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# Recurrent Sets for Dynamic Fuzzy Systems

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## 0. Introduction

This paper discusses a recurrent behavior of dynamic fuzzy systems defined by fuzzy relations on a Euclidean space. By introducing a recurrence for crisp sets, we give probability-theoretical properties for the fuzzy systems. When the fuzzy relations satisfy a contraction condition, the existence of the maximum recurrent set is shown. We also consider a monotonicity condition for the fuzzy relations as an extended case of a linear structure in one-dimensional fuzzy numbers. Then we present the existence of the arcwise connected maximal recurrent sets.

## 1. Notations

Let $S$ be a metric space. We write a fuzzy set on $S$ by its membership function $\tilde{s} : S \mapsto [0,1]$ and an ordinary set $A(\subset S)$ by its indicator function $1_A : S \mapsto \{0,1\}$. The $\alpha$-cut $\tilde{s}_\alpha$ is defined by

$$\tilde{s}_\alpha := \{x \in S \mid \tilde{s}(x) \geq \alpha\} \ (\alpha \in (0,1]) \quad \text{and} \quad \tilde{s}_0 := \mathrm{cl}\{x \in S \mid \tilde{s}(x) > 0\},$$

where cl denotes the closure of a set. $\mathcal{F}(S)$ denotes the set of all fuzzy sets $\tilde{s}$ on $S$ satisfying the following conditions (i) and (ii) :

(i) $\tilde{s}_\alpha \in \mathcal{E}(S)$ for $\alpha \in [0,1]$;

(ii) $\bigcap_{\alpha' < \alpha} \tilde{s}_{\alpha'} = \tilde{s}_\alpha$ for $\alpha \in (0,1]$,

where $\mathcal{E}(S) := \{A \mid A = \bigcup_{n=0}^{\infty} C_n,\ C_n \text{ are closed subsets of } S\ (n = 0,1,2,\cdots)\}$. We also define

$$\mathcal{G}(S) := \left\{ \text{fuzzy sets } \tilde{s} \text{ on } S \ \middle|\ \text{there exists } \{\tilde{s}_n\}_{n\in\mathbf{N}} \subset \mathcal{F}(S) \text{ satisfying } \tilde{s} = \bigvee_{n\in\mathbf{N}} \tilde{s}_n \right\},$$

where $\mathbf{N} := \{0,1,2,3,\cdots\}$ and for a sequence of fuzzy sets $\{\tilde{s}_n\}_{n\in\mathbf{N}}$ on $S$ we define

$$\bigwedge_{n\in\mathbf{N}} \tilde{s}_n(x) := \inf_{n\in\mathbf{N}} \tilde{s}_n(x) \quad \text{and} \quad \bigvee_{n\in\mathbf{N}} \tilde{s}_n(x) := \sup_{n\in\mathbf{N}} \tilde{s}_n(x) \qquad x \in S.$$

Let a time space by $\mathbf{N}$ and put $\overline{\mathbf{N}} := \mathbf{N} \cup \{\infty\}$. Let a state space $E$ be a finite-dimensional Euclidean space. We put a path space by $\Omega := \prod_{k=0}^{\infty} E$ and we write a sample path by $\omega = (\omega(0), \omega(1), \omega(2), \cdots) \in \Omega$. We define a map $X_n(\omega) := \omega(n)$ and a shift $\theta_n(\omega) := (\omega(n), \omega(n+1), \omega(n+2), \cdots)$ for $n \in \mathbf{N}$ and $\omega = (\omega(0), \omega(1), \omega(2), \cdots) \in \Omega$. We put

$\sigma$-fields by $\mathcal{M}_n := \sigma(X_0, X_1, \cdots, X_n)$ [1] for $n \in \mathbf{N}$ and $\mathcal{M} := \sigma(\bigcup_{n\in\mathbf{N}} \mathcal{M}_n)$ [2]. Let $\Delta$ be not a point of $E$ and put $E_\Delta := E \cup \{\Delta\}$. We can extend the state space $E$ to $E_\Delta$, setting $\tilde{s}(\Delta) := 0$ for $\tilde{s} \in \mathcal{G}(E_\Delta)$ and $X_\infty(\omega) := \Delta$ for $\omega \in \Omega$ ([10, Section 2]). Let $\tilde{q}$ be an upper semi-continuous binary relation on $E \times E$ satisfying the following normality condition :

$$\sup_{x\in E} \tilde{q}(x,y) = 1 \ (y \in E) \quad \text{and} \quad \sup_{y\in E} \tilde{q}(x,y) = 1 \ (x \in E).$$

We call $\tilde{q}$ a fuzzy relation. We define a fuzzy expectation : For an initial state $x \in E$ and an $\mathcal{M}$-measurable fuzzy set $h \in \mathcal{F}(\Omega)$,

$$E_x(h) := \oint_{\{\omega\in\Omega:\omega(0)=x\}} h(\omega)\, \mathrm{d}\tilde{P}(\omega),$$

where $\tilde{P}$ is the following possibility measure :

$$\tilde{P}(\Lambda) := \sup_{\omega\in\Lambda} \bigwedge_{n\in\mathbf{N}} \tilde{q}(X_n\omega, X_{n+1}\omega) \qquad \Lambda \in \mathcal{M}$$

and $\oint \mathrm{d}\tilde{P}$ denotes Sugeno integral (Sugeno [9]).

We need the first entry times (the first hitting times) of a set, which is adapted to the dynamic fuzzy system $X := \{X_n\}_{n\in\mathbf{N}}$, in order to define a recurrence of sets in Section 3. We define

$$\mathcal{E} := \{A \mid A \in \mathcal{E}(E) \text{ and } E \setminus A \in \mathcal{E}(E)\}$$

and we call a map $\tau : \Omega \mapsto \overline{\mathbf{N}}$ an $\mathcal{E}$-stopping time if

$$\{\tau = n\} \in \mathcal{M}_n \cap \mathcal{E}(\Omega) \qquad n \in \mathbf{N}.$$

For example, a constant stopping time i.e. $\tau = n_0$ for some $n_0 \in \mathbf{N}$, is an $\mathcal{E}$-stopping time. For $A \in \mathcal{E}$ we put

$$\tau_A(\omega) := \inf\{n \in \mathbf{N} \mid X_n(\omega) \in A\} \quad \omega \in \Omega;$$

$$\sigma_A(\omega) := \inf\{n \in \mathbf{N} \mid n \geq 1, X_n(\omega) \in A\} \quad \omega \in \Omega,$$

where the infimums of the empty set are understood to be $+\infty$. Then the first entry time $\tau_A$ of $A$ and the first hitting time $\sigma_A$ of $A$ are also $\mathcal{E}$-stopping times ([10, Lemma 1.5]).

Define a map $P : \mathcal{G}(E) \mapsto \mathcal{G}(E)$ by

$$P\tilde{s}(x) := E_x(\tilde{s}(X_1)) = \sup_{y\in E}\{\tilde{q}(x,y) \wedge \tilde{s}(y)\} \quad x \in E \qquad \text{for } \tilde{s} \in \mathcal{G}(E), \tag{1.1}$$

where we write binary operations $a \wedge b := \min\{a,b\}$ and $a \vee b := \max\{a,b\}$ for $a, b \in [0,1]$. We call $P$ a fuzzy transition defined by the fuzzy relation $\tilde{q}$. We also define $n$-steps fuzzy transitions $P_n : \mathcal{G}(E) \mapsto \mathcal{G}(E)$, $n \in \mathbf{N}$, by

$$P_n\tilde{s} := E_\cdot(\tilde{s}(X_n)) = \sup_{y\in E}\{\tilde{q}^n(\cdot,y) \wedge \tilde{s}(y)\} \quad \text{for } \tilde{s} \in \mathcal{G}(E),$$

[1] It denotes the smallest $\sigma$-field on $\Omega$ relative to which $X_0, X_1, \cdots, X_n$ are measurable.

[2] It denotes the smallest $\sigma$-field generated by $\bigcup_{n\in\mathbf{N}} \mathcal{M}_n$.

where for $n \in \mathbf{N}$

$$\tilde{q}^1(x,y) := \tilde{q}(x,y) \quad \text{and} \quad \tilde{q}^{n+1}(x,y) := \sup_{z\in E}\{\tilde{q}^n(x,z) \wedge \tilde{q}(z,y)\} \quad x,y \in E.$$

Further for an $\mathcal{E}$-stopping time $\tau$, a fuzzy transition $P_\tau : \mathcal{G}(E) \mapsto \mathcal{G}(E)$ is defined by

$$P_\tau \tilde{s} := E.(\tilde{s}(X_\tau)) \quad \text{for } \tilde{s} \in \mathcal{G}(E),$$

where $X_\tau := X_n$ on $\{\tau = n\}$, $n \in \overline{\mathbf{N}}$.

## 2. Transitive closures and $P$-superharmonic fuzzy sets

We define a partial order $\geq$ on $\mathcal{G}(E)$ : For $\tilde{s}, \tilde{r} \in \mathcal{G}(E)$

$$\tilde{s} \geq \tilde{r} \iff \tilde{s}(x) \geq \tilde{r}(x) \quad x \in E.$$

**Definition** ([10, Section 4]). A fuzzy set $\tilde{s}$ ($\in \mathcal{G}(E)$) is called $P$-harmonic ($P$-superharmonic) provided that

$$\tilde{s} = P\tilde{s} \quad (\tilde{s} \geq P\tilde{s} \text{ resp.}).$$

Clearly a constant fuzzy set, $\tilde{s} = \beta$ for some $\beta \in [0,1]$, is $P$-superharmonic. We represent the fuzzy set by $\beta$ simply.

**Theorem 2.1.** *Let $\tilde{s}$ be $P$-superharmonic and let a set $A \in \mathcal{E}$. Then $P_{\tau_A}\tilde{s}$ is the smallest $P$-superharmonic fuzzy set which dominates $\tilde{s} \wedge 1_A$.*

We define an operator $G := \bigvee_{n\in\mathbf{N}} P_n$ on $\mathcal{G}(E)$. Then we note that

$$PG1_{\{y\}}(x) = \bigvee_{n\geq 1} P_n 1_{\{y\}}(x) = \sup_{n\geq 1} \tilde{q}^n(x,y) \qquad x,y \in E.$$

This is called a transitive closure ([3, Section 3.3]). In this paper we also call $PG$ a transitive closure. Now we need to investigate the operator $G$ in order to analyse the transitive closure $PG := \bigvee_{n\geq 1} P_n$. We have the following properties regarding $G$.

**Lemma 2.1** ([10, Lemma 4.1(ii)]). *Let $\tilde{s} \in \mathcal{G}(E)$. Then :*

(i) *It holds that*

$$G\tilde{s} = \tilde{s} \vee P(G\tilde{s});$$

(ii) *$G\tilde{s}$ is the smallest $P$-superharmonic dominating $\tilde{s}$.*

For $A \in \mathcal{E}(E)$ we introduce an operator $I_A : \mathcal{G}(E) \mapsto \mathcal{G}(E)$ by

$$I_A\tilde{s} := \tilde{s} \wedge 1_A \qquad \tilde{s} \in \mathcal{G}(E).$$

We define a sequence of hitting times $\{\sigma_A^n\}_{n\in\mathbf{N}}$ of a set $A(\in \mathcal{E})$ by

$$\sigma_A^n := \begin{cases} 0 & \text{if } n = 0 \\ \sigma_A^{n-1} + \sigma_A \circ \theta_{\sigma_A^{n-1}} & \text{if } n \geq 1. \end{cases}$$

Then $\sigma_A^n$ means the first time to hit $A$ after time $\sigma_A^{n-1}$ (c.f. [8]).

**Proposition 2.1.** *Let $A \in \mathcal{E}$. Then :*

(i) $P_{\tau_A}\tilde{s} = GI_A\tilde{s}$ *for $P$-superharmonic $\tilde{s}$;*

(ii) $P_{\sigma_A^n}\tilde{s} = (PGI_A)^n\tilde{s}$ *for $P$-superharmonic $\tilde{s}$ and $n \in \mathbf{N}$.*

## 3. $\alpha$-recurrent sets

**Definition.** Let $\alpha \in (0,1]$. A set $A \in \mathcal{E}(E)$ is called $\alpha$-recurrent provided :

(a) $A$ is non-empty;

(b) $P_{\sigma_B^n}1 \geq \alpha$ on $A$ for all $n \in \mathbf{N}$ and all non-empty $B \in \mathcal{E}$ satisfying $B \subset A$.

The $\alpha$-recurrence of a set $A$ means that a possibility to transit infinite times from any point of $A$ to any point of $A$ is greater than $\alpha$.

We give simple necessary and sufficient criteria for $\alpha$-recurrence by the transitive closure $PG$.

**Proposition 3.1.** *Let $\alpha \in (0,1]$ and let non-empty $A \in \mathcal{E}(E)$. Then the following statements are equivalent :*

(i) *$A$ is $\alpha$-recurrent;*

(ii) $PG1_B \geq \alpha \wedge 1_A$ *for non-empty $B \in \mathcal{E}(E)$ satisfying $B \subset A$;*

(iii) $PG1_{\{y\}} \geq \alpha \wedge 1_A$ *for $y \in A$.*

We gives, by the fuzzy relation $\tilde{q}$, a representation of the union of all $\alpha$-recurrent sets.

**Theorem 3.1.** *It holds that*

$$\bigcup_{A\in\mathcal{E}(E)\,:\,\alpha-\text{recurrent sets}} A = \left\{x \in E \mid \sup_{n\geq 1} \tilde{q}^n(x,x) \geq \alpha\right\} \qquad \text{for } \alpha \in (0,1].$$

## 4. The contractive case

We consider the contractive case in [5] and we give the maximum $\alpha$-recurrent set for the dynamic fuzzy system $X$.

Let $E_c$ be a compact subset of $E$. We deal with a dynamic fuzzy system restricted on the compact space $E_c$ according to [5]. Let $\mathcal{C}(E_c)$ be the set of all closed subsets of $E_c$ and let $\rho$ be the Hausdorff metric on $\mathcal{C}(E_c)$. Let $\mathcal{F}^0(E_c)$ be the set of all fuzzy sets $\tilde{s}$ on $E_c$ which are upper semi-continuous and satisfy $\sup_{x\in E_c}\tilde{s}(x) = 1$. Then we note $\mathcal{F}^0(E_c) \subset \mathcal{F}(E_c)$. Let $\tilde{p}_0 \in \mathcal{F}^0(E_c)$ be a fuzzy set. Define a sequence of fuzzy sets $\{\tilde{p}_n\}_{n=0}^{\infty}$ by

$$\tilde{p}_{n+1}(y) = \sup_{x\in E_c}\{\tilde{p}_n(x) \wedge \tilde{q}(x,y)\} \quad y \in E_c \quad \text{for } n \geq 0. \tag{4.1}$$

The fuzzy set $\tilde{p}_0$, in [5], is called an initial fuzzy state and the sequence $\{\tilde{p}_n\}_{n=0}^{\infty}$ is called a sequence of fuzzy states. The fuzzy relation $\tilde{q}$ is also restricted on $E_c \times E_c$ and it is assumed to be continuous on $E_c \times E_c$ and satisfy $\tilde{q}(x,\cdot) \in \mathcal{F}^0(E)$. Define a map $\tilde{r}_\alpha : \mathcal{C}(E_c) \mapsto \mathcal{C}(E_c)$ ($\alpha \in [0,1]$) by

$$\tilde{r}_\alpha(D) := \begin{cases} \{y \mid \tilde{q}(x,y) \geq \alpha \ \text{ for some } x \in D\} & \text{for } \alpha > 0,\ D \in \mathcal{C}(E_c),\ D \neq \emptyset, \\ \mathrm{cl}\{y \mid \tilde{q}(x,y) > 0 \ \text{ for some } x \in D\} & \text{for } \alpha = 0,\ D \in \mathcal{C}(E_c),\ D \neq \emptyset, \\ E_c & \text{for } 0 \leq \alpha \leq 1,\ D = \emptyset. \end{cases}$$

In the sequel we assume the following contraction property for the fuzzy relation $\tilde{q}$ (see [5, Section 2]) : There exists a real number $\beta \in (0,1)$ satisfying

$$\rho(\tilde{r}_\alpha(A), \tilde{r}_\alpha(B)) \leq \beta\ \rho(A,B) \quad \text{for all } A, B \in \mathcal{C}(E_c) \text{ and all } \alpha \in [0,1].$$

**Lemma 4.1** ([5, Theorem 1]).

(i) *There exists a unique fuzzy state $\tilde{p} \in \mathcal{F}^0(E_c)$ satisfying*

$$\tilde{p}(y) = \max_{x \in E_c}\{\tilde{p}(x) \wedge \tilde{q}(x,y)\} \qquad y \in E_c. \tag{4.2}$$

(ii) *The sequence $\{\tilde{p}_n\}_{n=0}^{\infty}$ converges to a unique solution $\tilde{p} \in \mathcal{F}^0(E_c)$ of (4.2) independently of the initial fuzzy state $\tilde{p}_0$. Namely,*

$$\lim_{n\to\infty} \tilde{p}_n = \tilde{p},$$

*where the convergence means $\sup_{\alpha\in[0,1]} \rho(\tilde{p}_{n,\alpha}, \tilde{p}_\alpha) \to 0$ ($n \to \infty$) provided $\tilde{p}_{n,\alpha}, \tilde{p}_\alpha$ are $\alpha$-cuts ($\alpha \in [0,1]$) for the fuzzy states $\tilde{p}_n, \tilde{p}$ respectively.*

**Proposition 4.1.** *The $\alpha$-cut of the solution $\tilde{p}$ of (4.2) is*

$$\tilde{p}_\alpha = \mathrm{cl}\left\{x \in E_c \mid \sup_{n\geq 1} \tilde{q}^n(x,x) \geq \alpha\right\} \qquad \text{for } \alpha \in (0,1].$$

Finally we prove that the closure of the union of all $\alpha$-recurrent sets equals to $\alpha$-cuts of the limit fuzzy state $\tilde{p}$. Now we compare (1.1) and (4.1). Using the inverse fuzzy relation $\hat{q}$ ([3, Section 3.2]):

$$\hat{q}(x,y) := \tilde{q}(y,x) \qquad x, y \in E_c,$$

we find that (4.1) follows

$$\tilde{p}_{n+1}(x) = \sup_{x \in E_c}\{\hat{q}(x,y) \wedge \tilde{p}_n(y)\} \quad x \in E_c \quad \text{for } n \geq 0.$$

Therefore we can apply the results in Sections 1 – 3 to a dynamic fuzzy system defined by the inverse fuzzy relation $\hat{q}$.

**Theorem 4.1.**

$$\tilde{p}_\alpha = \mathrm{cl}\left\{x \in E_c \mid \sup_{n\geq 1} \tilde{q}^n(x,x) \geq \alpha\right\} = \mathrm{cl}\left(\bigcup_{A \in \mathcal{E}(E)\,:\,\alpha-\text{recurrent sets}} A\right) \qquad \text{for } \alpha \in (0,1].$$

*Further it is the maximum $\alpha$-recurrent set for $X$.*

## 5. The monotone case

In general, there does not always exist the maximum $\alpha$-recurrent set for the dynamic fuzzy system $X$, however we can consider the existence of the maximal $\alpha$-recurrent sets. In this section we deal with a case when the transition fuzzy relation $\tilde{q}$ has a certain monotone property (see Section 6 for numerical examples). Then we prove the existence of at most countable arcwise connected maximal $\alpha$-recurrent sets.

In this section we use the notations in Sections 1 – 3. Further we introduce the following notations of $\alpha$-cuts ([5, Section 2]) :

$$\tilde{q}_\alpha(x) := \{y \in E \mid \tilde{q}(x,y) \geq \alpha\} \quad \text{for } x \in E \text{ and } \alpha \in (0,1];$$

$$\tilde{q}_\alpha(A) := \bigcup_{x \in A} \tilde{q}_\alpha(x) \quad \text{for } A \in \mathcal{E}(E) \text{ and } \alpha \in (0,1];$$

$$\tilde{q}_0(A) := \mathrm{cl}(\bigcup_{\alpha>0} \tilde{q}_\alpha(A)) \quad \text{for } A \in \mathcal{E}(E).$$

For $\alpha \in (0,1]$ and $x \in E$ we define a sequence $\{\tilde{q}_\alpha^m(x)\}_{m=1,2,\cdots}$ :

$$\tilde{q}_\alpha^1(x) := \tilde{q}_\alpha(x); \quad \text{and} \quad \tilde{q}_\alpha^{m+1}(x) := \tilde{q}_\alpha(\tilde{q}_\alpha^m(x)) \quad \text{for } m = 1, 2, \cdots.$$

We also need some elementary notations in the finite dimensional Euclidean space $E$: $x + y$ denotes the sum of $x, y \in E$ and $\gamma x$ denotes the product of a real number $\gamma$ and $x \in E$. We put $A + B := \{x + y \mid x \in A, y \in B\}$ for $A, B \in \mathcal{E}(E)$. Then we define a half line on $E$ by

$$l(x,y) := \{\gamma(y - x) \mid \text{real numbers } \gamma \geq 0\} \quad \text{for } x, y \in E.$$

**Definition.** We call a transition fuzzy relation $\tilde{q}$ unimodal provided that $\tilde{q}_\alpha(x)$ are bounded closed convex subsets of $E$ for all $\alpha \in (0,1]$ and all $x \in E$.

**Definition.** We call a unimodal transition fuzzy relation $\tilde{q}$ monotone provided that

$$\tilde{q}_\alpha(y) \subset \tilde{q}_\alpha(x) + l(x,y) \quad \text{for all } \alpha \in (0,1] \text{ and all } x, y \in E.$$

From now on we deal with only unimodal fuzzy relations $\tilde{q}$, which is monotone and continuous on $E \times E$. The monotonicity is a natural extension of one-dimensional models with the linear structure in [12] and means that the fuzzy relations $\tilde{q}$ keeps the partial order of fuzzy numbers (see (C.iii') in Section 6).

**Theorem 5.1.** *Assume that $\tilde{q}$ is monotone. Let $\alpha \in (0,1]$. Then*

$$\bigcup_{A \in \mathcal{E}(E)\,:\,\alpha-\text{recurrent sets}} A = \{x \in E \mid \tilde{q}(x,x) \geq \alpha\}.$$

We need the following assumption on $\tilde{q}$, which is technical but not so strong. It means that the function $\tilde{q}$ does not have flat areas as a curved surface (Section 6).

**Assumption (A).** For $\alpha \in (0,1)$,

$$\text{int}\,\{(x,y) \in E \times E \mid \tilde{q}(x,y) \geq \alpha\} = \{(x,y) \in E \times E \mid \tilde{q}(x,y) > \alpha\},$$

where int denotes the interior of a set.

Since $\tilde{q}$ is continuous, $\{x \in E \mid \tilde{q}(x,x) \geq \alpha\}$ is represented by a disjoint sum of at most countable arcwise connected closed sets ([4]), we represent it by

$$\{x \in E \mid \tilde{q}(x,x) \geq \alpha\} = \bigcup_{n \in \mathbf{N}(\alpha)} F_{\alpha,n} \quad \text{for } \alpha \in (0,1),$$

where $F_{\alpha,n}$ are arcwise connected closed subsets of $E$ and we put the index set $\mathbf{N}(\alpha)$ $(\subset \mathbf{N})$.

**Theorem 5.2.** *We suppose Assumption (A). Let $\alpha \in (0,1)$. Then maximal $\alpha$-recurrent sets for $X$ are $F_{\alpha,n}$ $(n \in \mathbf{N}(\alpha))$.*

## 6. Numerical examples

Let a one-dimensional state space $E = \mathbf{R}$. We consider one-dimensional numerical examples. In Section 5 we have assumed the following conditions (C.i) — (C.iv):

(C.i) $\tilde{q}$ is continuous on $E \times E$;

(C.ii) $\tilde{q}$ is unimodal;

(C.iii) $\tilde{q}$ is monotone;

(C.iv) $\tilde{q}$ satisfies Assumption (A).

When $E = \mathbf{R}$, $\mathcal{F}^0(\mathbf{R})$ means all fuzzy numbers on $\mathbf{R}$. From (C.ii), $\tilde{q}_\alpha(x)$ are bounded closed intervals of $\mathbf{R}$ $(\alpha \in (0,1], x \in \mathbf{R})$. So we write $\tilde{q}_\alpha(x) = [\min \tilde{q}_\alpha(x), \max \tilde{q}_\alpha(x)]$, where $\min A$ ($\max A$) denotes the minimum (maximum resp.) point of a interval $A \subset \mathbf{R}$. Then (C.iii) is equivalent to the following (C.iii') :

(C.iii') $\min \tilde{q}_\alpha(\cdot)$ and $\max \tilde{q}_\alpha(\cdot)$ are non-decreasing functions on $\mathbf{R}$ for all $\alpha \in (0,1]$.

Next we consider the following partial order $\preceq$ on $\mathcal{F}^0(\mathbf{R})$ (Nanda [6]) : For $\tilde{s}, \tilde{r} \in \mathcal{F}^0(\mathbf{R})$,

$$\tilde{s} \preceq \tilde{r} \quad \text{means that} \quad \min \tilde{s}_\alpha \leq \min \tilde{r}_\alpha \text{ and } \max \tilde{s}_\alpha \leq \max \tilde{r}_\alpha \text{ for all } \alpha \in (0,1].$$

Then we can easily find that (C.iii) is equivalent to the following (C.iii") :

(C.iii") If $\tilde{s}, \tilde{r} \in \mathcal{F}^0(\mathbf{R})$ satisfy $\tilde{s} \preceq \tilde{r}$, then $Q(\tilde{s}) \preceq Q(\tilde{r})$,

where $Q : \mathcal{F}^0(\mathbf{R}) \mapsto \mathcal{F}^0(\mathbf{R})$, see (4.1), is defined by

$$Q\tilde{s}(y) = \max_{x \in \mathbf{R}} \{\tilde{s}(x) \wedge \tilde{q}(x,y)\}, \quad y \in \mathbf{R} \quad \text{for } \tilde{s} \in \mathcal{F}^0(\mathbf{R}).$$

(C.iii") means that $Q$ preserves the monotonicity on $\mathcal{F}^0(\mathbf{R})$ with respect to the order $\preceq$. Finally (C.iv) means that the $\alpha$-slice $\{x \in \mathbf{R} \mid \tilde{q}(x,x) = \alpha\}$ $(\alpha \in (0,1))$ are drawn by not areas but curved lines. We give an example of monotone fuzzy relations, which is not contractive and does not have the linear structure in [12]. Then we calculate its maximal $\alpha$-recurrent sets.

**Example 6.1 (monotone case).** We give a fuzzy relation by

$$\tilde{q}(x,y) = (1 - |y - x^3|) \vee 0, \quad x, y \in \mathbf{R}.$$

Then $\tilde{q}(x,y)$ satisfies the conditions (C.i) - (C.iv) (see Figure 6.1 for the fuzzy relation $\tilde{q}(x,y)$ and Figure 6.2 for the $\frac{3}{4}$-slice).

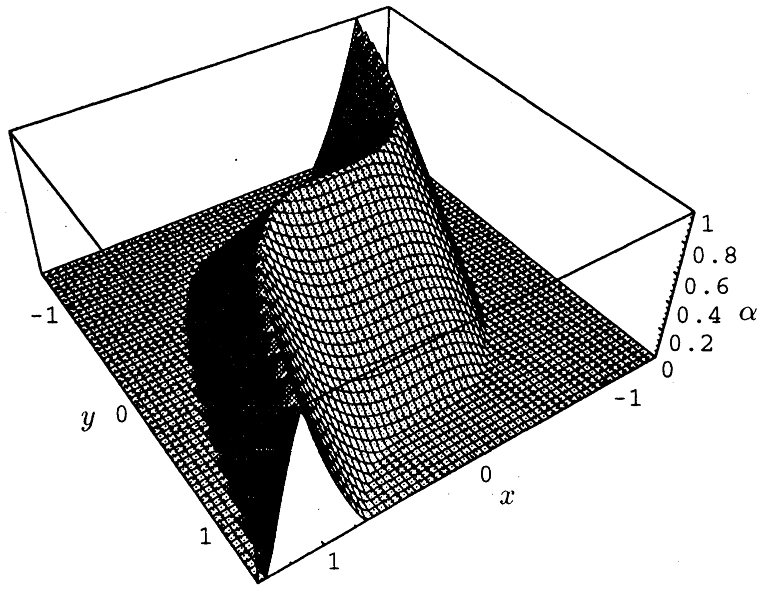


Fig. 6.1 : The monotone fuzzy relation $\tilde{q}(x,y)$.

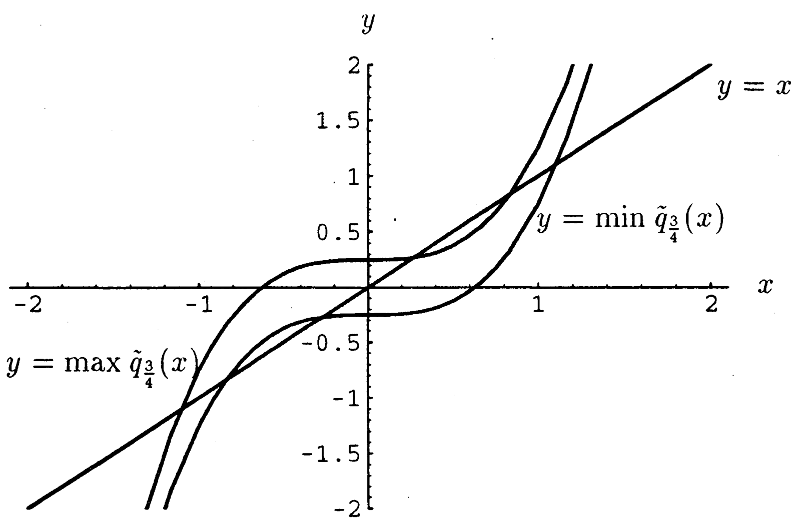


Fig. 6.2. The $\frac{3}{4}$-level sets $\{(x,y) \mid \tilde{q}(x,y) = \frac{3}{4}\}$.

Then we have

$$\tilde{q}(x,x) = (1 - |x - x^3|) \vee 0, \quad x \in \mathbf{R}.$$

Therefore $\mathbf{N}(\frac{3}{4}) = \{0,1,2\}$ and

$$\begin{aligned} \left\{x \in \mathbf{R} \mid \tilde{q}(x,x) \geq \tfrac{3}{4}\right\} &= F_{\frac{3}{4},0} \cup F_{\frac{3}{4},1} \cup F_{\frac{3}{4},2} \\ &\approx [-1.10716, -0.837565] \cup [-0.269594, 0.269594] \cup [0.837565, 1.10716]. \end{aligned}$$

By Theorem 5.2, the maximal $\frac{3}{4}$-recurrent sets are given by three intervals

$$\begin{aligned} F_{\frac{3}{4},0} &\approx [-1.10716, -0.837565], \\ F_{\frac{3}{4},1} &\approx [-0.269594, 0.269594], \\ F_{\frac{3}{4},2} &\approx [0.837565, 1.10716]. \end{aligned}$$

## References


[1] R.E.Bellman and L.A.Zadeh, Decision-making in a fuzzy environment, *Management Sci. Ser B.* **17** (1970) 141-164.

[2] A.O.Esogbue and R.E.Bellman, Fuzzy dynamic programming and its extensions, *TIMS / Studies in Management Sci.* **20** (North-Holland, Amsterdam, 1984) 147-167.

[3] G.J.Klir and T.A.Folger, *Fuzzy Sets, Uncertainty, and Information* (Prentice-Hall, London, 1988).

[4] K.Kuratowski, *Topology I* (Academic Press, New York, 1966).

[5] M.Kurano, M.Yasuda, J.Nakagami and Y.Yoshida, A limit theorem in some dynamic fuzzy systems, *Fuzzy Sets and Systems* **51** (1992) 83-88.

[6] S.Nanda, On sequences of fuzzy numbers, *Fuzzy Sets and Systems* **33** (1989) 123-126.

[7] J.Neveu, *Discrete-Parameter Martingales* (North-Holland, New York, 1975).

[8] D.Revuz, *Markov Chains* (North-Holland, New York, 1975).

[9] M.Sugeno, Fuzzy measures and fuzzy integral : a survey in M.M.Gupta, G.N.Saridis and B.R.Gaines, Eds., *Fuzzy Automata and Decision Processes* (North-Holland, Amsterdam, 1977) 89-102.

[10] Y.Yoshida, Markov chains with a transition possibility measure and fuzzy dynamic programming, *Fuzzy Sets and Systems* **66** (1994) 39-57.

[11] Y.Yoshida, The recurrence of dynamic fuzzy systems, *RIFIS Technical Report* **80** (1994), Research Institute of Fundamental Information Science, Kyushu University, Japan.

[12] Y.Yoshida, M.Yasuda, J.Nakagami and M.Kurano, A potential of fuzzy relations with a linear structure : The contractive case, *Fuzzy Sets and Systems* **60** (1993) 283-294.

[13] Y.Yoshida, M.Yasuda, J.Nakagami and M.Kurano, A potential of fuzzy relations with a linear structure : The unbounded case, *Fuzzy Sets and Systems* **66** (1994) 83-95.