

VASSILIEV TYPE INVARIANTS OF ORDER ONE OF GENRIC MAPPINGS FROM A SURFACE TO THE PLANE

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ABSTRACT. In this note we give some isotopy invariants of C^∞ stable mappings from a closed surface M to $\mathbb{R}^2$ in the similar way as Vassiliev, Arnol'd and Goryunov [13], [2], [3], [7]. The detailed argument and applications will appear in the forthcoming paper.

§1 INTRODUCTION

V.A.Vassiliev introduced in [13] graded modules of knot invariants (the so-called Vassiliev knot invariants or knot invariants of finite type) by using appropriate stratifications of the mapping space from S^1 to $\mathbb{R}^3$. Later, his method was used to produce Arnold's invariants of immersed plane curves, denoted by $J^\pm$ and St (cf. [2], [3]), and Goryunov's invariants of generic mappings from a closed oriented surface into $\mathbb{R}^3$ (cf. [7]). In this note, we will describe in a formal way *Vassiliev type invariants of order one for isotopy classes of C^∞ stable mappings*, that is mostly based on Goryunov's description. When the target manifold of mappings is Euclidean space, we will see that such invariants corresponds to 1-cocycles of the "Vassiliev complex" for $\mathcal{A}$ -classes of multi-germs (graded by $\mathcal{A}_e$ codimension). And next, as a concrete example, we will give Vassiliev type invariants for C^∞ stable mappings from a closed surface to the plane. Throughout this paper, we assume that all manifolds and mappings are of class C^∞ .

Let N be a closed C^∞ manifold of dimension n and P a C^∞ manifold of dimension p . Recall that f is C^∞ -stable (simply called *stable*) if there is a neighborhood $\mathcal{U}$ of f in the W^∞ topology on $C^\infty(N, P)$ such that $g \in \mathcal{U}$ implies that there is $h \in \text{Diff}(N)$ and $h' \in \text{Diff}(P)$ such that $g = h' \circ f \circ h$ (i.e., g is $\mathcal{A}$ -equivalent to f). In other words, the $\mathcal{A}$ -orbit of f is open in $C^\infty(N, P)$. We shall say that two C^∞ stable maps f and g from N to P are C^∞ stably isotopic (or simply, *isotopic*) if there exist a C^∞ mapping $F : N \times [0, 1] \rightarrow P$ such that

- (1) for each $0 \leq t \leq 1$, the map $F_t : N \rightarrow P$ sending x to $F(x, t)$ is C^∞ stable ;
- (2) $F_0 = f$ and $F_1 = g$.

It can be shown that the isotopic relation is an equivalence relation among all C^∞ stable mappings in $C^\infty(N, P)$, and also that any two isotopic C^∞ stable maps are $\mathcal{A}$ -equivalent to each other (see §2). We shall often write by $[f]$ the isotopy equivalent class of a C^∞ stable mapping f .

We assume that N and P are connected. Let $\mathcal{M}$ denote the mapping space $C^\infty(N, P)$ and Γ the subset of $\mathcal{M}$ consisting of all C^∞ maps which are not C^∞ stable. The complement $\mathcal{M} - \Gamma$ consists of all C^∞ stable mappings. When $p \leq 2n + 1$ and the codimension $\sigma(n, p)$ of moduli spaces of $\mathcal{A}$ -orbits is greater than $n + 1$ (cf. [9]), it turns out that Γ can be regarded to have "codimension one in $\mathcal{M}$ ". In particular, the regular part Γ_{Reg} of Γ consists of C^∞ mappings which have only a (multi-)singularity with codimension one except for C^∞ stable singularities (namely, there is a finite set S of N such that the germ at S , $f : N, S \rightarrow P, f(S)$ has $\mathcal{A}_e$ -codimension one, and also that $f|_{N-S}$ is proper and C^∞ stable).

We are interested in numerical invariants of C^∞ stable mappings. Let R be a commutative ring with unit. A locally constant function $V : \mathcal{M} - \Gamma \rightarrow R$ is said a R valued *isotopy invariant of C^∞ stable mappings* : for any $f, g \in \mathcal{M} - \Gamma$ stably isotopic each other, $V(f) = V(g)$. It may be worthy to note that the 0-th cohomology group $H^0(\mathcal{M} - \Gamma; R)$ can be regarded as the module consisting of all G valued isotopy invariants. Let a C^∞ stable map $f_0 \in \mathcal{M} - \Gamma$ be fixed such as it defines an argumentation $\epsilon : S_0(\mathcal{M} - \Gamma) \rightarrow R$ of the singular chain complex $S_*(\mathcal{M} - \Gamma; R)$, and then each element of the reduced 0-th cohomology group $\bar{H}^0(\mathcal{M} - \Gamma; R)$ corresponds to an isotopy invariant which vanishes on the isotopy class of f_0 .

Definition 1.1. Assume that R has no elements of order 2. An isotopy invariant $V : \mathcal{M} - \Gamma \rightarrow R$ is called *Vassiliev type of order one* if V can be extended to a function $\mathcal{M} \rightarrow R$ satisfying the following condition : there is a locally finite partition $\mathcal{G}$ of Γ_{Reg} consisting of some cooriented strata $\{\Xi_i\}$ and non-coorientable strata such that

- (i) V is constant on each stratum of $\mathcal{G}$, and especially, constantly zero over non-coorientable strata ;
- (ii) V is constantly zero over $\Gamma - \Gamma_{Reg}$;
- (iii) (the difference equation) for each cooriented stratum Ξ_i and for any family of C^∞ maps $\phi = \phi_t : (-a, a) \rightarrow \mathcal{M}$, $\phi_0 \in \Xi_i$, which is transversal to Ξ_i with the positive direction compatible to the coorientation, it holds that

$$V(\Xi_i) = V([\phi_{+\epsilon}]) - V([\phi_{-\epsilon}]), \quad (\epsilon > 0)$$

- (iv) (normalization condition) V is constantly zero on the isotopy class of the distinguished map f_0 .

In particular, according to Goryunov's terminology [7], we state one more definition :

Definition 1.2. A Vassiliev type invariant V of order one is called *local* if each stratum of the partition of Γ_{Reg} corresponds to a singularity type (i.e., $\mathcal{A}$ -equivalent class of germs) with codimension one, and the coorientation of a stratum is determined by the coorientation of the corresponding singularity type (that is the coorientation of the parameter space of its versal deformation, see §2).

In §3 we will introduce *Vassiliev cycle of order one for $\mathcal{A}$ -classes of multi-germs* , and we will see in Proposition 4.2 in §4 that for the case of $P = \mathbb{R}^p$ there is one-to-one correspondence between order one local invariants and Vassiliev cycles.

Remark 1.3. (1) We can also define $\mathbb{Z}_2$ valued invariants of order one, by ignoring the coorientability of strata in the above definition. (2) Given any Vassiliev type invariants V and V' of order one, by taking a refinement of both of associated partitions of Γ_{Reg} , any linear combination $aV + bV'$ ($a, b \in \mathbb{R}$) also becomes an invariant of order one. Thus all Vassiliev type invariants of order one form a submodule of $\bar{H}^0(\mathcal{M} - \Gamma; \mathbb{R})$. (3) As in [2], [3], [14], there may be several way to coorient strata by using the data of configurations of singular point sets of maps in N .

Remark 1.4. In the above, as the mapping space $\mathcal{M}$, we consider the space of all C^∞ mappings, but it is also possible to consider the space of C^∞ mappings with several constraint as $\mathcal{M}$ (for example, the space of immersed plane curves with a fixed winding number [2], the space of plane fronts with a fixed Maslov index [3], the space of algebraic projective plane curves [15], etc).

Now let us consider a special case where N is a connected closed surface and P is the 2-plane $\mathbb{R}^2$. Elements of $\mathcal{M} - \Gamma$, i.e., C^∞ stable maps f , can be characterized as follows : f admits singularities only of type (1) fold, (2) cusp (3) double fold (bi-germ of fold types whose contours are transverse to each other). Besides, generic 1-parameter local bifurcations of multi-singularities, $N \times \mathbb{R}, S \times \{0\} \rightarrow \mathbb{R}^2, 0$, S being a finite set, can be also classified. The classification (for uni-germs, the case where S is a single point) is due to Arnold [1], Rieger [10], and Rieger-Ruas [11]. These bifurcations of apparent contours and images are depicted in Figure 1 below, and normal forms are given in Table 1 on the end of §5.

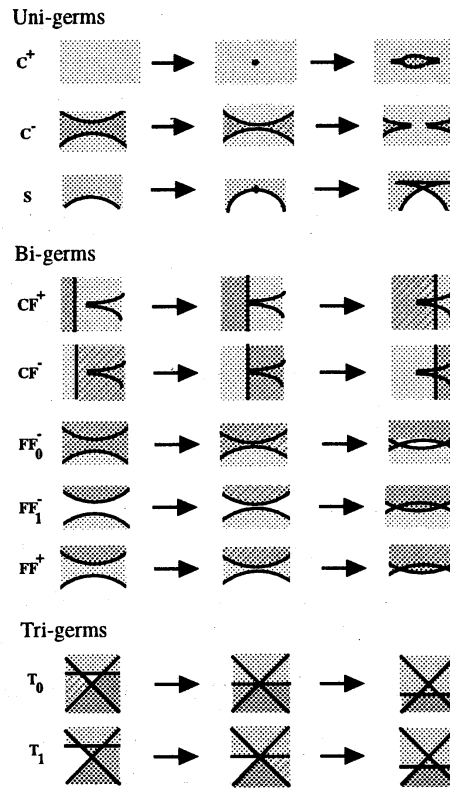


Figure 1

Generic 1-parameter bifurcations of map-germs of the plane to the plane
The bracket lines are the apparent contours and the darked areas are the images.

The main result is the following theorem :

Theorem 1.5. *The submodule of $\bar{H}^0(\mathcal{M} - \Gamma, \mathbb{Z})$ consisting of local Vassiliev type invariants of order one are generated by the following three invariants :*

$$\begin{aligned} I_C &:= C + S, \\ I_D &:= S + 2CF + 2FF^+ + 2FF^-, \\ I_F &:= 2FF^- + CF. \end{aligned}$$

Theorem 1.6. *The submodule of $\bar{H}^0(\mathcal{M} - \Gamma, \mathbb{Z}_2)$ consisting of local Vassiliev type invariants of order one are generated by the following three invariants :*

$$I_{C;2} := C, I_{D;2} := S, I_{F;2} := CF.$$

Remark 1.7. (1) The choice of f_0 is of course not unique, and there is no standard way to choose it. (2) The value of the invariant I_C is equal to a half of the difference

between the (geometric) number of cusps of f and one of the distinguished map f_0 . Also the value of the invariant I_S is equal to the difference between the (geometric) number of transverse double folds points of f and one of f_0 .

§2 PRELIMINARY : MULTI-GERMS AND $\mathcal{A}$ -EQUIVALENCE

In this section, we quickly review the most fundamental notions in Singularity Theory, which will be used later. For the detail, see, e.g., [16], [6], [8], [9], [4], [5].

Multi-germs, deformations and $\mathcal{A}$ -equivalences.

Two maps f and g between N and P is said to define the same *germ* at a compact subset S of N if there is a neighborhood of S on which f coincides to g . Usually we are concerned with the case when S consists of finitely many points and $f(S)$ is one point, and we shall simply write the germ of f at S like as $f : N, S \rightarrow P, y$. In particular, we often say it a *multi-germ* if S is not one point. A deformation of a multi-germ $f : N, S \rightarrow P, y$ with a parameter space $\mathbb{R}^s$ centered at 0 means a germ $F : N \times \mathbb{R}^s, S \times \{0\} \rightarrow P, y$ satisfying that $F(x, 0) = f(x)$. We often write $F_p(x)$ to be $F(x, p)$. Let $\pi : N \times \mathbb{R}^s \rightarrow \mathbb{R}^s$ denote the projection onto the parameter space.

Map-germs $f : N, S \rightarrow P, y$ and $g : N', S' \rightarrow P', y'$ are called *$\mathcal{A}$ -equivalent* if there exist germs of diffeomorphisms $\sigma : N, S \rightarrow N', S'$ and $\varphi : P, y \rightarrow P', y'$ such that $g \circ \sigma = \varphi \circ f$. Deformations F of f and G of g with the same dimension of parameters are called *$\mathcal{A}$ -equivalent* if F and G are $\mathcal{A}$ -equivalent as map-germs by the diffeomorphism-germs letting the following diagram commute :

$$\begin{array}{ccccc} (N \times \mathbb{R}^s, S \times \{0\}) & \xrightarrow{(F, \pi)} & (P \times \mathbb{R}^s, (y, 0)) & \xrightarrow{\pi} & (\mathbb{R}^s, 0) \\ \downarrow R & & \downarrow L & & \downarrow \phi \\ (N' \times \mathbb{R}^s, S' \times \{0\}) & \xrightarrow{(G, \pi)} & (P' \times \mathbb{R}^s, (y', 0)) & \xrightarrow{\pi} & (\mathbb{R}^s, 0). \end{array}$$

Two deformations F and G of f with the same dimension of parameter spaces are called *f -isomorphic* if F and G are $\mathcal{A}$ -equivalent by a triplet (R, L, ϕ) , where R and L are deformations of identity maps id_N and id_P , respectively.

Let $F : N \times \mathbb{R}^s, S \times \{0\} \rightarrow P, y$ be a deformation of f and $g : \mathbb{R}^t, 0 \rightarrow \mathbb{R}^s, 0$ a map-germ, then we define the induced deformation $g^*F : N \times \mathbb{R}^t, S \times \{0\} \rightarrow P, y$, by $g^*F(x, w) = F(x, g(w))$. A deformation F of f is called *versal* if any deformation G of f is isomorphic to a deformation induced from F . An versal deformation of a germ f is called *miniversal* if the parameter space has the minimal dimension in all versal deformations of f .

For a germ $f : N, S \rightarrow P, y$, let $\theta(f)_S$ denote the set of C^∞ vector fields along f , i.e., germs of C^∞ maps $\zeta : N, S \rightarrow TP$ such that $\zeta(x) \in TP_{f(x)}(x \in N)$. We set $\theta(N)_S = \theta(1_N)_S$, $\theta(P)_y = \theta(1_P)_{\{y\}}$ and let $tf : \theta(N)_S \rightarrow \theta(f)_S$ and $\omega f : \theta(P)_y \rightarrow \theta(f)_S$ be defined as $tf(\xi) = Tf \circ \xi$ and $\omega f(\eta) = \eta \circ f$. The extended tangent space $T\mathcal{A}_\epsilon f$ is given by

$$T\mathcal{A}_\epsilon f := tf[\theta(N)_S] + \omega f[\theta(P)_y] \subset \theta(f)_S,$$

and the dimension of the quotient space $\theta(f)_S/T\mathcal{A}_\epsilon f$ is called $\mathcal{A}_\epsilon$ -codimension of f .

When $\mathcal{A}_\epsilon$ -codimension of f is finite, letting $\{g_i\}$ be a $\mathbb{R}$ -basis of $\theta(f)_S/T\mathcal{A}_\epsilon f$ and set $F := f + \sum_i u_i g_i$ by using a local coordinate systems of P . Then the deformation F becomes a versal deformation of f . Besides, it also holds that for any versal deformation F of f , the set of the derivatives $\partial_i F := \frac{\partial F}{\partial u_i}(x, 0)$ with respect to the parameter coordinates form a basis of $\theta(f)_S/T\mathcal{A}_\epsilon f$. A germ $f : N, S \rightarrow P, y$ is called $\mathcal{A}_\epsilon$ -finite if $\dim \theta(f)_S/T\mathcal{A}_\epsilon f < \infty$. It should be noted that every $\mathcal{A}_\epsilon$ -finite multi-germ is *finitely determined*, that is its $\mathcal{A}$ equivalent class is determined by its jet of finite order, and hence it is represented as polynomial map-germs whose images are in general position.

Coorientability.

Definition 2.1. An $\mathcal{A}_\epsilon$ -finite germ $f : N, S \rightarrow P, y$ is said to be *non-coorientable* if for any miniversal deformation F of f there is a triplet (R, L, ϕ) which makes an f -isomorphism from F to itself where ϕ is a germ of an orientation-reversing diffeomorphism of the parameter space.

Note that the (non) coorientability of $\mathcal{A}$ -finite germs are preserved under $\mathcal{A}$ -equivalence, thus we can say that an $\mathcal{A}$ -class is coorientable or non-coorientable.

Multi-jets, Transversality and Stability.

Let $N^{(r)}$ be the set of ordered r -tuples of distinct elements of N , denoted by $\mathbf{x} = \langle x_1, \dots, x_r \rangle$ with $x_i \neq x_j$ for $i \neq j$. Let $\pi_N : J^l(N, P) \rightarrow N$ denote the projection, where $J^l(N, P)$ is the bundle of l -jets. Define ${}_r J^l(N, P) = (\pi_N^r)^{-1}[N^{(r)}]$, where $\pi_N^r : J^l(N, P)^r \rightarrow N^r$ is the r fold Cartesian product of π_N with itself. A C^∞ mapping $f : N \rightarrow P$ defines a C^∞ section ${}_r j^l f : N^{(r)} \rightarrow {}_r J^l(N, P)$ sending $\langle x_1, \dots, x_r \rangle$ to $\langle j^l f(x_1), \dots, j^l f(x_r) \rangle$, which is called *the multi- l -jet extension of f* . Here are various characterizations of C^∞ stability of mappings :

Theorem 2.2. [Mather; V] *Let $r \geq p + 1$ and $l \geq p$, where p is the dimension of P . Let f be a proper C^∞ mapping from N to P . Then the following conditions are equivalent :*

- (1) f is C^∞ stable;
- (2) f is infinitesimally stable, i.e., $tf[\theta(N)] + \omega f[\theta(P)] = \theta(f)$;
- (3) $rj^l f$ is transversal to every $\mathcal{A}$ -orbit in ${}_r J^l(N, P)$;
- (4) For any point $y \in P$ and any multi-germ f_S of f at any finite subset S of $f^{-1}(y)$ consisting of r or less than r points, we have

$$\theta(f)_S = T\mathcal{A}_\epsilon f_S + m_S^{l+1} \theta(f)_S.$$

Let $F : N \times W \rightarrow P$ a C^∞ mappings, which is considered as a family of C^∞ maps from N to P with a manifold W of parameters. Such a family F defines a family of C^∞ sections

$${}_r J_1^l F : N^{(r)} \times W \rightarrow {}_r J^l(N, P), \quad {}_r J_1^l F(\mathbf{x}, p) := {}_r J^l F_p(\mathbf{x}).$$

Theorem 2.3. cf. [Mather, V] *Let $F : N \times W \rightarrow P$ a smooth family with a parameter manifold W of dimension s . Then the following conditions are equivalent :*

- (1) ${}_r j_1^l F$ is transversal to every $\mathcal{A}$ -orbit in ${}_r J^l(N, P)$;
- (2) For every $p \in W$ and every finite subset S of N consisting of r or less than r points, such that $F_p(S)$ is a single point, we have

$$\theta(f)_S = T\mathcal{A}_\epsilon F_p + \{\partial_1 F|_{u=p}, \dots, \partial_s F|_{u=p}\} \mathbb{R}.$$

A parametrized version of Thom's multi-transversality theorem are stated as follows :

Theorem 2.4. cf. [Mather, V] *Let Θ be a $\mathcal{A}$ -invariant subset of ${}_r J^l(N, P)$, and $F : N \times W \rightarrow P$ a C^∞ mapping as a family of C^∞ maps from N to P . Then F can be approximated by those families $G : N \times W \rightarrow P$ that the parametrized jet extension ${}_r j_1^l G : N^{(r)} \times W \rightarrow {}_r J^l(N, P)$ is transversal to Θ .*

§3 VASSILIEV CYCLES OF ORDER ONE FOR $\mathcal{A}$ -CLASSES

In this section, we describe a formal set-up of the first degree part of the so-called Vassiliev complex for simple $\mathcal{A}$ -equivalent classes of multi-germs of C^∞ mappings (

cf. [12], [4]). We assume that the pair of dimensions (n, p) satisfies that there are finitely many $\mathcal{A}$ -classes with $\mathcal{A}_\epsilon$ -codimension less than or equal to 2.

For each coorientable $\mathcal{A}$ -equivalent classes of multi-germs with $\mathcal{A}_\epsilon$ -codimension 1, we take a miniversal deformation of a multi-germ representing the class :

$$F_i : \mathbb{R}^n \times \mathbb{R}, S_i \times \{0\} \rightarrow \mathbb{R}^p, 0, \quad (i = 1, \dots, l).$$

For each $\mathcal{A}$ -class of multi-germs with $\mathcal{A}_\epsilon$ -codimension 2, we also take a miniversal deformation

$$G_j : \mathbb{R}^n \times \mathbb{R}^2, S'_j \times \{0\} \rightarrow \mathbb{R}^p, 0, \quad (j = 1, \dots, l').$$

We can assume that every F_i (resp. G_j) is presented at each point of S_i (resp. S'_j) as a polynomial map-germ. We fix the orientation of the parameter space $\mathbb{R}$ of each germ F_i , by which the corresponding class are cooriented. We also fix the orientation of the parameter space $\mathbb{R}^2$ of each germ $G_{j(k)}$, although the corresponding class is not necessarily coorientable. We simply write $(F_i)_t(x) = F_i(x, t)$ and $(G_j)_p(x) = G_j(x, p)$.

Then we set as a formal way

$$\begin{aligned} C^1(\mathcal{A}_{n,p}^{ori}) &:= \text{the } R\text{-module generated by } \{F_1, \dots, F_l\}, \\ C^2(\mathcal{A}_{n,p}) &:= \text{the } R\text{-module generated by } \{G_1, \dots, G_{l'}\}, \end{aligned}$$

We should remark that for each F_i the $\mathcal{A}$ -class of the induced deformation $\iota^* F_i$, where $\iota : \mathbb{R}, 0 \rightarrow \mathbb{R}, 0$ is a germ of an orientation-reversing diffeomorphism, is identified with $-F_i$ as an element in $C^1(\mathcal{A}_{n,p}^{ori})$. We don't require such identification for G_j .

Next we shall define an operator $\delta : C^1(\mathcal{A}_{n,p}^{ori}) \rightarrow C^2(\mathcal{A}_{n,p})$. To do this, for every pairs of F_i and G_j we define an integer $[F_i : G_j]$ as follows. Simply, we write F and G instead of F_i and G_j . Let $\tilde{G} : U \times W \rightarrow \mathbb{R}^p$ be a representative of the germ G , U an open neighborhood of the source points $S \subset \mathbb{R}^n$ and W an open neighborhood of the origin in $\mathbb{R}^2$. We let $W_F(\tilde{G})$ denote the set consisting of $p \in W$ satisfying that there is a point $y \in \mathbb{R}^p$ near 0 and a subset $S_p \subset \tilde{G}_p^{-1}(y)$ such that the multi-germ $\tilde{G}_p : U, S_p \rightarrow \mathbb{R}^p, y$ is equivalent to F . If $W_F(\tilde{G})$ is empty, define $[F : G]$ to be zero. Otherwise, by the multi-transversality theorem, taking U and W sufficiently small if nessacary, the closure of $W_F(\tilde{G})$ is one dimensional semialgebraic set in W whose closure contains the origin (since the closure of a $\mathcal{A}$ -finite orbit in a multi-jet space becomes a semi-algebraic set). In particular, it turns out that there is $\epsilon > 0$ such that for any $0 < \epsilon' \leq \epsilon$, the circle $S_{\epsilon'}^1$, centered at the origin with radius ϵ' is transverse to $W_F(\tilde{G})$. According to the fixed orientation of the parameter space of G , we let the circles be anti-clockwise oriented. Since the class equivalent to F_i is oriented,

the stratum $W_F(\tilde{G})$ has cooriented. Thus an intersection index of S_ϵ^1 and $W_F(\tilde{G})$ is well-defined, and we denote it by $[F : G]$. Obviously the integer is independent of the choice of the representative $\tilde{G}$, and if we take another orientation of the parameter space of G , the index has opposite sign.

Now we can define a R -homomorphism

$$\delta : C^1(\mathcal{A}_{n,p}^{ori}; R) \rightarrow C^2(\mathcal{A}_{n,p}; R), \quad \text{by} \quad \delta F_i := \sum_{j=1}^{l'} [F_i : G_j] G_j.$$

Definition 3.1. Let c be a non-trivial element of $C^1(\mathcal{A}_{n,p}^{ori})$ such that $\delta c = 0$, then we call c a *Vassiliev cycle of order one for $\mathcal{A}$ -equivalent classes of multi-germs with the pair of dimensions (n, p)* .

In the next section, for such a Vassiliev cycle we will define an invariants of isotopy classes of generic maps.

§4 INVARIANTS OF ISOTOPY CLASSES OF C^∞ STABLE MAPPINGS TO EUCLIDEAN SPACE

In this section we treat with the case that $P = \mathbb{R}^p$. As in the previous section we here assume the pair (n, p) to satisfy that there are finitely many $\mathcal{A}$ -classes with $\mathcal{A}_\epsilon$ -codimension less than or equal to 2. As in §1, we let $\mathcal{M}$ denote the mapping space $C^\infty(N, \mathbb{R}^p)$, Γ the subset of all non-generic (C^∞ unstable) mappings, and f_0 a fixed generic mapping in $\mathcal{M} - \Gamma$.

First, since the target space is a linear space $\mathbb{R}^p$, it is easily seen that the mapping space $\mathcal{M}(= C^\infty(N, \mathbb{R}^p))$ is contractible. In particular, any generic mapping f can be joined to f_0 by a smooth homotopy $\tau : N \times I \rightarrow \mathbb{R}^p$ with $\tau(x, 0) = f_0(x)$, $\tau(x, 1) = f(x)$, for instance, which can be achieved by $f_0 + t(f - f_0)$. For $t \in I$ we simply set $\tau_t : N \rightarrow \mathbb{R}^p$ to be the map sending x to $\tau(x, t)$. It is convenient to regard a smooth homotopy as a continuous path in the mapping space $\mathcal{M}$ with Whitney C^∞ topology, and when we distinguish them, we will often write $\bar{\tau} : I \rightarrow \mathcal{M}$ (i.e., $\bar{\tau}(t) := \tau_t$).

By using the parametrized transversality theorem, we can assume τ to satisfy that there is a finite subset A of I such that

- (1) at each point t outside A the map τ_t is a C^∞ stable mpping ;
- (2) at each point t of A there is a point y of $\mathbb{R}^p$ and $S \subset \tau_t^{-1}(y)$ so that the germ $\tau_i : N, S \rightarrow \mathbb{R}^p, y$ is $\mathcal{A}$ -equivalent to an oriented class in $C^1(\mathcal{A}_{n,p}^{ori})$.

For a smooth homotopy τ satisfying the property, we say roughly that the path $\bar{\tau}$ is *transverse to the discriminant Γ* . For such a path τ , we define an integer $\epsilon_i(\tau)$ to

be the number (taking accounts of sign) of events of local bifurcations of type F_i moving along the path $\bar{\tau}$. Namely, if the germ τ at $S \times \{t\}$ is equivalent to the normal form of the class F_i compatibly on the orientation of parameter lines, we count $+1$, and otherwise -1 . Summing up the signs at all events, the amount is just $\epsilon_i(\tau)$. It is reasonable to regard $\epsilon_i(\tau)$ as the intersection index of the strata of type F_i in Γ and the path $\bar{\tau}$.

Let $c \in \ker \delta$, a Vassiliev cycle of order one, and assume that c is written as a linear form $\sum_{i=1}^s \lambda_i F_i$ where F_i are generators of $C^1(\mathcal{A}_{n,p}^{ori})$ and $\lambda_i \in R$. For c , f and τ , we define an integer $I_c(f; \tau)$ by

$$I_c(f; \tau) := \sum_{i=1}^s \lambda_i \epsilon_i(\tau).$$

Lemma 4.1. *The value $I_c(f; \tau)$ is independent of the choice of τ .*

Proof of Lemma. Take another path $\bar{\tau}' : I \rightarrow \mathcal{M}$ from f_0 to f transverse to the discriminant Γ . Then we have a continuous homotopy $\eta : N \times I \rightarrow \mathbb{R}^p$ which is defined by $\eta(x, t) = \tau(x, 2t)$ for $0 \leq t \leq 1/2$ and $\eta(x, t) = \tau'(x, 2-2t)$ for $1/2 \leq t \leq 1$. The homotopy η is smooth off $t = 0$ and 1 , and we can slightly modify η to be a C^∞ mapping over $N \times I$ using the partition of unity if necessary. Since η defines a continuous loop in $\mathcal{M}$ and $\mathcal{M}$ is contractible, there is a C^∞ mapping $\Xi : N \times D^2 \rightarrow \mathbb{R}^p$, where D^2 is the unit closed disc in $\mathbb{C}$ centered at the origin satisfying that $\Xi(x, e^{2\pi t}) = \eta(x, t)$.

By the transversality theorem, it can be assumed that the parametrized jet extension of Ξ is transversal to all $\mathcal{A}$ -orbits of $\mathcal{A}_e$ -codimension less than or equal to two. Hence there is a Whitney stratification $\mathcal{W}$ of D^2 satisfying the following properties :

- (1) for any point p in the top strata of dimension 2, $\Xi_p : N \rightarrow \mathbb{R}^p$ is C^∞ stable;
- (2) each 1-dimensional stratum consists of such points $p \in D^2$ which satisfy that there is a point y of $\mathbb{R}^p$ such that the germ $\Xi_p : N, S \rightarrow \mathbb{R}^p, \{y\}$, $S \subset \Xi_p^{-1}(y)$ is $\mathcal{A}$ -equivalent to a class $(F_i)_0$ in $C^1(\mathcal{A}_{n,p}^{ori})$, and such a stratum is denoted by W_{F_i} ;
- (3) for each point of the 0-dimensional strata $\{p_1, \dots, p_s\}$, $\Xi_{p_k} : N \times p_k \rightarrow \mathbb{R}^p$ has a (multi-) singularity equivalent to a class in $C^2(\mathcal{A}_{n,p})$, denoted by $G_{j(k)}$ ($k = 1, \dots, s$).

We take small disjoint k discs $B(p_k)$ ($k = 1, \dots, s$) centered at p_k in the interior $\text{int} D^2$ transverse to the stratification $\mathcal{W}$. Let ∂D^2 and every $\partial B(p_k)$ be anti-clockwise oriented. It can be easily verified that the intersection index of ∂D^2 and W_{F_i} is equal to the sum of the intersection indices $\partial B(p_k)$ and W_{F_i} over all $k = 1, \dots, s$. Hence,

by definitions, we have that $\epsilon_i(\eta) = \sum_k \pm [F_i : G_{j(k)}]$ where the sign $\pm$ depends on the fixed orientation of the parameters of $G_{j(k)}$. Thus,

$$\begin{aligned}
 I_c(f; \tau) - I_c(f; \tau') &= \sum_i \lambda_i (\epsilon_i(\tau) - \epsilon_i(\tau')) \\
 &= \sum_i \lambda_i \epsilon_i(\eta) \\
 &= \sum_i \lambda_i \sum_k \pm [F_i : G_{j(k)}] \\
 &= \sum_k \pm \sum_i \lambda_i [F_i : G_{j(k)}] \\
 &= \sum_k \pm (\text{the coefficient of } \delta c \text{ with respect to } G_{j(k)}) = 0.
 \end{aligned}$$

This completes the proof.

In the same way of the above proof, we can see that the integer $I_c(f; \tau)$ depends only on the isotopy classes of f and f_0 . So we shall write it by $I_c(f; f_0)$ or simply $I_c(f)$. This defines a homomorphism $I : \ker \delta \rightarrow \bar{H}^0(\mathcal{M} - \Gamma; R)$. In particular, we can show that the following proposition :

Proposition 4.2. *For each cycle $c \in \ker \delta$, I_c is an isotopy invariant of local Vassiliev type of order one described as in §1. Furthermore, when $\dim N$ is greater than 1, every order one local invariant can be expressed as I_c for some $c \in \ker \delta$.*

The second assertion comes from the fact that the subset of $\mathcal{M}$ consisting of C^∞ maps of N to $\mathbb{R}^p$ which have singularity of type F_i (the closure of the strata of Γ_{Reg} corresponding to the class F_i) is connented.

§5 $\mathcal{A}$ -CLASSES FOR MAPPINGS FROM THE PLANE TO THE PLANE AND THEOREMS

From now on we treat with C^∞ mappings from a closed surface N to 2-plane $\mathbb{R}^2$. The lists at the end of this section show all $\mathcal{A}$ -equivalent classes of multi-germs from the plane to the plane with $\mathcal{A}_\epsilon$ -codimension less than or equal to 2. The classification of uni-germs is due to Rieger [10] and Rieger-Ruas [11], and we use their notation for uni-germs. For 1-parameter deformations, we consider $\mathcal{A}$ -equivalent classes of oriented deformations. In the list, every multi-germ $N, S \rightarrow P, y, S = \{p_k\}_k$, is described as the set consisting of k germs $\mathbb{R}^n, 0 \rightarrow \mathbb{R}^p, 0$ taking local coordinate systems of N centered at p_k and a local coordinate system of P centered at y .

Coorientation.

As to the coorientation, we define the orientation of the parameter line as the direction such that the number of cusp points and double fold points increase for uni-germs and bi-germs, and the number of sheets covering the “vanishing triangle” increases for triple fold points, T_0 and T_1 . Figure 1 in §1 depicts local bifurcations of apparent contours and shadows (the image) of the map in these direction.

Vassiliev complex.

The module $C^1(\mathcal{A}_{2,2}^{ori}; \mathbb{Z})$ (and $C^1(\mathcal{A}_{2,2}^{ori}; \mathbb{Z}_2)$) is generated by ten elements

$$C_{\pm}, S, CF^{\pm}, FF^+, FF_0^-, FF_1^-, T_{\pm},$$

and $C^2(\mathcal{A}_{2,2}; \mathbb{Z})$ (and $C^2(\mathcal{A}_{2,2}; \mathbb{Z}_2)$) is generated by

$$[6^{\pm}], [4_3^{\pm}], [11_5], I_{2,2}^{1,1}, II_{2,2}^{1,1}, \tilde{C}_{\pm}, \tilde{S}, Q_{\pm}, \tilde{F}F^+, \tilde{F}F_0^-, \tilde{F}F_1^-, CC, FC, \tilde{C}F_{\pm,\pm}, \tilde{T}_{\pm}.$$

Proposition 5.1. *The coboundary operation $\delta : C^1(\mathcal{A}_{2,2}^{ori}; \mathbb{Z}) \rightarrow C^2(\mathcal{A}_{2,2}; \mathbb{Z})$ is determined as follows (in the case coefficients in $\mathbb{Z}_2$, these equalities valid modulo 2)*

$$\begin{aligned} \delta C_+ &= [4_3^+] + [4_3^-], & \delta C_- &= -[4_3^+] - [4_3^-] - 2[11_5], & \delta S &= 2[11_5], \\ \delta FF^+ &= -[11_5] + FC, & \delta FF_0^- &= -Q_-, & \delta FF_1^- &= Q_- + FC, \\ \delta CF^+ &= \tilde{C}_+ + \tilde{C}_- + \tilde{S} - FC, & \delta CF^- &= -\tilde{C}_+ - \tilde{C}_- - \tilde{S} - FC, \\ \delta T_+ &= -\tilde{F}F_0^- - \tilde{F}F_1^- + \tilde{C}F_2 + \tilde{C}F_3, \\ \delta T_- &= -\tilde{S} + \tilde{F}F_0^- + \tilde{F}F_1^- + \tilde{C}F_2 + \tilde{C}F_3. \end{aligned}$$

This proposition follows from direct computation. Solving the equation $\delta c = 0$, we have Theorem 1.5 which is introduced in §1. In the case of coefficients in $\mathbb{Z}_2$, considering the equalities in (2) of the above Proposition modulo 2, we get Theorem 1.6.

Table of the Classification

Stable-germs	
Type	normal form $f(x, y)$
<i>regular</i>	(x, y)
<i>fold</i>	(x, y^2)
<i>cusp</i>	$(x, y^3 + xy)$
<i>double fold</i>	$(x, y^2), (x'^2, y')$
1-parameter deformations	
Type	versal deformation $F(x, y, a)$
$C^\pm(4_2)$	$(x, y^3 \pm y(x^2 - a))$
$S, (5)$	$(x, y^4 + xy - ay^2)$
$CF^\pm$	$(x, y^3 + xy), (\pm y'^2 - a, x')$
FF^+	$(x, y^2 + a), (x', x'^2 + y'^2)$
FF_0^-, FF_1^-	$(x, \mp y^2 + a), (x', x'^2 \pm y'^2)$
T_0, T_1	$(x + y^2, x - y^2 + a), (x', y'^2), (\mp x''^2, y'')$
2-parameter deformations	
Type	versal deformation $G(x, y, a, b)$
$4_3^\pm$	$(x, y^3 \pm x^3y + ax^2y + bxy)$
$6^\pm$	$(x, xy + y^5 \pm y^7 + ay^3 + by^2)$
11_5	$(x, xy^3 + y^4 + y^5 + axy + by)$
$I_{2,2}^{1,1}$	$(x^2 + y^3 + ay, y^2 + x^3 + bx)$
$II_{2,2}^{1,1}$	$(x^2 - y^2 + x^3 + ay, xy + bx)$
$\tilde{C}^\pm$	$C^\pm$ and $(x'^2 + b, y')$
$\tilde{S}$	S and $(x'^2 + b, y')$
$Q_\pm$	$(x, x^3 - ax + y^2), (x', \pm y'^2 + b)$
$\tilde{F}F^+$	$(x, x^2 + y^2), (x', y'^2 + a), (x''^2 + b, y'')$
$\tilde{F}F_0^-, \tilde{F}F_1^-$	$(x, x^2 \pm y^2), (x', \mp y'^2 + a), (x''^2 + b, y'')$
CC	$(x + a, y^3 + xy), (y'^3 + x'y', x' + b)$
FC	$(x + a, y^3 + xy), (x', y^2 + ax + b)$
$\tilde{C}F_{\epsilon_1, \epsilon_2}$	$(x, y^3 + xy), (x' + \epsilon_1 y'^2, x' - \epsilon_1 y'^2 + a),$
$(\epsilon_1, \epsilon_2 = \pm 1)$	$(x'' + \epsilon_2 y''^2, -x'' + \epsilon_2 y''^2 + b)$
$\tilde{T}_0, \tilde{T}_1$	$T_{0,1}$ and $(x''' + y'''^2, -x''' + y'''^2 + b)$

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