

Existence Theorems in Conditional Gauss Variational Problems

M. YAMASAKI (OKAYAMA UNIV.)

§ 1. Introduction

Let Ω be a locally compact Hausdorff space and G be a Borel measurable function on $\Omega \times \Omega$ which takes values in $(-\infty, \infty]$. We assume that $G(u, v) = G(v, u)$ for all $u, v \in \Omega$ and G is bounded below on every compact set. Such a function G is called a kernel. A non-negative Radon measure μ with compact support S_μ will be called simply a measure. Denote by $M^+(\Omega)$ the totality of measures on Ω and put $M^+(S_\mu) = \{v \in M^+(\Omega); S_v \subset S_\mu\}$ for $\mu \in M^+(\Omega)$. Given measures μ, v on Ω , we define $G(u, \mu)$ and (v, μ) by

$$G(u, \mu) = \int G(u, v) d\mu(v),$$

$$(v, \mu) = \int G(u, \mu) dv(u),$$

and call them the potential of μ and the mutual energy of μ and v respectively. We call (μ, μ) simply the energy of μ . Denote by E the set of measures with finite energy.

We shall say that a property holds n.e. (= nearly everywhere) on a set $B \subset \Omega$ if it holds on B' such that $B' \subset B$, $\mu(K) = 0$ for all compact sets $K \subset B - B'$ and $\mu \in E$. A kernel G is termed to be of positive type or positive semi-definite if

$$(\mu - v, \mu - v) = (\mu, \mu) + (v, v) - 2(v, \mu) \geq 0$$

for all $\mu, v \in E$. In case g and h are extended real-valued functions on Ω which are μ -summable for all $\mu \in E$, we set

$$g(u) - h(u) = 0$$

at points u where $g(u) = h(u) = \infty$ or $g(u) = h(u) = -\infty$.

Let B be a set in Ω which is measurable with respect to every $\mu \in E$ and satisfies the condition that $E'_B \neq \{0\}$, where

$$E'_B = \{\mu \in E; \mu(\Omega - B) = 0\}.$$

Let f and g_k , $k = 1, \dots, n$, be real-valued functions on B which are μ -summable for every $\mu \in E'_B$ and $\{c_k; k = 1, \dots, n\}$ be a set of real numbers. For simplicity, we shall write

$$\int f d\mu = \langle f, \mu \rangle.$$

A mutual energy (μ, ν) can be written as $\langle G(\cdot, \mu), \nu \rangle$.

We shall consider the problem of minimizing the expression (Gauss integral)

$$I(\mu) = (\mu, \mu) - 2 \langle f, \mu \rangle$$

subject to $\mu \in E'_B$ and $\langle g_k, \mu \rangle = c_k$ for each k . This is called the conditional Gauss variational problem. Let A be the transformation from E'_B into the n -dimensional Euclidean space R^n defined by

$$A\mu = (\langle g_1, \mu \rangle, \dots, \langle g_n, \mu \rangle)$$

and let $z_0 = (c_1, \dots, c_n)$. Let us put

$$V = \inf\{I(\mu); \mu \in S\},$$

$$S = \{\mu \in E'_B; A\mu = z_0\},$$

$$S_0 = \{\mu \in S; V = I(\mu)\}.$$

In case S_0 is nonempty, V is finite by our assumption for f .

Let $\mu^* \in E'_B$. $\{g_k; k = 1, \dots, n\}$ is called μ^* -independent if there exists a set $\{\mu_k; k = 1, \dots, n\}$ in $M^+(S\mu^*)$ such that $\mu^* - \mu_k \in M^+(S\mu^*)$ for each k and $\det(\langle g_j, \mu_k \rangle) \neq 0$, where $\det(a_{ij})$ means the determinant of a matrix (a_{ij}) . The set $\{\mu_k; k = 1, \dots, n\}$ is called a system of components of μ^* .

The following existence theorem was established by M. Ohtsuka ([1], p. 213, Theorem 2. 1).

Theorem 1. Assume that $\mu^* \in S_0$ and that $\{g_k; k = 1, \dots, n\}$ is μ^* -independent. Let $\{\mu_k; k = 1, \dots, n\}$ be a system of components of μ^* and $\{r_j; j = 1, \dots, n\}$ be the solution of the equations

$$(1) \quad \sum_{j=1}^n r_j \langle g_j, \mu_k \rangle = \langle G(\cdot, \mu^*) - f, \mu_k \rangle.$$

Then it is valid that

$$(2) \quad G(\cdot, \mu^*) \geq f + \sum_{k=1}^n r_k g_k \quad \text{n.e. on } B,$$

$$(3) \quad G(\cdot, \mu^*) \leq f + \sum_{k=1}^n r_k g_k \quad \mu_k\text{-a.e.}$$

for each k .

The aim of this paper is to study the roles of μ^* -independence in the conditional Gauss variational problem. We improve Theorem 1 and related results in [1]. For applications of our results and the existence of a measure of S_0 , we refer to the forthcoming paper [2].

§ 2. Existence theorems

A system of components $\{\mu_k; k = 1, \dots, n\}$ of μ is called to be full if $\mu = \sum_{k=1}^n \mu_k$. In this case we say that μ has a full system of components.

For $\mu \in E$, we define $C[\mu]$ by

$$C[\mu] = \{v \in M^+(S\mu); \mu - v \in M^+(S\mu)\}.$$

It is shown by the symmetry of G that $C[\mu]$ is a convex subset of E .

Let $Q(\mu)$ be the convex cone generated by $A(C[\mu])$, i.e., $z \in Q(\mu)$ if and only if there exist $v \in C[\mu]$ and a non-negative number t such that $z = tAv$. Denote by F^0 the interior of a set F in R^n and by $((x, y))_2$ the usual inner product of $x, y \in R^n$.

We have

Lemma 1. Let $\mu^* \in S$. If $\{g_k; k = 1, \dots, n\}$ is μ^* -independent, then $z_0 \in Q(\mu^*)^0$.

Proof. Supposing the contrary, we see that z_0 is a boundary point of $Q(\mu^*)$. There exists a nonzero $w = (r_1, \dots, r_n) \in R^n$ by a well-known separation theorem such that

$$((z_0, w))_2 = 0 \leq ((z, w))_2$$

for all $z \in Q(\mu^*)$. Let $\{\mu_k; k = 1, \dots, n\}$ be a system of components of μ^* . From $\mu_k \in C[\mu^*]$ and $\mu^* - \mu_k \in C[\mu^*]$ for each k , it follows that

$$0 \leq ((A\mu_k, w))_2 \leq ((A\mu^*, w))_2 = ((z_0, w))_2 = 0,$$

so that

$$0 = ((A\mu_k, w))_2 = \sum_{j=1}^n r_j \langle g_j, \mu_k \rangle.$$

Since $\det(\langle g_j, \mu_k \rangle) \neq 0$, we conclude that $r_j = 0$ for each j .

This is a contradiction. Therefore $z_0 \in Q(\mu^*)^0$.

Lemma 2. Let $\mu^* \in S$ and assume that $\{g_k; k = 1, \dots, n\}$ is μ^* -independent. Then μ^* has a full system of components.

Proof. Since $z_0 \in Q(\mu^*)^0$ by Lemma 1, we can find a set $\{v_k; k = 1, \dots, n\}$ of measures of $C[\mu^*]$ and a set $\{s_k; k = 1, \dots, n\}$ of strictly positive numbers such that $\{Av_k; k = 1, \dots, n\}$ is linearly independent and

$$z_0 = \sum_{k=1}^n s_k A v_k.$$

In case $s_0 = \sum_{k=1}^n s_k \leq 1$, we have $\bar{\mu} = \sum_{k=1}^n s_k v_k \in S$ and $v = \mu^* - \bar{\mu} \in$

$M^+(S\mu^*)$. Choosing $\mu_k^* = s_k v_k + v/n$ for each k , we see that $\{\mu_k^*; k = 1, \dots, n\}$ is a full system of components of μ^* . In case $s_0 > 1$,

let us put $t_k = s_k/s_0$ for each k and consider $\mu_0 = \sum_{k=1}^n t_k v_k$. Then

$\mu_0 \in C[\mu^*]$ and $A\mu_0 = z_0/s_0$. Taking $v_0 = \mu^* - \mu_0$ and $\mu_k^* = t_k v_k + v_0/n$, we can show that $\{\mu_k^*; k = 1, \dots, n\}$ is a full system of components of μ^* .

For $\mu^* \in S_0$, we denote by $S_0^*(\mu^*)$ the set of points $w = (r_1, \dots, r_n)$ which satisfy the following relations (4) and (5):

$$(4) \quad G(\cdot, \mu^*) \geq f + \sum_{k=1}^n r_k g_k \quad \text{n.e. on } B,$$

$$(5) \quad G(\cdot, \mu^*) \leq f + \sum_{k=1}^n r_k g_k \quad \mu^*\text{-a.e.}$$

If $w \in S_0^*(\mu^*)$, then it is valid that

$$(6) \quad \langle G(\cdot, \mu^*) - f, \mu^* \rangle = ((z_0, w))_2,$$

$$(7) \quad V = I(\mu^*) = ((z_0, w))_2 - \langle f, \mu^* \rangle = 2((z_0, w))_2 - (\mu^*, \mu^*).$$

It is easily seen that (4) and (6) imply (5).

We shall prove

Theorem 2. Assume that $\mu^* \in S_0$ and that $\{g_k; k = 1, \dots, n\}$ is μ^* -independent. Then $S_0^*(\mu^*)$ consists of only one point $\bar{w}(\mu^*) = (r_1, \dots, r_n)$. If $\{\mu_k; k = 1, \dots, n\}$ is a system of components of μ^* , then $\{r_j; j = 1, \dots, n\}$ is the solution of the equations

$$(8) \quad \sum_{j=1}^n r_j \langle g_j, \mu_k \rangle = \langle G(\cdot, \mu^*) - f, \mu_k \rangle.$$

Proof. Let $\{\mu_k^*; k = 1, \dots, n\}$ be a full system of components of μ^* and define $\{r_j; j = 1, \dots, n\}$ by

$$\sum_{j=1}^n r_j \langle g_j, \mu_k^* \rangle = \langle G(\cdot, \mu^*) - f, \mu_k^* \rangle.$$

It follows from Theorem 1 that

$$(9) \quad G(\cdot, \mu^*) \geq f + \sum_{j=1}^n r_j g_j \quad \text{n.e. on } B,$$

$$(10) \quad G(\cdot, \mu^*) \leq f + \sum_{j=1}^n r_j g_j \quad \mu^*\text{-a.e.},$$

since $\mu^* = \sum_{k=1}^n \mu_k^*$. Therefore $\bar{w} = (r_1, \dots, r_n) \in S_0^*(\mu^*)$. If $w =$

$(s_1, \dots, s_n) \in S_0^*(\mu^*)$, then

$$\sum_{j=1}^n s_j \langle g_j, \mu_k^* \rangle = \langle G(\cdot, \mu^*) - f, \mu_k^* \rangle = \sum_{j=1}^n r_j \langle g_j, \mu_k^* \rangle,$$

so that

$$\sum_{j=1}^n (s_j - r_j) \langle g_j, \mu_k^* \rangle = 0.$$

Since $\det(\langle g_j, \mu_k^* \rangle) \neq 0$, we have $w = \bar{w}$. Namely $S_0^*(\mu^*)$ consists of only one point. Let $\{\mu_k; k = 1, \dots, n\}$ be a system of components of μ^* . Then we have (8) by (9) and (10). This completes the proof.

Theorem 3. Assume that E_B' is convex and that $\mu^* \in S_0$. If $\{g_k; k = 1, \dots, n\}$ is μ -independent for some $\mu \in S$, then $S_0^*(\mu^*)$ is nonempty.

Proof. From our assumption that E_B' is convex, it follows that S is convex. It is shown that

$$\langle G(\cdot, \mu^*) - f, \mu^* \rangle \leq \langle G(\cdot, \mu^*) - f, v \rangle$$

for all $v \in S$. Writing $g_0 = G(\cdot, \mu^*) - f$, we have

$$M = \langle g_0, \mu^* \rangle = \min\{\langle g_0, v \rangle; v \in S\},$$

$$V = \langle g_0, \mu^* \rangle - \langle f, \mu^* \rangle.$$

Our assumption that $\{g_k; k = 1, \dots, n\}$ is μ -independent for some $\mu \in S$ is equivalent to the condition $z_0 \in A(E'_B)^0$. By means of a duality theorem in semi-infinite programs, we see that there exists $\bar{w} \in R^n$ such that

$$((Av, \bar{w}))_2 \leq \langle g_0, v \rangle$$

for all $v \in E'_B$ and $M = ((z_0, \bar{w}))_2$. It is shown that $\bar{w} \in S^*_0(\mu^*)$.

E'_B is convex if and only if (μ, v) is finite for all $\mu, v \in E'_B$. It is clear that E'_B is convex whenever G is of positive type or G is bounded on $B \times B$.

Theorem 4. Assume that G is of positive type and that μ^* and v^* are elements of S_0 . Then it is valid that $S^*_0(\mu^*) = S^*_0(v^*)$.

Proof. From $I(\mu^*) = I(v^*) = V$ and $(\mu^* + v^*)/2 \in S$, it follows that

$$I(\mu^*) \leq I((\mu^* + v^*)/2) = I(\mu^*) - (\mu^* - v^*, \mu^* - v^*)/4,$$

and hence $(\mu^* - v^*, \mu^* - v^*) \leq 0$. Since G is of positive type, we have $(\mu^* - v^*, \mu^* - v^*) = 0$ and $G(\cdot, \mu^*) = G(\cdot, v^*)$ n.e. in Ω .

Consequently $(\mu^*, \mu^*) = (\mu^*, v^*) = (v^*, v^*)$ and $\langle f, \mu^* \rangle = \langle f, v^* \rangle$.

Assume that $\bar{w} = (r_1, \dots, r_n) \in S^*_0(\mu^*)$. By the above observation, we see that

$$G(\cdot, v^*) - f = G(\cdot, \mu^*) - f \geq \sum_{k=1}^n r_k g_k \quad \text{n.e. on } B,$$

$$\langle G(\cdot, v^*) - f, v^* \rangle = \langle G(\cdot, \mu^*) - f, \mu^* \rangle = \sum_{k=1}^n r_k c_k$$

$$= \sum_{k=1}^n r_k \langle g_k, v^* \rangle = \langle \sum_{k=1}^n r_k g_k, v^* \rangle,$$

and hence

$$G(\cdot, v^*) - f = \sum_{k=1}^n r_k g_k \quad v^*\text{-a.e.},$$

so that $\bar{w} \in S_0^*(v^*)$. Therefore $S_0^*(\mu^*) \subset S_0^*(v^*)$. Since the discussion is symmetric, we have $S_0^*(v^*) \subset S_0^*(\mu^*)$ and hence $S_0^*(\mu^*) = S_0^*(v^*)$.

Corollary. Assume that G is of positive type and let μ^* and v^* be elements of S_0 . If $\{g_k; k = 1, \dots, n\}$ is μ^* -independent and v^* -independent, then it is valid that $\bar{w}(\mu^*) = \bar{w}(v^*)$.

We observe that Theorem 4 and its corollary are not always valid if G is not of positive type.

References

- [1] M. Ohtsuka: On potentials in locally compact spaces, J. Sci. Hiroshima Univ. Ser. A-I Math., 25 (1961), 135 - 352.
- [2] M. Yamasaki: Semi-infinite programs and conditional Gauss variational problems, Hiroshima Math. J., 1 (1971).