

MULTIPLE STONE-ČECH EXTENSIONS

(DUAL STONE-ČECH EXTENSIONS)

AKIO KATO

ABSTRACT. For a nowhere (locally) compact space we iterate Stone-Čech compactification ω_1 many times to get a compact space where two or more disjoint dense subsets are C^* -embedded. The corresponding compact spaces we get for $\mathbb{Q}$ (the rationals), $\mathbb{P}$ (the irrationals) and $\mathbb{S}$ (the Sorgenfrey line) are not extremally disconnected, hence different from their absolutes.

1. INTRODUCTION

This talk originates from van Douwen's question in his paper "Remote points" (see §19 of [4]) that:

What happens if we repeat taking remainders of Stone-Čech compactifications of the rationals

$$\mathbb{Q}^* = \beta \mathbb{Q} \setminus \mathbb{Q}, \quad \mathbb{Q}^{**} = \beta \mathbb{Q}^* \setminus \mathbb{Q}^*, \quad \mathbb{Q}^{***}, \quad \dots$$

He remarks that "it might be interesting to define $\mathbb{Q}^{(\alpha)}$'s, for $\alpha \geq \omega$, using inverse limits at limit stages" and that "there must be a γ for which the natural map from $\mathbb{Q}^{(\gamma+2)}$ to $\mathbb{Q}^{(\gamma)}$ is a homeomorphism." We will show in this paper that the least such γ is the first uncountable ordinal ω_1 (which we will denote by Ω for notational convenience).

Let K be a compact space of countable π -weight, partitioned as a disjoint union of two dense Lindelöf subspaces $K = K^- \cup K^+$. Then, in this paper, iterating Stone-Čech compactification $\omega_1 = \Omega$ many times, we will construct a compact space $\Omega(K) = K_\Omega^- \cup K_\Omega^+$ satisfying the following conditions:

(1) $\Omega(K)$ admits a perfect irreducible map $g : \Omega(K) \rightarrow K$ such that $g(K_\Omega^-) = K^-$, $g(K_\Omega^+) = K^+$.

(2) Both of K_Ω^- , K_Ω^+ are C^* -embedded in $\Omega(K)$.

Though, as is well known, the absolute (or the projective cover) of K also satisfies the corresponding conditions as above (1), (2), we can show, in most cases we deal with, that our compact space $\Omega(K)$ is not extremally disconnected, hence different from the absolute.

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Typical cases we are going to deal with are the following partitions.

Example 1. $K = [0, 1], K^- = Q, K^+ = P$ where $Q = (0, 1) \cap \mathbb{Q}$ and $P = [0, 1] \setminus Q$. Obviously, Q is a homeomorphic copy of the rationals $\mathbb{Q}$, and P is that of the irrationals $\mathbb{P}$.

Example 2. $K =$ the Alexandroff double arrow space $\mathbb{A}$, i.e., the lexicographically ordered space $\mathbb{A} = [0, 1] \times \{0, 1\} \setminus \{(0, 0), (1, 1)\}$ which is the union of two dense sets $K^- = (0, 1] \times \{0\}, K^+ = [0, 1) \times \{1\}$, each of which is a copy of the Sorgenfrey line $\mathbb{S}$.

In this talk we show how to construct such an extension $\Omega(K)$ in general. The proofs and the details of its properties will appear in the forthcoming paper [6].

All spaces are assumed to be completely regular and Hausdorff, and maps are always continuous, unless otherwise stated. "Partition" is synonymous with "disjoint union."

As a suitable class for our purpose we consider the following class $\mathcal{L}$ consisting of Lindelöf spaces X such that

- (i) X is *nowhere compact* (or nowhere locally compact), i.e., X has no compact neighborhood, and
- (ii) every compact subset of X is included in some compact zero-set of X .

In terms of compactifications the condition (i) is equivalent to say that the remainder $cX \setminus X$ of any/some compactification cX of X is dense in cX , while the second one (ii) is equivalent to say that $cX \setminus X$ is Lindelöf for any/some compactification cX . The subclass of $\mathcal{L}$ consisting only of first countable spaces will be denoted by $\mathcal{L}(1st)$.

The rationals $\mathbb{Q}$, the irrationals $\mathbb{P} = \mathbb{R} \setminus \mathbb{Q} \approx \omega^\omega$, the Sorgenfrey line $\mathbb{S}$ (i.e., the real line with the half-open interval topology) are the typical members of $\mathcal{L}(1st)$. That $\mathbb{S}$ belongs to $\mathcal{L}(1st)$ can be seen by regarding the double arrow space $\mathbb{A}$ in Example 2 as a compactification of $\mathbb{S}$. All of

$$\mathbb{P} \times \mathbb{Q}, \mathbb{S} \times \mathbb{Q}, \mathbb{Q} \times \mathbb{C}, \mathbb{S} \times \mathbb{C}, \mathbb{S} \times \mathbb{P}$$

belong to $\mathcal{L}$. Note that $\mathbb{P} \times \mathbb{C}$ is nothing but $\mathbb{P}$ because

$$\mathbb{P} \times \mathbb{C} \approx \omega^\omega \times 2^\omega \approx (\omega \times 2)^\omega \approx \omega^\omega \approx \mathbb{P}.$$

For topological characterization of $\mathbb{P} \times \mathbb{Q}$ and $\mathbb{Q} \times \mathbb{C}$ see [7] and [8].

As a basic tool we use perfect irreducible maps, so we will list their properties needed here. Let g be a map from X onto Y . For a subset $U \subseteq X$ define $g^\circ(U) \subseteq Y$ by

$$y \in g^\circ(U) \text{ if and only if } g^{-1}(y) \subseteq U,$$

i.e., $g^\circ(U) = Y \setminus g(X \setminus U) \subseteq g(U)$. Note an obvious, but useful, formula

$$g^\circ(U \cap V) = g^\circ(U) \cap g^\circ(V)$$

for any sets $U, V \subseteq X$, which especially implies that $g^\circ(U) \cap g^\circ(V) = \emptyset$ whenever $U \cap V = \emptyset$. An onto map g is called *irreducible* if $g^\circ(U) \neq \emptyset$ for every non-empty open set U . A collection $\mathcal{B}$ of nonempty open sets of X is called a π -base for X if every nonempty open set in X contains some member of $\mathcal{B}$. The minimal cardinality of such a π -base is called the π -weight of X . Observe that any dense subspace of X has the same π -weight as X , and that any space of countable π -weight is separable. Consequently, any dense or open subset of a space of countable π -weight is also of countable π -weight, and hence separable. So, for example, all of $\mathbb{Q}$, $\beta\mathbb{Q}$, $\mathbb{Q}^* = \beta\mathbb{Q} \setminus \mathbb{Q}$ are of countable π -weight, and hence separable. A closed map with compact fibers are called *perfect*. We assume a perfect map is always onto.

Fact 1.1. (Properties of Closed Irreducible Maps)

Let $g : X \rightarrow Y$ be any closed irreducible map. Then

(1) $g^\circ(U)$ is non-empty and open whenever U is. Moreover,

$$\text{cl}_Y g^\circ(U) = \text{cl}_Y g(U) = g(\text{cl}_X U)$$

for every open subset $U \subseteq X$, i.e., g carries a regular closed set $\text{cl}_X U$ to a regular closed set $\text{cl}_Y g^\circ(U)$.

(2) g preserves ccc, i.e., X is ccc if and only if Y is. Similarly, g preserves density and π -weight. In case g is perfect irreducible, it also preserves nowhere compactness.

Next lemma shows how we can produce perfect irreducible maps.

Lemma 1.2. Let $\phi : X \rightarrow Y$ be a perfect map and let $\Phi : bX \rightarrow cY$ be its extension where bX and cY are some compactifications of X and Y respectively. Then Φ maps the remainder of X onto that of Y , i.e., $\Phi(bX \setminus X) = cY \setminus Y$. Moreover,

(1) ϕ is perfect irreducible if and only if Φ is.

(2) If ϕ is perfect irreducible and X (hence Y also) is nowhere compact, then the restriction of Φ to the remainders

$$bX \setminus X \rightarrow cY \setminus Y$$

is also perfect irreducible. $\square$

Perfect irreducible maps we encounter frequently in this paper are those induced by some homeomorphisms, i.e., when the above ϕ is an identity map.

For an open set U of X we can define its maximal open extension to βX by

$$\text{Ex}(U) = \beta X \setminus \text{cl}_{\beta X}(X \setminus U).$$

We denote the boundary of a subset W in Y by $\text{Bd}_Y W$ so that $\text{Bd}_Y W = \text{cl}_Y W \setminus W$ if W is open in Y . Van Douwen [4] proved the following quite useful formula:

$$(1-0) \quad \text{Bd}_{\beta X} \text{Ex}(U) = \text{cl}_{\beta X} \text{Bd}_X(U) \text{ for every open set } U \text{ in } X.$$

A space with a clopen base is called *0-dimensional*, and most spaces we deal with in this paper are 0-dimensional. As is well known (cf. 16.16 in [5]), for a Lindeöf space X the 0-dimensionality of X is equivalent with that of βX ; in other words, the collection of $\text{Ex}(U)$'s where U ranges over all clopen sets in X forms a clopen base for βX .

2. CONSTRUCTION OF DUAL EXTENSIONS

We use inverse systems only of the form

$$\{X_\xi, g_{\alpha,\beta}, \xi\}$$

where ξ is an ordinal, and $g_{\alpha,\beta} : X_\beta \rightarrow X_\alpha$ ($\alpha < \beta < \xi$) are bonding maps, and denote its inverse limit as $X_\xi = \varprojlim \{X_\alpha, g_{\alpha,\beta}, \xi\}$. Projections are denoted by $\pi_\alpha : X_\xi \rightarrow X_\alpha$, or $\pi_\alpha = \pi_\alpha^\xi = g_{\alpha,\xi}$. We assume all inverse systems in this paper are *continuous*, i.e.,

$$X_\eta = \varprojlim \{X_\alpha, g_{\alpha,\beta}, \eta\}$$

for each limit $\eta < \xi$. Recall that, if we take a base $\mathcal{B}_\alpha$ for each X_α , the collection $\bigcup_{\alpha < \xi} \pi_\alpha^{-1}(\mathcal{B}_\alpha)$ forms a base for X_ξ .

The next lemma is well known for a system of compact spaces (cf. §11 in [1]); what we need here is for a system of Lindelöf spaces.

Lemma 2.1. (Factorization Lemma) *Suppose $\text{cof}(\xi) > \omega$, and $X_\xi = \varprojlim \{X_\alpha, g_{\alpha,\beta}, \xi\}$ is Lindelöf. Then every map $f : X_\xi \rightarrow \mathbb{R}$ can be factorized as $f = \hat{f} \circ \pi_\alpha$ for some $\alpha < \xi$ and some map $\hat{f} : X_\alpha \rightarrow \mathbb{R}$.*

Proof. Let $\mathcal{B}$ be a countable open base of $\mathbb{R}$, and $f : X_\xi \rightarrow \mathbb{R}$. Take any $U \in \mathcal{B}$. Then, since $f^{-1}(U)$ is a cozero-set of X_ξ , it can be expressed that $f^{-1}(U) = \pi_{\alpha(U)}^{-1}(W)$ for some cozero-set W of $X_{\alpha(U)}$ with $\alpha(U) < \xi$. Put $\alpha = \sup\{\alpha(U) : U \in \mathcal{B}\} < \xi$. Then this α has the property that for every $U \in \mathcal{B}$ there exists an open set W of X_α such that $f^{-1}(U) = \pi_\alpha^{-1}(W)$. Therefore Lemma 2.1 follows from the next lemma. $\square$

Lemma 2.2 (Yong [9]). *Let $\pi : X \rightarrow Y$, $f : X \rightarrow Z$ and suppose π is onto. Then f is factorized as $f = \hat{f} \circ \pi$ for some map $\hat{f} : Y \rightarrow Z$ if and only if the space Z has an open base $\mathcal{B}$ with the property that: For every $U \in \mathcal{B}$ the open set $f^{-1}(U)$ takes the form $f^{-1}(U) = \pi^{-1}(W)$ for some open set $W \subseteq Y$. $\square$*

Now let $K = X^{(0)} \cup X^{(1)}$ be a compact space with a partition into nowhere compact spaces $X^{(0)}, X^{(1)}$. Since both of $X^{(0)}, X^{(1)}$ are dense in K , we can see K as a compactification of either of $X^{(0)}$ or $X^{(1)}$. Put $X_0 = K$, $X_1 = \beta X^{(1)}$, $X^{(2)} = \beta X^{(1)} \setminus X^{(1)}$, and let

$$\Phi_0 : X_1 = \beta X^{(1)} = X^{(1)} \cup X^{(2)} \rightarrow X_0 = X^{(0)} \cup X^{(1)}$$

be the Stone extension of the identity map $id : X^{(1)} \rightarrow X^{(1)}$. Denote by

$$\phi_0 : X^{(2)} \rightarrow X^{(0)}$$

the restriction of Φ_0 . Next, putting $X_2 = \beta X^{(2)}$, $X^{(3)} = \beta X^{(2)} \setminus X^{(2)}$, let

$$\Phi_1 : X_2 = \beta X^{(2)} = X^{(2)} \cup X^{(3)} \rightarrow X_1 = \beta X^{(1)} = X^{(1)} \cup X^{(2)}$$

be the Stone extension of the identity map $id : X^{(2)} \rightarrow X^{(2)}$. Denote by

$$\phi_1 : X^{(3)} \rightarrow X^{(1)}$$

the restriction of Φ_1 . Repeating these procedures of Stone-Čech compactifications infinitely many times, we get mappings Φ_n, ϕ_n ($n \in \omega$) such that

$$\Phi_n : X_{n+1} = X^{(n+1)} \cup X^{(n+2)} \rightarrow X_n = X^{(n)} \cup X^{(n+1)},$$

where $X_m = \beta X^{(m)}$, $X^{(m+1)} = \beta X^{(m)} \setminus X^{(m)}$ for $m \geq 1$, is the Stone extension of the identity map $id : X^{(n+1)} \rightarrow X^{(n+1)}$, and

$$\phi_n : X^{(n+2)} \rightarrow X^{(n)}$$

is the restriction of Φ_n . Then all of Φ_n, ϕ_n ($n \in \omega$) are perfect irreducible. We can consider the system $\{X_n, \Phi_n\}_{n \in \omega}$ and its induced ones $\{X^{(2m)}, \phi_{2m+1}\}_{m \in \omega}$, $\{X^{(2m+1)}, \phi_{2m+2}\}_{m \in \omega}$ as inverse sequences, and take their limits

$$X_\omega = \varprojlim \{X_n, \Phi_n\}_{n \in \omega},$$

$$X_\omega^- = \varprojlim \{X^{(2m)}, \phi_{2m+1}\}_{m \in \omega}, \quad X_\omega^+ = \varprojlim \{X^{(2m+1)}, \phi_{2m+2}\}_{m \in \omega}.$$

Then it is easy to see that the projections $\pi_n^\omega : X_\omega \rightarrow X_n$ are perfect irreducible, and so, X_ω^-, X_ω^+ are nowhere compact and $X_\omega = X_\omega^- \cup X_\omega^+$ can be seen as a compactification of X_ω^- . Therefore, just replacing the starting $X_0 = X^{(0)} \cup X^{(1)}$ by $X_\omega = X_\omega^- \cup X_\omega^+$, we can repeat the Stone-Čech extensions as before to get $\{X_{\omega+n}, \Phi_{\omega+n}\}_{n \in \omega}$ and $X_{\omega+\omega} = \varprojlim \{X_{\omega+n}, \Phi_{\omega+n}\}_{n \in \omega}$. Let us do these extensions up to $\Omega = \omega_1$. (For notational simplicity we use Ω for the first uncountable ordinal ω_1 .) Then we finally get a continuous inverse system of length Ω

$$(2-0) \quad X_\Omega = \varprojlim \{X_\alpha, \Phi_{\alpha, \beta}, \Omega\}$$

with the following properties:

(1) Each X_α ($\alpha \leq \Omega$) is partitioned as $X_\alpha = X_\alpha^- \cup X_\alpha^+$ into two disjoint dense subsets, and

$X_\alpha^+ = X_{\alpha+1}^+$ for even α , while $X_\alpha^- = X_{\alpha+1}^-$ for odd α .

(An ordinal of the form $\gamma + 2m$ where γ is a limit ordinal and $m \in \omega$ is called “even,” while an ordinal not even is “odd.” Note that limit ordinals are even.)

(2) For any $\alpha < \beta < \Omega$ the bonding map $\Phi_{\alpha,\beta}$ is such that

$$\Phi_{\alpha,\beta} : X_\beta = X_\beta^- \cup X_\beta^+ \rightarrow X_\alpha = X_\alpha^- \cup X_\alpha^+$$

$$\Phi_{\alpha,\beta}(X_\beta^-) = X_\alpha^-, \quad \Phi_{\alpha,\beta}(X_\beta^+) = X_\alpha^+.$$

Moreover, $\Phi_{\alpha,\alpha+1}$ is the Stone extension of the following identity map:

$id : X_{\alpha+1}^+ = X_\alpha^+$ for even α , and $id : X_{\alpha+1}^- = X_\alpha^-$ for odd α .

So, to be compatible with our beginning notation, we need to set

$$X_{2m}^+ = X_{2m+1}^+ = X^{(2m+1)}, \quad X_{2m+1}^- = X_{2m+2}^- = X^{(2m+2)}, \quad \Phi_{\alpha,\alpha+1} = \Phi_\alpha$$

for $m \in \omega$ and $\alpha < \omega + \omega$. In particular, $X_0 = X^{(0)} \cup X^{(1)} = X_0^- \cup X_0^+$, and we call any one of spaces X_0, X_0^-, X_0^+ the *starting space*.

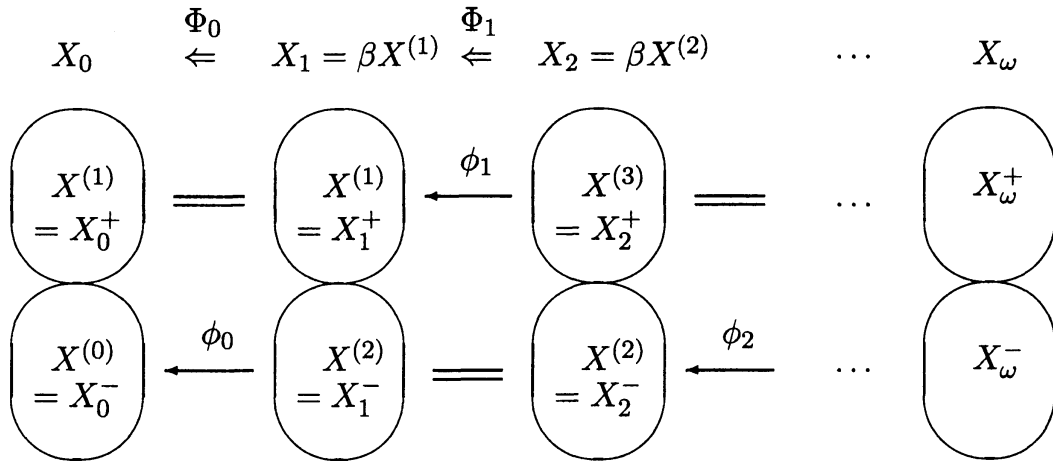


FIG. 1. The first ω steps

Naturally this inverse system $\{X_\alpha, \Phi_{\alpha,\beta}, \Omega\}$ has two subsystems

$$\{X_\alpha^-, \Phi_{\alpha,\beta}^-, \Omega\}, \quad \{X_\alpha^+, \Phi_{\alpha,\beta}^+, \Omega\}$$

with limits X_Ω^- , X_Ω^+ respectively, where

$$\Phi_{\alpha,\beta}^- : X_\beta^- \rightarrow X_\alpha^-, \quad \Phi_{\alpha,\beta}^+ : X_\beta^+ \rightarrow X_\alpha^+$$

are restrictions of $\Phi_{\alpha,\beta}$. The corresponding projections will be denoted by

$$\pi_\alpha : X_\Omega \rightarrow X_\alpha, \quad \pi_\alpha^- : X_\Omega^- \rightarrow X_\alpha^-, \quad \pi_\alpha^+ : X_\Omega^+ \rightarrow X_\alpha^+.$$

All maps $\Phi_{\alpha,\beta}$, $\Phi_{\alpha,\beta}^-$, $\Phi_{\alpha,\beta}^+$, π_α , π_α^- , π_α^+ are perfect irreducible. Consequently, if one of the beginning spaces X_0^-, X_0^+ belongs to the class $\mathcal{L}$, so do all of X_α^-, X_α^+ ($\alpha \leq \Omega$). Note also that if one of X_0^-, X_0^+, X_0 has a countable

π -base, all of $X_\alpha^-, X_\alpha^+, X_\alpha$ ($\alpha \leq \Omega$) have countable π -bases.

The factorization lemma implies

Theorem 2.3. (Dually C^* -embedded Extension)

Assume $X_0^- \in \mathcal{L}$, i.e., $X_0^+ \in \mathcal{L}$. Then $X_\Omega^-, X_\Omega^+ \in \mathcal{L}$, and both of them are C^* -embedded in X_Ω , i.e., symbolically,

$$\beta(X_\Omega^-) = \beta(X_\Omega^+) = X_\Omega.$$

Proof. By symmetry it suffices to show that $X_\Omega^- = \varprojlim \{X_\alpha^-, \Phi_{\alpha,\beta}^-, \Omega\}$ is C^* -embedded in X_Ω . Let $f : X_\Omega^- \rightarrow [0, 1]$ be any continuous function on X_Ω^- . Then, by the factorization lemma, we can find some $\alpha < \Omega$ and a continuous function $\hat{f}$ on X_α^- such that $f = \hat{f} \circ \pi_\alpha^-$. Once such an α is chosen, any $\beta > \alpha$ plays the same role as α . Therefore we can assume that α is odd. Then our construction assures that X_α^- is C^* -embedded in X_α , so that the bounded function $\hat{f}$ can be extended to $h : X_\alpha \rightarrow [0, 1]$. The function $h \circ \pi_\alpha : X_\Omega \rightarrow [0, 1]$ is the desired extension of f . $\square$

We call the space X_Ω in Theorem 2.3

the dual Stone-Ćech Ω -extension of the partition $\mathcal{P} : X_0 = X_0^- \cup X_0^+$.

In general let $Y = Y^- \cup Y^+$ be a partition of a space Y into two dense subsets. Then we call $Y = Y^- \cup Y^+$ as a *dually C^* -embedded* partition of Y , if both of Y^-, Y^+ are C^* -embedded in Y . With this terminology Theorem 2.3 can be rephrased that

$X_\Omega = X_\Omega^- \cup X_\Omega^+$ is a dually C^* -embedded partition if $X_0^- \in \mathcal{L}$.

We can show that the space X_Ω of (2-0) depends only on the partition $\mathcal{P}$, so that in particular we get the same space $X_\Omega = \Omega(X_0)$ if we exchange the role of X_0^- and X_0^+ in the above construction. For the proof of this fact see the forthcoming paper [6]. So, let us denote X_Ω by $\Omega(\mathcal{P})$, or simply by $\Omega(X_0)$ when the partition $\mathcal{P}$ is clear.

Now suppose a nowhere compact space $X \in \mathcal{L}$ is given. Then, regarding $X = X_0^-$, we get the subspace X_Ω^- of X_Ω which is uniquely determined by the given space X . Let us denote this X_Ω^- by $\Omega(X)$. Then Theorem 2.3 implies

$$\Omega(\beta X) = \beta(\Omega(X))$$

for $X \in \mathcal{L}$. For example, we have

$$\Omega([0, 1]) = \Omega(\beta\mathbb{Q}) = \beta(\Omega(\mathbb{Q})) = \Omega(\beta\mathbb{P}) = \beta(\Omega(\mathbb{P}))$$

for the partition of $[0, 1]$ in Example 1, and

$$\Omega(\mathbb{A}) = \Omega(\beta\mathbb{S}) = \beta(\Omega(\mathbb{S}))$$

for the partition of $\mathbb{A}$ in Example 2. We can show that $\Omega(\mathbb{A})$ is not homeomorphic with $\Omega([0, 1])$, by proving that $\Omega(\mathbb{A})$ contains no dense set of first category which is C^* -embedded (see [6]).

Note that our construction becomes trivial if the given partition $X_0 = X_0^- \cup X_0^+$ itself is dually C^* -embedded. Fortunately we can prove that is not the case if $X_0^- \in \mathcal{L}(1st)$, i.e.,

Theorem 2.4. ([6]) *Assume $X^{(0)} = X_0^- \in \mathcal{L}(1st)$. Then no bonding map*

$$\Phi_{\alpha,\beta} : X_\beta \rightarrow X_\alpha \quad (\alpha < \beta < \Omega)$$

is one to one.

3. COMMON BOUNDARY POINTS

Let S be a dense subset of T . A point $p \in T \setminus S$ is called *remote from S* , or a *remote point w.r.t. (S, T)* , if $p \notin \text{cl}_T F$ for every nowhere dense closed subset F of S . In case $T = \beta S$ we simply call such a point p as a *remote point of S* . Van Douwen [3, 4], and independently Chae and Smith [2], have shown that:

Fact 3.1. *Every non-pseudocompact space of countable π -weight has 2^c many remote points.*

A space T is said to be *extremally disconnected at a point $p \in T$* (see [4]) if $p \notin \text{cl}_T U_1 \cap \text{cl}_T U_2$ for every pair of disjoint open sets U_1, U_2 in T . We call such a point p an *extremally disconnected point of T* , or simply, an *e.d. point of T* . Obviously a space T is extremally disconnected if every point of T is an e.d. point. If S is dense in T , we always have $\text{cl}_T U = \text{cl}_T (U \cap S)$ for every open set U of T . So, an equivalent definition of an e.d. point is given using only open subsets of any dense subset $S \subseteq T$:

$p \in T$ is an e.d. point if and only if $p \notin \text{cl}_T V_1 \cap \text{cl}_T V_2$ for every pair of disjoint open sets V_1, V_2 in S .

Note that this definition does not depend on the choice of the dense subset S , while it is clear that the notion of remote points depends on the choice of the dense subset S . Note also that in case T, S are ccc (e.g., of countable π -weight), we can choose the above U_1, U_2 as cozero-sets of T , and V_1, V_2 as cozero-sets of S . The next fact proved by van Douwen [4] tells that “remote” implies “e.d.” implies “ C^* -embedded.”

Fact 3.2. (1) *If $p \in \beta X \setminus X$ is remote from X , then p is an e.d. point of βX .*
 (2) *Let X be dense in Y , and $p \in Y \setminus X$. If p is an e.d. point of Y , then X is C^* -embedded in $X \cup \{p\}$ ($\subseteq Y$).*

The proof of the above (1) uses the formula (1-0) in §1.

Let us call a non-e.d. point of T as a “common boundary point” of T , that is, $p \in T$ is a *common boundary point of T* if $p \in \text{cl}_T U_1 \cap \text{cl}_T U_2$ for some pair of disjoint open sets U_1, U_2 in T . Similarly, a closed subset $A \subseteq T$

is called a *common boundary set* in T if $A \subseteq \text{cl}_T U_1 \cap \text{cl}_T U_2$ for some pair of disjoint open sets U_1, U_2 in T . Let us abbreviate “common boundary” to “co-boundary.” (Such p, A are called “2-point” or “2-set” in [4]. We prefer geometric terminology.) Let $\text{Ed}(T)$ denote the set of all e.d. points of T , and put $\text{Cob}(T) = T \setminus \text{Ed}(T)$ which is the set of all co-boundary points of T .

Theorem 3.3. ([6]) *Assume $X_0^-, X_0^+ \in \mathcal{L}$ and that the starting space $X_0 = X_0^- \cup X_0^+$ contains a compact co-boundary set F_0 such that $F_0^- = F_0 \cap X_0^-$, $F_0^+ = F_0 \cap X_0^+$ are nowhere compact and $F_0 \subseteq \text{cl} U_0 \cap \text{cl} V_0$ in X_0 for some disjoint open sets U_0, V_0 in X_0 . Then we can find a compact co-boundary set F_Ω in $X_\Omega = \Omega(X_0)$ such that*

$$\pi_0(F_\Omega) = F_0 \text{ and } F_\Omega \subseteq \text{cl}_{X_\Omega}(U_\Omega) \cap \text{cl}_{X_\Omega}(V_\Omega)$$

for disjoint open sets $U_\Omega = \pi_0^{-1}(U_0)$, $V_\Omega = \pi_0^{-1}(V_0)$ in $X_\Omega = \Omega(X_0)$. Hence, for each $x \in F_0$ we get

$$\pi_0^{-1}(x) \cap \text{Cob}(X_\Omega) \neq \emptyset.$$

Consequently, $\text{Cob}(X_\Omega) = X_\Omega \setminus \text{Ed}(X_\Omega)$ is not empty, i.e., $X_\Omega = \Omega(X_0)$ is not extremally disconnected.

Next easy lemma tells when the hypothesis of Theorem 3.3 is satisfied.

Lemma 3.4. *Suppose $Y \in \mathcal{L}(1\text{st})$, and that Y contains a nowhere dense closed subset $F \in \mathcal{L}(1\text{st})$. Then we can find disjoint open subsets U, V such that $F \subseteq \text{cl} U \cap \text{cl} V$ in Y . $\square$*

From this lemma it is easy to see that the typical examples $\mathbb{Q}, \mathbb{P}, \mathbb{S} \in \mathcal{L}(1\text{st})$ satisfy the hypothesis of Theorem 3.3. Let us illustrate a specific simple partition of $\mathbb{Q}$, as in Lemma 3.4, into the form $U \cup F \cup V$ where $F = \text{cl} U \cap \text{cl} V$, using the standard Cantor set. Consider the standard middle-thirds Cantor set

$$\mathbb{C} = [0, 1] \setminus \bigcup_{n \in \omega} (a_n, b_n)$$

where (a_n, b_n) ($n \in \omega$) are disjoint open intervals in $(0, 1)$ with end points $a_n, b_n \in \mathbb{Q}$. Choose $c_n \in (a_n, b_n) \cap \mathbb{P}$ for each $n \in \omega$ and put

$$U = \mathbb{Q} \cap \bigcup_{n \in \omega} (a_n, c_n), \quad V = \mathbb{Q} \cap \bigcup_{n \in \omega} (c_n, b_n), \quad F = \mathbb{Q} \cap \mathbb{C}.$$

Then $\mathbb{Q}$ is partitioned as $\mathbb{Q} = U \cup F \cup V$, and $F = \text{cl}_\mathbb{Q} U \setminus U = \text{cl}_\mathbb{Q} V \setminus V \approx \mathbb{Q}$ is nowhere dense closed in $\mathbb{Q}$.

We can conclude from Theorem 3.3 and Lemma 3.4 that neither $\Omega([0, 1])$ nor $\Omega(\mathbb{A})$ is extremally disconnected.

4. GENERALIZATION TO MULTIPLE EXTENSIONS

Now let us consider more general partitions. Suppose a compact space K has a partition $\mathcal{P}$ such that

$$(4-0) \quad \mathcal{P} : \quad K = \left(\bigcup_{i \in A} L^i \right) \cup S$$

where $A \subseteq \omega$, $2 \leq |A| \leq \omega$, and each L^i ($i \in A$) is dense in K . We put no particular condition on $S = K \setminus \bigcup_{i \in A} L^i$; for example, S need not be dense, or it may happen $S = \emptyset$. The case of §2 is

$$L^0 = X^-, L^1 = X^+, A = \{0, 1\}, S = \emptyset.$$

Using inverse limits similar to §2, we can construct

$$(4-1) \quad \Omega(\mathcal{P}) = \left(\bigcup_{i \in A} L_\Omega^i \right) \cup S_\Omega,$$

where $L_\Omega^i = \pi^{-1}(L^i)$, $S_\Omega = \pi^{-1}(S)$, and $\pi : \Omega(\mathcal{P}) \rightarrow K$ is a perfect irreducible projection, with the following property similar to Theorem 2.3.

Theorem 4.1. ([6]) *Suppose a partition $\mathcal{P}$ of (4-0) is such that each dense subset L^i ($i \in A$) is Lindeöf. Then the corresponding Lindeöf dense subset L_Ω^i in (4-1) is C^* -embedded in $\Omega(\mathcal{P})$, i.e., $\Omega(\mathcal{P}) = \beta(L_\Omega^i)$ for each $i \in A$.*

In view of this theorem we can call $\Omega(\mathcal{P})$

the multiple Stone-Čech Ω -extension w.r.t. the dense sets L^i ($i \in A$) of the partition $\mathcal{P}$.

We may think of various partitions $\mathcal{P}$, and accordingly various multiple extensions. See [6] for further details.

5. CONCLUSION

As is well known, for every space X there exists an extremely disconnected space $\mathbf{E}(X)$ called the “absolute,” with a perfect irreducible map onto X . Our space $\Omega(X)$ lies in between X and $\mathbf{E}(X)$, and will serve as a useful device to mediate X and $\mathbf{E}(X)$.

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DEPARTMENT OF MATHEMATICS, NATIONAL DEFENSE ACADEMY, YOKOSUKA 239-8686, JAPAN

E-mail address: akiokato@nda.ac.jp