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A Typical Lower Bound for Odds Problem in Markov-dependent Trials [1]

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1 Introduction

We study a stopping problem for Markov-dependent trials of the odds problem. It may be described as follows. For a positive integer N, let $X_1, X_2, \ldots, X_N$ denote $0/1$ random variables defined on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$. These $0/1$ random variables appears according to non-homogenous Markov chain with the transition probability such that

$$\mathbf{P}_i = \begin{pmatrix} 1-\beta_i & \beta_i \\ \alpha_i & 1-\alpha_i \end{pmatrix}, \tag{1}$$

where $\beta_i := \mathrm{P}(X_{i+1} = 1|X_i = 0)$, $\alpha_i := \mathrm{P}(X_{i+1} = 0|X_i = 1)$ $\beta_0 := \mathrm{P}(X_1 = 0)$ and $\alpha_0 := \mathrm{P}(X_1 = 1) = 1 - \beta_0$. Each α_i and β_i are given. We assume $0 < \alpha_i, \beta_i < 1$ for all i. We observe these X_i's sequentially and claim that the ith trial is a *success* if $X_i = 1$. We want to find the optimal stopping rule that maximize the probability of obtaining the last success (we call this event *win*) and the probability of win.

If $0 < \alpha_i + \beta_i < 1$ for all $i = 1, 2, \ldots, N$ in the transition probability, then Hsiau and Yang [7] found the optimal rule, but it was not of odds form. Bruss [5] proved that the lower bound for odds problem in Bernoulli trials is $1/e$ for any sequence of success probabilities, $P(X_i = 1),\ i = 1, 2, \ldots, N$.

The main result of this paper is that the asymptotic lower bound of probability of win is also $1/e$ for any transition probability of Markov chain under a certain condition. I think it is wonderful!

2 Main result

Recall that Ano, Kakie and Miyoshi [3] proved that even thought it is for Markov-dependent trials, the optimal stopping rule can be expressed as of odds form. Let

$$p_{ij} := \begin{cases} \mathrm{P}(X_{i+1} = 1|X_i = 1, X_{i+2} = 0) = (1-\alpha_i)\alpha_{i+1}, & j = i+1, \\ \mathrm{P}(X_{i+1} = 1|X_{j-1} = 0, X_{j+1} = 0) = \beta_{j-1}\alpha_j, & j > i+1, \end{cases}$$

and $r_{ij} = p_{ij}/(1-p_{ij})$. This is our key setting inspired by the incredible insight in Ferguson [6] who studied the general dependent sequence of X_i in odds problem.

[1] This is an abbreviation of the original version.

Theorem 1 (Ano, Kakie and Miyoshii [3]) *Assume that* $0 < \alpha_i, \beta_i < 1$ *for each* $i \in \mathcal{N}$. *The optimal single selecting strategy for the non-homogeneous Markov-dependent trials is given by*

$$\tau^* = \min\left\{ i \in \mathcal{N} : X_i = 1 \ \& \ \sum_{j=i+1}^{N} r_{ij} < 1 \right\} = \min\left\{ i \geq i^* : X_i = 1 \right\}.$$

Assume that $X_1 = 1$ *a.s., then the probability of win holds the inequality*

$$\mathrm{P}_N(\text{win}) = \mathrm{P}_N(\alpha_0, \ldots, \alpha_{N-1}, \beta_0, \ldots, \beta_{N-1}) \geq R_{i^*-1} V_{i^*-1},$$

where $R_s = \sum_{j=s+1}^{N} r_{sj}$ *and* $V_s = \alpha_s \prod_{k=s+1}^{N-1}(1-\beta_k)$.

Next theorem is the main result of this paper.

Theorem 2 *Assume that* $X_1 = 1$, *a.s. If* $R_s = \sum_{j=s+1}^{N} r_{sj}$ *with* $s = i^* - 1$, *then*

(i) $\mathrm{P}_N(\text{win}) \geq R_s V_s > R_s e^{-R_s}$.

(ii) *If* $R_s = R_{s(N)} \to 1$ *as* $N \to \infty$, *then* $\lim_{N\to\infty} \mathrm{P}_N(\text{win}) > 1/e$.

Proof.

(i) By the optimality equation $M_i = \max\left\{V_i, \sum_{j=i+1}^{N} \mathbf{P}_{ij} M_j\right\}$ and since $\sum_{j=i+1}^{N} \mathbf{P}_{ij} M_j$ is decreasing in i, we have

$$\begin{aligned}
\mathrm{P}_N(\text{win}) &= \mathrm{P}_N(\text{win}|X_1 = 1) \\
&=: M_1 = \max\left\{ V_1, \sum_{j=2}^{N} \mathbf{P}_{2j} M_j \right\} \\
&\geq \sum_{j=s}^{N} \mathbf{P}_{sj} M_j \geq \sum_{j=s}^{N} \mathbf{P}_{sj} V_j,
\end{aligned} \tag{2}$$

where $V_i = \mathrm{P}_N(\text{win by stop at } X_i = 1 | X_1 = 1) = \alpha_i \prod_{j=i+1}^{N-1}(1-\beta_j)$. Using $r_{ij} = (1-\alpha_i)\alpha_{i+1}/\alpha_i(1-\beta_{i+1})$ for $j = i+1$; $= \alpha_{j-1}\beta_j \,/(1-\beta_{j-1})\,(1-\beta_j)$ for $j > i+1$, we have

$$\mathrm{P}_N(\text{win}) \geq \sum_{j=s+1}^{N} \frac{\mathbf{P}_{sj} V_j}{V_s} V_s = \sum_{j=s+1}^{N} r_{sj} V_s = R_s V_s.$$

(ii) Note that $V_s = \prod_{k=s+1}^{N-1} q_{sk} / (\prod_{k=s+1}^{\tilde{N}-1}(1-\beta_k))$, where $\tilde{N} = N$ if N is an even integer, and $\tilde{N} = N - 1$ if N is an odd integer. Since $1 - \beta_k < 1$,

$$\mathrm{P}_N(\text{win}) \geq R_s V_s = \frac{R_s \prod_{k=s+1}^{N-1} q_{sk}}{\prod_{k=s+1}^{\tilde{N}-1}(1-\beta_k)} > R_s \prod_{k=s+1}^{N-1} q_{sk}.$$

From $R_s = \sum_{k=s+1}^{N}(1/q_{sk} - 1)$, we have $\sum_{k=s+1}^{N}(1/q_{sj}) = R_s + N - s$. By the inequality for arithmetic mean and geometric mean, then

$$\left(\prod_{k=s+1}^{N} \frac{1}{q_{sk}}\right)^{\frac{1}{N-s}} = \left(\frac{1}{\prod_{k=s+1}^{N} q_{sk}}\right)^{\frac{1}{N-s}} \le \frac{\sum_{k=s+1}^{N} \frac{1}{q_{sk}}}{N-s} = 1 + \frac{R_s}{N-s}$$

and thus $\prod_{k=s+1}^{N} q_{sk} \ge (1 + R_s/(N-s))^{-(N-s)}$. From $(1 + R_s/(N-s))^{-(N-s)} \downarrow e^{-R_s}$ as $N \to \infty$, it follows that

$$\mathrm{P}_N(\text{win}) > R_s \prod_{k=s+1}^{N-1} q_{sk} \ge R_s \left(1 + \frac{R_s}{N-s}\right)^{-(N-s)} > R_s e^{-R_s} \to 1/e,$$

as $N \to \infty$.

□

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