ}{e}^u}{r} , \quad v \neq u \quad (\text{B.47})$$

$$\overset{\circ}{\omega}^{Bisv}_0 = -\frac{\overset{\circ}{e}^{\hat{v}}}{r} , \quad (\text{B.48})$$

$$\overset{\circ}{\omega}^{Bis\hat{v}}_0 = \frac{\overset{\circ}{e}^v}{r} . \quad (\text{B.49})$$

Note that if we restrict the connection 1-forms $\overset{\circ}{\omega}^{Bis\alpha}_{\beta}$ on the hyperplane of $r = 1$, then we obtain the connection 1-form $\omega^{Ta}_b = \overset{\circ}{\omega}^{Bisa}_b|_{r=1}$ with respect to the original metric g_{2n+1}

and the torsion T . Since the curvature 2-form with torsion $\mathring{\mathcal{R}}^{Bis\alpha}{}_{\beta}$ and the Ricci form with torsion $\rho^{Bis}(X, Y)$ are calculated as [106]

$$\mathring{\mathcal{R}}^{Bis\alpha}{}_{\beta}(X, Y) = g(\mathring{R}^{Bis}(X, Y)\mathring{e}_{\alpha}, \mathring{e}_{\beta}) , \quad (\text{B.50})$$

$$\rho^{Bis}(X, Y) = \frac{1}{2} \sum_{\alpha} \mathring{\mathcal{R}}^{Bis}(X, Y, \mathring{e}_{\alpha}, J(\mathring{e}_{\alpha})) , \quad (\text{B.51})$$

where $\mathring{R}^{Bis}(X, Y)$ is the curvature defined by (4.24) with respect to $\mathring{\nabla}^{Bis}$, we have the curvature 2-form with torsion as

$$\mathring{\mathcal{R}}^{Bisr}{}_0 = -2 \sum_{v=1}^n \partial_v H e^v \wedge e^{\hat{v}} , \quad (\text{B.52})$$

$$\begin{aligned} \mathring{\mathcal{R}}^{Bisv}{}_{\hat{v}} = & -\frac{1}{2} \left(\partial_v^2 \mathcal{Q} + 4 \right) e^v \wedge e^{\hat{v}} + \frac{1}{2} \sum_{u \neq v} \left(-\frac{\partial_v \mathcal{Q}}{x_v - x_u} + \frac{\partial_u \mathcal{Q}}{x_v - x_u} \right) e^u \wedge e^{\hat{u}} \\ & - 2 \sum_{u=1}^n \sqrt{Q_u} \partial_v \partial_u H e^u \wedge e^0 - 2 \sum_{u=1}^n (1 + 2\partial_v H)(1 + \partial_u H) e^u \wedge e^{\hat{u}} , \end{aligned} \quad (\text{B.53})$$

and the non-zero components of the Ricci form as

$$\rho^{Bis}(e_v, e_{\hat{v}}) = -\frac{1}{2} \partial_v^2 \mathcal{Q} + \frac{1}{2} \sum_{u \neq v} \left(-\frac{\partial_v \mathcal{Q}}{x_v - x_u} + \frac{\partial_u \mathcal{Q}}{x_v - x_u} \right) - 2(n+1)(1 + \partial_v H) . \quad (\text{B.54})$$

Thus we find that $\rho^B(X, Y) = 0$ for all vector fields X, Y , when provided that the functions X_v and N_v take the form

$$X_v(x_v) = -4x_v^{n+1} + \sum_{j=1}^n c_j x_v^j + b_v - 4(n+1)q_v x_v , \quad (\text{B.55})$$

$$N_v(x_v) = \sum_{i=1}^{n-1} a_i x_v^i + q_v , \quad (\text{B.56})$$

where a_i , b_j , m_v and q_v are constant parameters. This gives a Calabi-Yau with torsion metric on a cone. The function (B.55) in five dimensions is different from (4.94).

Appendix C

Deviation field for AD construction

C.1 Deviation field for AD construction

In this appendix we provide the explicit calculation of the deviation field in Chap. 5. Owing to the low cohomogeneity, even by the component expression the deviation field can be written down fairly simple form.

Considering the background AdS metric (5.110), we introduce the inverse of its vielbein:

$$\bar{g}^{\mu\nu}\partial_\mu\partial_\nu = -\frac{1}{(\frac{r^2}{l^2} + \frac{\gamma^2}{4})}\partial_t^2 + (\frac{r^2}{l^2} + \frac{\gamma^2}{4})\partial_r^2 + \frac{1}{r^2}(\check{g}^{ij}(\partial_i - \check{A}_i\partial_\phi)(\partial_j - \check{A}_j\partial_\phi) + \gamma^{-2}\partial_\phi^2), \quad (\text{C.1})$$

$$\equiv \eta^{AB}\bar{E}_A\bar{E}_B, \quad (\text{C.2})$$

with

$$\bar{E}_0 = \frac{1}{\sqrt{f(r)}}\partial_t, \quad \bar{E}_1 = \sqrt{f(r)}\partial_r, \quad \bar{E}_2 = \frac{1}{\gamma r}\partial_\phi, \quad (\text{C.3})$$

$$\bar{E}_a = \frac{1}{r}(\check{E}_a - \check{A}_a\partial_\phi), \quad \check{E}_a^i\check{E}_a^j\partial_i\partial_j = \partial_s^2. \quad (\text{C.4})$$

In the component expression, the spin connection is given by

$$\bar{\omega}_\mu{}^A{}_B = \bar{E}_B^\nu\partial_{[\nu}\bar{e}_{\mu]}^A - \bar{E}^{A\nu}\partial_{[\nu}\bar{e}_{\mu]B} + \bar{e}_\mu^C\bar{E}^{A\nu}\bar{E}_B^\rho\partial_{[\rho}\bar{e}_{\nu]C}. \quad (\text{C.5})$$

They result in

$$\bar{\omega}_0{}^1{}_0 = \frac{f'}{2\sqrt{f}}, \quad \bar{\omega}_0{}^\alpha{}_0 = \bar{\omega}_0{}^\alpha{}_1 = \bar{\omega}_0{}^\alpha{}_\beta = 0, \quad (\text{C.6})$$

$$\bar{\omega}_\alpha{}^1{}_\beta = -\frac{\sqrt{f}}{r}\delta_{\alpha\beta}, \quad \bar{\omega}_1{}^B{}_A = 0, \quad (\text{C.7})$$

$$\bar{\omega}_\alpha{}^\gamma{}_\beta = \frac{1}{r}\bar{\bar{\omega}}_\alpha{}^\gamma{}_\beta, \quad (\text{C.8})$$

$$\bar{\omega}_2{}^a{}_b = \frac{\gamma}{2r}\check{J}^a{}_b, \quad \bar{\omega}_a{}^2{}_b = -\frac{\gamma}{2r}\check{J}_{ab}, \quad (\text{C.9})$$

$$\bar{\omega}_a{}^b{}_c = \frac{1}{r}\check{\bar{\omega}}_a{}^b{}_c, \quad \bar{\omega}_2{}^a{}_2 = 0. \quad (\text{C.10})$$

We examine the deviation field in the extent of the following ansatz,

$$h_{\mu\nu}dx^\mu dx^\nu = h_{00}(r)\bar{e}^0\bar{e}^0 + h_{11}(r)\bar{e}^1\bar{e}^1 + 2h_{02}(r)\bar{e}^0\bar{e}^2 + h_{22}(r)\bar{e}^2\bar{e}^2. \quad (C.11)$$

Then their first-order derivatives are calculated to

$$\bar{\nabla}_{[0}h_{1]}^0 = -\frac{1}{2}(\partial(\sqrt{f}h_0^0) - \frac{f'}{2\sqrt{f}}h_1^1), \quad (C.12)$$

$$\bar{\nabla}_{[0}h_{1]}^2 = -\frac{1}{2}\partial(\sqrt{f}h_0^2), \quad \bar{\nabla}_{[0}h_{1]}^1 = \bar{\nabla}_{[0}h_{1]}^a = 0, \quad (C.13)$$

$$\bar{\nabla}_{[1}h_{2]}^0 = \frac{\sqrt{f}}{2r}\partial(rh_2^0), \quad (C.14)$$

$$\bar{\nabla}_{[1}h_{2]}^2 = \frac{\sqrt{f}}{2r}(\partial(rh_2^2) - h_1^1), \quad (C.15)$$

$$\bar{\nabla}_{[1}h_{2]}^1 = \bar{\nabla}_{[1}h_{2]}^a = 0, \quad (C.16)$$

$$\bar{\nabla}_{[1}h_{a]}^b = -\frac{\sqrt{f}}{2r}h_1^1\delta_a^b, \quad (C.17)$$

$$\bar{\nabla}_{[1}h_a^0 = \bar{\nabla}_{[1}h_a^1 = \bar{\nabla}_{[1}h_a^2 = 0, \quad (C.18)$$

$$\bar{\nabla}_{[0}h_{2]}^1 = \frac{\sqrt{f}}{2}(\frac{f'}{2f} - \frac{1}{r})h_2^0, \quad (C.19)$$

$$\bar{\nabla}_{[0}h_{2]}^0 = \bar{\nabla}_{[0}h_{2]}^2 = \bar{\nabla}_{[0}h_{2]}^a = 0, \quad (C.20)$$

$$\bar{\nabla}_{[0}h_{a]}^b = -\frac{\gamma}{4r}h_0^2\check{J}_a^b, \quad (C.21)$$

$$\bar{\nabla}_{[0}h_a^0 = \bar{\nabla}_{[0}h_a^1 = \bar{\nabla}_{[0}h_a^2 = 0, \quad (C.22)$$

$$\bar{\nabla}_{[2}h_{a]}^b = -\frac{\gamma}{4r}h_2^2\check{J}_a^b, \quad (C.23)$$

$$\bar{\nabla}_{[2}h_{a]}^0 = \bar{\nabla}_{[2}h_{a]}^1 = \bar{\nabla}_{[2}h_{a]}^2 = 0, \quad (C.24)$$

$$\bar{\nabla}_{[a}h_{b]}^0 = \frac{\gamma}{2r}h_2^0\check{J}_{ab}, \quad \bar{\nabla}_{[a}h_{b]}^2 = \frac{\gamma}{2r}h_2^2\check{J}_{ab}, \quad (C.25)$$

$$\bar{\nabla}_{[a}h_{b]}^1 = \bar{\nabla}_{[a}h_{b]}^c = 0. \quad (C.26)$$

These give the second-order derivative terms as

$$R_{(1)01}^{01} = -\frac{f}{2}h_0^{0''} - \frac{3r}{2l^2}h_0^{0'} + \frac{r}{2l^2}h_1^{1'} + \frac{1}{l^2}h_1^1, \quad (C.27)$$

$$R_{(1)12}^{01} = \frac{f}{2}h_2^{0''} + (\frac{r}{l^2} + \frac{f}{2r})h_2^{0'} + (\frac{3}{2l^2} - \frac{f}{2r^2} - \frac{r^2}{2l^4f})h_2^0, \quad (C.28)$$

$$R_{(1)02}^{01} = -\frac{1}{2l^2}(rh_2^{0'} + h_2^0), \quad (C.29)$$

$$R_{(1)ab}^{01} = -\frac{\sqrt{f}}{r}(h_2^{0'} + \frac{f'}{2f}h_2^0 - \frac{1}{r}h_2^0)\check{J}_{ab}, \quad (C.30)$$

$$R_{(1)1a}^{01} = R_{(1)0a}^{01} = R_{(1)2a}^{01} = 0, \quad (C.31)$$

$$R_{(1)01}^{12} = \frac{f}{2}h_0^{2''} + (\frac{r}{l^2} + \frac{f}{2r})h_0^{2'} + (\frac{3}{2l^2} - \frac{f}{2r^2} - \frac{r^2}{2l^4f})h_0^2, \quad (C.32)$$

$$R_{(1)12}^{12} = -\frac{f}{2}h_2^{2''} - \frac{f}{2r}h_2^{2'} + \frac{1}{l^2}h_1^1, \quad (C.33)$$

$$R_{(1)ab}^{12} = -\frac{\sqrt{f}}{r}h_2^{2'}\check{J}_{ab}, \quad (C.34)$$

$$R_{(1)1a}^{12} = R_{(1)02}^{12} = R_{(1)0a}^{12} = R_{(1)2a}^{12} = 0, \quad (C.35)$$

$$R_{(1)1a}^{1c} = \delta_a^c \left(\frac{f}{2r}h_1^{1'} + \frac{3}{2l^2}h_1^1 \right), \quad (C.36)$$

$$R_{(1)0a}^{1c} = \frac{\sqrt{f}}{2r}h_0^{2'}\check{J}_a^c, \quad (C.37)$$

$$R_{(1)2a}^{1c} = \frac{\sqrt{f}}{2r}h_2^{2'}\check{J}_a^c, \quad (C.38)$$

$$R_{(1)01}^{1c} = R_{(1)12}^{1c} = R_{(1)02}^{1c} = R_{(1)ab}^{1c} = 0, \quad (C.39)$$

$$R_{(1)02}^{02} = -\frac{f}{2r}h_0^{0'} + \frac{1}{2l^2}(h_1^1 + h_2^2), \quad (C.40)$$

$$R_{(1)01}^{02} = R_{(1)12}^{02} = R_{(1)1a}^{02} = 0, \quad (C.41)$$

$$R_{(1)0a}^{02} = R_{(1)2a}^{02} = R_{(1)ab}^{02} = 0, \quad (C.42)$$

$$R_{(1)1a}^{0c} = -\frac{\sqrt{f}}{r^2}h_2^0\check{J}_a^c, \quad (C.43)$$

$$R_{(1)0a}^{0c} = \delta_a^c \left(\frac{1}{2l^2}h_1^1 - \frac{f}{2r}h_0^{0'} \right), \quad (C.44)$$

$$R_{(1)2a}^{0c} = \delta_a^c \left(-\frac{f}{2r}h_2^{0'} + \frac{1}{2r^2}h_2^0 \right), \quad (C.45)$$

$$R_{(1)01}^{0c} = R_{(1)12}^{0c} = R_{(1)02}^{0c} = R_{(1)ab}^{0c} = 0, \quad (C.46)$$

$$R_{(1)1a}^{2c} = -\frac{\sqrt{f}}{2r}h_2^{2'}\check{J}_a^c, \quad (C.47)$$

$$R_{(1)0a}^{2c} = \delta_a^c \left(-\frac{f}{2r}h_0^{2'} + \frac{1}{2r^2}h_0^2 \right), \quad (C.48)$$

$$R_{(1)2a}^{2c} = \delta_a^c \left(-\frac{f}{2r}h_2^{2'} + \frac{1}{r^2}h_2^2 + \frac{f}{r^2}h_1^1 \right), \quad (C.49)$$

$$R_{(1)01}^{2c} = R_{(1)12}^{2c} = R_{(1)02}^{2c} = R_{(1)ab}^{2c} = 0, \quad (C.50)$$

$$R_{(1)01}^{ce} = \frac{\sqrt{f}}{r}(h_0^{2'} + \frac{f'}{2f}h_0^2 - \frac{1}{r}h_0^2)\check{J}^{ce}, \quad (C.51)$$

$$R_{(1)12}^{ce} = -\frac{\sqrt{f}}{r}h_2^{2'}\check{J}^{ce}, \quad (C.52)$$

$$R_{(1)ab}^{ce} = -\frac{2}{r^2}h_2^2(\check{J}^{ce}\check{J}_{ab} + \check{J}_{[a}^{[c}\check{J}_{b]}^{e]}) + \frac{2f}{r^2}h_1^1\delta_{[a}^{[c}\delta_{b]}^{e]}, \quad (C.53)$$

$$R_{(1)1a}^{ce} = R_{(1)02}^{ce} = R_{(1)0a}^{ce} = R_{(1)2a}^{ce} = 0. \quad (C.54)$$

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