

Celestial Holography meets dS/CFT

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Abstract

The holographic principle suggests that a consistent theory of quantum gravity may admit a description in terms of degrees of freedom localized on a boundary, rather than in the bulk spacetime itself. Its most concrete and quantitatively precise realization is the AdS/CFT correspondence, in which gravitational dynamics in a $(D+1)$ -dimensional anti-de Sitter (AdS) spacetime is encoded in a d -dimensional conformal field theory (CFT) defined on its boundary. Motivated by the question of whether such a holographic framework can be extended beyond AdS, this thesis investigates holographic structures in spacetimes closer to our universe, in particular asymptotically flat spacetimes.

We first review the role of asymptotic symmetries in gravity and gauge theories. In asymptotically flat spacetimes, the Bondi–van der Burg–Metzner–Sachs (BMS) group acts nontrivially on physical observables and can be naturally interpreted as the symmetry of a putative holographic dual. This perspective leads to celestial holography, in which scattering amplitudes in Minkowski spacetime are reorganized as correlation functions of a two-dimensional conformal theory living on the celestial sphere.

To clarify the conceptual structure underlying these ideas, we adopt a stepwise strategy. We first analyze a $U(1)$ gauge theory with a boundary at finite location, formulating it in the language of path integrals and carefully identifying the physical symmetries and observables. We then generalize the analysis to gravity, and subsequently extend the discussion to flat spacetime, culminating in an introduction to celestial holography from the viewpoint of asymptotic symmetries.

Finally, following and extending the results of [1], we construct a concrete correspondence between cosmological correlators in $(D+1)$ -dimensional de Sitter space and celestial amplitudes. By employing slicing constructions and analytic continuation from Euclidean theories on the sphere, we derive an explicit identification between celestial operators defined via Mellin transforms of flat-space scattering states and the extrapolate operators that appear in dS/CFT. This correspondence unifies celestial holography, dS/CFT, and cosmological correlators within a single framework, and suggests a systematic route for importing the rich results of (A)dS holography into the study of quantum gravity in asymptotically flat spacetimes.

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Chapter 1

Introduction

1.1 Holographic Principle and AdS/CFT

The origin of the holographic principle traces back to black hole thermodynamics. Within the framework of purely classical general relativity, it was first shown that black holes satisfy thermodynamic relations, suggesting that they may be treated as thermodynamic systems [2]. In particular, the fact that the area of the event horizon increases monotonically in classical processes [3] naturally led to the idea that the entropy of a black hole should be proportional to the area of its event horizon.

Subsequently, within the framework of quantum field theory in curved spacetime, it was shown, by treating the black hole geometry as a fixed background, that black holes emit thermal radiation [4]. This firmly established that black holes are genuine thermodynamic objects endowed with both temperature and entropy. In particular, the previously undetermined proportionality constant in the entropy–area relation was fixed, leading to the Bekenstein–Hawking formula [5]

$$S_{\text{BH}} = \frac{A}{4G_N\hbar}, \tag{1.1.1}$$

which demonstrates that the entropy of a black hole is precisely tied to the area of its event horizon, with the Planck constant characterizing the quantum world and Newton’s constant characterizing the strength of gravitational interactions.

This relation suggests not merely a property of black holes as specific systems, but more generally that, in gravitational theories, the number of degrees of freedom scales with the area of the boundary rather than with the volume of the bulk [6, 7]. Equivalently, in a quantum theory of gravity it is natural to expect that all information contained in a given spatial region can be completely encoded in the degrees of freedom on its boundary. Pushing this viewpoint further, one arrives at the holographic principle: any consistent quantum theory of gravity should admit a “holographic” formulation in which bulk physics can be described purely in terms of boundary

data [8, 9, 10].

The most concrete and quantitatively precise realization of the holographic principle is provided by the AdS/CFT correspondence [11]. Within the framework of string theory, it was proposed that a gravitational theory on a $(D+1)$ -dimensional anti-de Sitter (AdS) spacetime is exactly dual to a D -dimensional conformal field theory (CFT) defined on its boundary. This duality was formulated in the context of type IIB string theory on $\text{AdS}_5 \times S^5$ and $\mathcal{N} = 4$ super Yang–Mills theory in four dimensions [11].

This correspondence is often summarized by the so-called GKPW relation [12, 13],

$$Z_{\text{grav}}[\phi_0] = Z_{\text{CFT}}[\phi_0], \quad (1.1.2)$$

where ϕ_0 denotes the boundary value of a bulk field, which simultaneously plays the role of a source for an operator in the boundary CFT. On the gravitational side, Z_{grav} is the partition function evaluated on classical solutions with prescribed boundary conditions, while on the CFT side Z_{CFT} is the generating functional for correlation functions. By taking functional derivatives with respect to ϕ_0 , one reproduces the CFT correlators.

A complementary viewpoint is provided by the BDHM-type “extrapolate dictionary,” which relates bulk fields to boundary operators according to

$$\mathcal{O}(x) \sim \lim_{r \rightarrow \infty} r^\Delta \phi(r, x). \quad (1.1.3)$$

This prescription was originally proposed in [14].¹

These successes naturally lead us to the following question: *Can a holographic correspondence also be constructed for spacetimes that more closely resemble our universe, such as asymptotically flat spacetimes?*

1.2 Asymptotic Symmetries and Flat Holography

Assuming that a holographic dual for gravity exists, it is natural to expect that the symmetries of the boundary theory correspond to the “true” symmetries of bulk gravity. In general relativity, most of the diffeomorphism invariance is merely gauge redundancy, introduced as an auxiliary symmetry in order to express the theory in terms of a local Lagrangian. Among these, however, there exist transformations that act nontrivially on boundary data; these define the *asymptotic symmetry group* (ASG). Since the ASG acts nontrivially on observables, it can be interpreted as a genuine physical symmetry (we will confirm it in detail in Chapter 2).²

A paradigmatic example is provided by three-dimensional gravity with a negative cosmological constant. In this case, the asymptotic symmetry group is realized as two copies of the Virasoro

¹In asymptotically AdS spacetimes, the GKPW and BDHM prescriptions are known to be equivalent [15, 16].

²For a review of asymptotic symmetries in gravity and their relation to infrared physics, see e.g. [17].

algebra with a nonvanishing central charge [18]. This asymptotic symmetry coincides with the conformal symmetry of the dual two-dimensional CFT in the $\text{AdS}_3/\text{CFT}_2$ correspondence, and thus offers a concrete illustration of how asymptotic symmetries can control the structure of a holographic dual [19].

In four-dimensional asymptotically flat spacetimes, the ASG is the Bondi–van der Burg–Metzner–Sachs (BMS) group [20, 21]. The BMS group extends the Poincaré group by an infinite set of angle-dependent translations (*supertranslations*), and has the semidirect product structure

$$\text{BMS} = \text{Lorentz} \ltimes \text{Supertranslations} . \quad (1.2.1)$$

The Lorentz subgroup $SL(2, \mathbb{C})$ acts as the group of global conformal transformations on the celestial sphere³ [22]. Moreover, it has been shown that this action can be locally extended to an infinite-dimensional symmetry, known as *superrotations* [23, 24], suggesting the emergence of a Virasoro-like symmetry algebra on the celestial sphere.

Supertranslations (superrotations) are associated with Noether currents, which are defined in terms of the flux of gravitational soft radiation through future null infinity $\mathcal{I}^+$ [25, 26]. The Ward identities for the conserved currents corresponding to supertranslations (superrotations) have been shown to be equivalent to the leading (subleading) soft graviton theorems in scattering amplitudes [27, 28], which were originally derived independently in the context of quantum field theory. This equivalence provides nontrivial support for the above line of reasoning.

From these considerations it is suggested that, at least at the level of symmetries, a two-dimensional CFT endowed with a Virasoro-like symmetry⁴ may serve as a holographic dual of four-dimensional flat spacetime. Such a framework is commonly referred to as *celestial holography* [29, 22, 17, 30, 31, 32].

In addition, there exists a complementary line of research toward the formulation of flat-space holography. This approach exploits the fact that Minkowski spacetime can be foliated by de Sitter or anti-de Sitter slices, and investigates the relation between comparatively well-understood frameworks such as AdS/CFT and dS/CFT and celestial holography (see e.g. [33, 34, 35]).

In this thesis, Chapter 2 organizes the basic facts leading up to celestial holography. In particular, it reviews and analyzes the structure of Ward identities and observables in spacetimes with boundaries and in asymptotically flat spacetimes, as well as their implications for holography. In Chapter 3, based on Ref. [1], we focus on the relation between dS/CFT (cosmological correlators) and celestial holography (celestial amplitudes), employing slicing constructions to demonstrate that there exists a simple correspondence between the two frameworks.

³This is obtained by quotienting future null infinity by the generator direction, namely, it is the manifold in the angular directions.

⁴Together with the symmetries associated with supertranslations.

Conventions in this thesis For integration over a manifold M , we use the shorthand

$$\int_M [d^D x] \equiv \int_M d^D x \sqrt{|g_M|}, \quad (1.2.2)$$

where g_M denotes the absolute value of the determinant of the metric on M . In addition, we adopt the normalization $32\pi G = 1$ for the gravitational constant and $e = 1$ for the electromagnetic coupling.

Chapter 2

Symmetry aspect of Gauge theories

The aims of this chapter can be roughly summarized as follows:

- To show that gravity is already holographic at a purely kinematical level, before entering dynamics, thereby providing motivation for considering holographic frameworks that are not restricted to AdS (see, for example, Refs. [36, 37] and references therein).
- To introduce the symmetry structure of gravity in asymptotically flat spacetimes (AFS) and to present an overview of celestial holography.

However, if we try to address gravity directly from the outset, we are forced to confront technical issues and conceptual issues simultaneously, and the logical flow of the discussion becomes unnecessarily obscure. For this reason, the strategy of this chapter is stepwise. We first analyze a $U(1)$ gauge theory with a boundary at finite location, and then generalize to gravity in the same setting. We next move on to a $U(1)$ theory in flat spacetime, and finally extend the discussion to gravity in AFS, culminating in an introduction to celestial holography.

Formally, this chapter is presented as a review of existing results. In practice, however, many of these results are rederived in the language of path integrals, which in itself lends a certain degree of originality to the discussion. Moreover, in several places we provide (modest) generalizations of known results.

Conventions in this chapter Bulk coordinates are denoted by X with indices $I, J, K, \dots$. Boundary coordinates are denoted by x with indices $i, j, k, \dots$. For codimension-two submanifolds embedded in the boundary, we use coordinates z and indices $\alpha, \beta, \gamma, \dots$.

Unless otherwise stated, the path-integral measure is understood to be divided by the volume of the gauge orbit. Furthermore, all path integrals are implicitly normalized by a factor of $1/Z$, where Z denotes the partition function. Even when the path integral is written only over the gauge field A or the metric g , it should be understood that, whenever appropriate, the integration also includes any matter fields with suitable degrees of freedom.

2.1 $U(1)$ gauge theory with boundary

In this section, we systematically organize the kinematical aspects of the simplest gauge theory, namely $U(1)$ gauge theory, on a $D + 1$ Lorentzian manifold M with boundary ∂M ¹. The aim of this chapter is to clarify that the boundary action, which may at first sight appear to play only a modest role, in fact plays a crucial role in determining observables, symmetries, and the associated Ward identities (see e.g. [38, 39]). We also wish to make precise the usage of several terms whose meanings often vary across the literature, by stating clearly how they are used in this work.

2.1.1 Set up and defining terminologies

In this subsection, we give definitions of the various concepts that will be used throughout this chapter.

Boundary action, LGT and SGT

In this subsubsection, we introduce the boundary action as well as LGT and SGT.

A *gauge transformation* (GT) is a transformation that leaves the bulk action strictly invariant². In the case of a $U(1)$ gauge theory, for example, for a gauge field A we have

$$\delta_\Lambda A = d\Lambda, \quad (2.1.1)$$

and, for later convenience, we classify GT into the following two classes:

- Those whose gauge parameter takes a nontrivial (finite) value on the boundary ∂M are called *large gauge transformations* (LGT).
- All others, *i.e.* those that become trivial on the boundary, are called *small gauge transformations* (SGT).

We denote the boundary value of the gauge field by

$$a_i(x) := A_i(X)|_{\partial M}. \quad (2.1.2)$$

Next, we define the full action obtained by adding a boundary term to the usual Maxwell action:

$$S[A] = S_{\text{bulk}}[A] + S_{\text{bdy}}[a]. \quad (2.1.3)$$

Here $S_{\text{bdy}}[a]$ is called the *boundary action*, and we allow $S_{\text{bdy}}[a]$ to break LGT invariance³.

¹Euclidean extension seems to be not so difficult.

²Here, “strictly” means that the action is invariant including possible boundary terms, not merely up to a boundary term.

³We also allow the corresponding boundary Lagrangian to be nonlocal in the boundary field a .

One might wonder why we consider a setup in which gauge invariance is broken at the boundary. However, this is in fact a quite standard situation. One concrete example is Eq. (2.1.6), which impose Dirichlet boundary condition. As is well known, boundary conditions of this type are imposed on timelike boundaries in order to make the Cauchy problem well defined (see, for example, [39]). In the context of AdS/CFT, this Dirichlet boundary condition is understood as fixing the sources in the boundary theory [12, 13].

In the path-integral formulation, boundary actions of this kind are often implicitly encoded in the integration measure. We will likewise absorb LGT-invariant boundary conditions that do not play an essential role in our discussion into the measure without writing them explicitly. By contrast, our main interest in this paper lies in boundary conditions that explicitly break LGT invariance, and for such cases it is technically convenient to write them as explicit boundary actions. This is the reason why we introduce terms that explicitly break LGT invariance.

It should be emphasized that this choice is not uniquely determined by the choice of bulk action (that is, by the choice of the theory itself), but rather depends on what physical setup one intends to describe. In this paper we will occasionally comment on this broader issue, but we will not attempt a systematic classification. Instead, we will focus on how the presence of a given boundary action modifies the symmetries and the structure of observables.

Symmetry and Observable

In this subsection, we define symmetries and observables⁴ in gauge theory.

Let us first define *observables*.⁵ The key point is that, in gauge theories, many operators (or functionals in the path-integral formalism) do not carry any physical information. Namely, within the path integral, any operator is projected onto its gauge-invariant component by local gauge averaging argument⁶ (A.1.6), and it is always this gauge-invariant component that is actually probed through correlation functions. Based on this observation, we regard operators that are mapped to the same object under gauge averaging as physically indistinguishable, and we define the algebra of observables to be the algebra consisting only of the fixed points of the local gauge averaging, i.e. of gauge-invariant operators.

In the setup considered in this paper, however, the presence of boundary terms implies that the action S is in general not invariant under LGT. On the other hand, we assume that it is invariant under SGT. As a result, the above “local gauge averaging argument” can be applied only to SGT.

⁴The set of symmetries and observables obtained in this paper partly agrees with, and partly differs from, those in the existing literature. An overview of the various definitions employed there, and of how they differ, can be found, for example, in Sec. 3 and Sec. 4 (in particular Sec. 4.3) of Ref. [40].

⁵Gauge theory and gravity theory observables are discussed in depth in Ref. [41].

⁶By this we mean the averaging over local gauge transformations that appears as an intermediate step in the proof of Elitzur’s theorem (see appendix A.1).

Thus, in our setting:

- Operators that carry charge under SGT (i.e. that transform non-trivially under SGT) are still projected to zero under gauge averaging.
- In contrast, operators that carry charge only under LGT are not required to vanish after gauge averaging.

Taking this into account, observables in $U(1)$ gauge theory can be classified according to the choice of boundary action as follows:

- **LGT-invariant boundary action:** When the action is invariant under the full set of gauge transformations including LGTs (see subsection A.2 for explicit examples), the independent observables are restricted to neutral operators such as Wilson loops and the field strength F_{IJ} .
- **LGT-non-invariant boundary action:** When the boundary action S_{bdy} is not invariant under LGT (see appendix 2.1.2 for explicit examples), the set of independent observables includes, in addition to the neutral operators above, operators that carry charge under LGT. A typical example is a Wilson line that pierces the boundary manifold ∂M .⁷; such an operator carries the LGT charge living on the boundary.⁸

In this way, within our framework the notion of “observables” in a gauge theory crucially depends on the choice of boundary action.

Once the set of observables has been fixed by this procedure, GT can further be classified into those that act non-trivially on observables (*symmetries*) and those that do not act non-trivially on them (*redundancies*).⁹ In other words, only symmetries can give rise to non-trivial constraints on observables, namely Ward identities.

⁷Boundary-piercing Wilson lines are sometimes referred to as edge modes. However, in this work we reserve the term edge modes for degrees of freedom obtained by promoting the boundary frame itself to a dynamical variable. This terminology coincides with that adopted in Ref. [40], where the differences in usage of these terms across the literature are also systematically summarized.

⁸The fact that LGT cease to be mere redundancies once we impose a Dirichlet problem is consistent with the discussion in Ref. [39]. In this reference [39], the authors argue, from a different perspective, that these transformations should not be regarded as redundancies by showing that, within the canonical formalism, the pre-symplectic form does not degenerate when one substitutes field variations along the directions of LGT.

⁹In the present context, LGT, SGT, symmetry, and redundancy are closely related but distinct notions. SGT always correspond to redundancies, whereas LGT may be treated as partly or entirely redundant, or as physical symmetries, depending on the choice of boundary action.

2.1.2 Ward identity

We now wish to derive the Ward identity. For an arbitrary variation, a change of integration variables implies the following identity:

$$\delta \langle \mathcal{O} \rangle = \int \delta (\mathcal{D}A e^{iS[A]} \mathcal{O}) = 0. \quad (2.1.4)$$

Taking the variation δ to be a gauge transformation δ_Λ , and assuming that this $U(1)$ gauge theory is anomaly-free so that $\delta_\Lambda \mathcal{D}A = 0$, we obtain

$$\delta_\Lambda \langle \mathcal{O} \rangle = \int \mathcal{D}A \delta_\Lambda (e^{iS[A]} \mathcal{O}) = 0. \quad (2.1.5)$$

By evaluating this expression explicitly, one derives the Ward identity.

Let us now consider the following Dirichlet boundary condition.¹⁰ In this case, we choose the boundary action as

$$e^{iS_{\text{bdy}}[a]} = \delta[a + d\bar{\phi}], \quad (2.1.6)$$

where δ denotes the functional delta function. Equivalently, one may regard this as restricting the integration domain of the path integral to configurations satisfying $a = -d\bar{\phi}$. This requirement not only imposes $F_{ij} = 0$ on the boundary, but also fixes the $U(1)$ gauge frame on the boundary.¹¹ In this setup, the correlator can be written as

$$\langle \mathcal{O} \rangle = \int \mathcal{D}A e^{iS_{\text{bulk}}[A]} e^{iS_{\text{bdy}}[a]} \mathcal{O}. \quad (2.1.7)$$

Using Eq. (2.1.5), we obtain

$$\delta_\Lambda \left\langle \prod_n \mathcal{O}_n(x_n) \right\rangle = \int \mathcal{D}A \delta_\Lambda (e^{iS_{\text{bulk}}[A]} e^{iS_{\text{bdy}}[a]} \mathcal{O}) = 0. \quad (2.1.8)$$

Since $\delta_\Lambda S_{\text{bulk}} = 0$, the Leibniz rule splits this into two terms: one where δ_Λ acts on the boundary factor, and one where it acts on the operator $\mathcal{O}$:

$$\left(\int \mathcal{D}A e^{iS_{\text{bulk}}[A]} \left(\int d^D x \nabla_i \Lambda(x) \frac{\delta}{\delta a(x)} e^{iS_{\text{bdy}}[a]} \right) \mathcal{O} \right) + \langle \delta_\Lambda \mathcal{O} \rangle = 0. \quad (2.1.9)$$

Let us focus on the first term. Integrating by parts with respect to the boundary variable a , the functional derivative with respect to the boundary field is transferred onto $e^{iS_{\text{bulk}}}$ and $\mathcal{O}$, yielding

$$\left\langle \left(\int_{\partial M} [d^D x] \nabla_i J^i \Lambda \right) \mathcal{O} \right\rangle = -i \left\langle \left(\delta_\Lambda - \int_{\partial M} [d^D x] \nabla_i \Lambda(x) \frac{\delta}{\delta a(x)} \right) \mathcal{O} \right\rangle \quad (2.1.10)$$

¹⁰It is not necessary to impose this on the entire boundary; in fact, it is more natural not to impose it on the Cauchy surfaces.

¹¹This somewhat artificial feature is discussed in a bit more detail when we consider edge modes in appendix A.2.

where we have used

$$\int_{\partial M} [d^D x] \nabla_i \Lambda(x) \frac{\delta}{\delta a_i(x)} S_{\text{bulk}} = - \int_{\partial M} [d^D x] \nabla_i J^i \Lambda, \quad (2.1.11)$$

and defined the *boundary $U(1)$ current J^j* by

$$J^i = \frac{\delta S_{\text{bulk}}}{\delta a_i(x)}. \quad (2.1.12)$$

This is the general form of the Ward identity for $U(1)$ gauge theory, although it is somewhat unfamiliar in this form. To obtain a “local” Ward identity, we need to make the following additional assumptions:

- We assume that $\mathcal{O}$ does not depend on a . This is required to cancel the contribution

$$- \int d^D x \partial_i \Lambda(x) \frac{\delta}{\delta a_i(x)} \mathcal{O} \quad (2.1.13)$$

appearing in the Ward identity (2.1.10). When imposing Dirichlet boundary conditions, we are in fact always implicitly making this assumption. Indeed, in the definition of the correlator (2.1.7), the integration has support only on configurations with $a = d\bar{\phi}$, so any dependence of $\mathcal{O}$ on a can be replaced by a dependence on $d\bar{\phi}$.¹²

- The observable $\mathcal{O}$ can be written as a product $\prod_n \mathcal{O}_n(x_n)$, where each $\mathcal{O}_n(x_n)$ is labeled by a boundary point x_n . Moreover, we assume that the gauge transformation of $\mathcal{O}_n(x_n)$ depends linearly on Λ , i.e. $\delta_\Lambda \mathcal{O}_n(x_n) \propto \Lambda(x_n) \delta \mathcal{O}_n(x_n)$. Examples of such observable $\mathcal{O}(x)$ include boundary electronic field $n^I F_{Ij}|_{\partial}(x)$ ¹³ and Wilson line operators that pierce the boundary at two points x_n, x_m , which can be viewed as a composite operator as following :

$$\mathcal{O}_\Gamma(x) \mathcal{O}_\Gamma(x') = \exp \left(i \int_{\Gamma(x, x')} A \right). \quad (2.1.14)$$

If matter operators Φ are present, one may also consider observable such as $n^I D_I \Phi(x_n)$, or a “dressed” observable of the form

$$\mathcal{O}_{\Phi, \Gamma}(x) = \exp \left(i \int_{\Gamma(x, X)} A \right) \Phi(X). \quad (2.1.15)$$

¹²Equivalently, one may say that the term $-\int d^D x \partial_i \Lambda(x) \delta / \delta a_i(x)$ serves to compensate for the difference between the variation δ_Λ evaluated *after* substituting $a = d\bar{\phi}$ into $\mathcal{O}$, and the variation evaluated *without* making that substitution.

¹³This assumption includes gauge-invariant operators as the special case where the proportionality constant vanishes.

Under these assumptions, Eq. (2.1.10) reduces, as a special case, to the local Ward identity

$$\begin{aligned} \left\langle \left(\int_{\partial M} d^D x \partial_j J^j \Lambda \right) \prod_n \mathcal{O}_n(x_n) \right\rangle &= -i \left\langle \Lambda(x_m) \delta \mathcal{O}_m(x_m) \prod_{n \neq m} \mathcal{O}_n(x_n) \right\rangle \\ \Leftrightarrow \left\langle \partial_j J^j \prod_n \mathcal{O}_n(x_n) \right\rangle &= -i \delta^{(D)}(x - x_m) \left\langle \delta \mathcal{O}_m(x_m) \prod_{n \neq m} \mathcal{O}_n(x_n) \right\rangle. \end{aligned} \quad (2.1.16)$$

Finally, let us comment on the generalization to other choices of boundary action. The essential step in generating the current term is that, upon integrating by parts, the functional derivative that originally acted on the boundary action is transferred to the other factors. This step relies only on the fact that the boundary action is not gauge invariant, and does not depend on its detailed form. In other words, the essential point is that the boundary action is not gauge invariant. Therefore, any boundary action that breaks gauge symmetry produces a similar effect. For example, one may choose

$$S_{\text{bdy}} = \frac{m^2}{2} \int_{\partial M} [d^D x] (a + d\bar{\phi})^2, \quad (2.1.17)$$

or

$$S_{\text{bdy}} = \int_{\partial M} [d^D x] \mathcal{J} \cdot (a + d\bar{\phi}). \quad (2.1.18)$$

The physical meaning of the former is to provide a mass term for the boundary gauge field, while the latter describes the coupling of the boundary gauge field to an external current $\mathcal{J}$.¹⁴

2.2 Gravitational theory with boundary

The quantization of a gravitational theory — that is, a diffeomorphism -invariant theory with a dynamical metric g — is extremely challenging and continues to involve many unresolved problems (see for recent review [42]). Nevertheless, it is known that even without delving into the full dynamical details, and already at the level of almost purely “kinematical” considerations, gravity exhibits holographic features (See, for example, Refs. [36, 37] and references therein). In this work, we focus on a series of studies that have clarified such properties and revisit them in the setting of a gravitational theory with Dirichlet boundary.

In this section, following the previous section 2.1, we organize and review the kinematical properties of gravitational theories defined on manifolds with boundary. More concretely, we elucidate the structure of Ward identities (see appendix A.3) and observables in these theories. Finally, we examine the implications of these results for holography.

¹⁴By taking the limit $m \rightarrow \infty$, or by integrating over $\mathcal{J}$ (i.e. promoting the external current to a dynamical field), one can recover the Dirichlet boundary condition.

2.2.1 Observable

In this subsection, we review the structure of observables following Ref. [43]. A crucial difference from the $U(1)$ case is that a *strictly bulk local* operator cannot, in general, be defined as an observable.

Closed manifold

Let us first consider a Euclidean closed manifold. A diffeomorphism (now we call GT) acts on a scalar operator as

$$\delta_\zeta \mathcal{O}(X) = \zeta^I \partial_I \mathcal{O}(X). \quad (2.2.1)$$

If we assume that local averaging argument can be straightforwardly generalized to a gravitational theory, we immediately reach the conclusion that any scalar quantity $\mathcal{O}(X)$ is not an observable.¹⁵

This becomes more intuitive if we think of coordinates in a gravitational theory as playing a role analogous to the color indices in a non-abelian gauge theory. In an ordinary non-abelian gauge theory, spacetime coordinates and color indices are completely distinct. In gravity, by contrast, the “color” and the coordinates are, in a sense, degenerate with one another. This is one source of the conceptual difficulty that sets gravity apart from ordinary gauge theories.

If we wish to construct observables, we must, for example, consider quantities in which the coordinates have been completely “traced out”, such as

$$\mathcal{O}_M = \int_M d^D X \sqrt{g} \mathcal{O}(X). \quad (2.2.2)$$

This is manifestly invariant under GT. However, it is also highly non-local.

Nevertheless, one may still be tempted, in some circumstances, to regard a geometric scalar quantity $\mathcal{O}(X)$ as a “local observable”. In order to state precisely what is actually gauge invariant in such discussions, it is useful to mimic a “local observable” by the following construction. We introduce D scalar fields, or reference fields, $Z^A(X)$, treat them as dynamical “frame” degrees of freedom, and integrate over them in the path integral, just like any other fields. One may think of these as, for example, a lattice of clocks distributed throughout spacetime. Here $A, B, \dots$ are not spacetime indices, but merely labels distinguishing the different scalar fields. Namely, under an infinitesimal GT generated by ζ^I , we take

$$\delta_\zeta Z^A(X) = \zeta^I \partial_I Z^A(X). \quad (2.2.3)$$

¹⁵This is not an intrinsically quantum statement; the same conclusion holds already at the classical level (see e.g. [41, 44]) It is simply the general statement that gauge non-invariant quantities are not observable.

We then define

$$\mathcal{O}_{\text{loc}}(\sigma) \equiv \int d^D X \sqrt{g} \mathcal{O}(X) \delta(Z^A(X) - \sigma^A), \quad (2.2.4)$$

where σ^A is a fixed label in the “reference space”. Since $\sqrt{g} \mathcal{O}$ is a scalar density and the delta functional is built out of the reference fields Z^A themselves, $\mathcal{O}_{\text{loc}}(\sigma)$ is invariant under GT as long as Z^A transforms as above. By contrast, if we treated Z^A as a non-transforming background reference field, the same expression would *not* be invariant under GT, and the operator would be non-observable.

Moreover, in a semiclassical regime where the state of the combined (g, Z) system is sharply peaked around a classical solution (g_*, Z_*) , and where $Z_*^A(X)$ defines a good coordinate system, we may choose coordinates such that

$$Z_*^A(X) = X^A. \quad (2.2.5)$$

On such a background we then have

$$\mathcal{O}_{\text{loc}}(\sigma)[g_*, Z_*] = \mathcal{O}[g_*](X = \sigma), \quad (2.2.6)$$

so that the expectation value of $\mathcal{O}_{\text{loc}}(\sigma)$ coincides, up to corrections suppressed by the fluctuations and backreaction of the reference fields Z^A , with the naive local scalar $\mathcal{O}(X)$ evaluated at $X = \sigma$.

In this restricted semiclassical sense, it is therefore harmless to refer to $\mathcal{O}(X)$ as an “approximately local observable”. However, one should always keep in mind that the truly GT-invariant quantity is $\mathcal{O}_{\text{loc}}(\sigma)$, not $\mathcal{O}(X)$ itself.

Compact manifold with boundary

Let us consider a gravitational theory on a manifold with a boundary, where Dirichlet boundary conditions are imposed. We introduce a smooth non-dynamical scalar function $\Omega(X)$ whose zero locus $\Omega(X) = 0$ describes the boundary. For later convenience, we also use the bulk metric g_{IJ} in a neighborhood of the boundary¹⁶ and the normal one-form $(d\Omega)|_{\partial M}$ to define the unit normal vector

$$n^I[g; \Omega] = \frac{g^{IJ}(d\Omega)_J}{\sqrt{g^{IJ}(d\Omega)_J(d\Omega)_I}}. \quad (2.2.7)$$

We consider observables $\mathcal{O}(x)$ constructed from a finite number of normal derivatives of fields evaluated at the boundary. By analogy with the extrapolate operators in AdS/CFT [14], we refer

¹⁶Since this is not the induced metric, we will not refer to it as “the metric on the boundary.” In fact, in order to define this quantity we need a small neighborhood that contains the boundary.

to these as “extrapolate observables”.¹⁷ For example, using a bulk scalar field $\Phi(X)$, the observable

$$\mathcal{O}(x) = n^I \partial_I \Phi(X(x)) \Big|_{\partial M} \quad (2.2.8)$$

is an example of an extrapolate observable. One can in fact check that this operator is invariant under SGT.^{18,19} Moreover, the BY tensor (A.3.6) is, by definition, invariant under SGT, and hence is included among the extrapolate observables in this sense.²⁰

By contrast, it is known that quantities involving higher normal derivatives are in general not invariant under SGT. For instance, if we consider

$$\mathcal{O}(x) = n^J \partial_J (n^I \partial_I \Phi(X(x))) \Big|_{\partial M}, \quad (2.2.10)$$

which contains higher derivatives along the normal direction, then this operator transforms non-trivially even under SGT and thus ceases to be an observable.

We can construct observables by dressing bulk operators inserted in the interior as

$$\mathcal{O}_{\text{bulk}}^{\text{dr}}(x) := \mathcal{O}(X^A = F^A(x)). \quad (2.2.11)$$

Here F^A is a dynamical field that depends on the gravitational field g and other dynamical data, and is chosen such that, under an infinitesimal GT δ_ζ , it satisfies

$$\delta_\zeta F^A(x) = -\zeta^A (X^A = F^A(x)) + \zeta^i \partial_i F^A(x), \quad (2.2.12)$$

where ζ^i denotes the restriction of ζ^A to the boundary manifold ∂M . In this way, the point $X^A = F^A(x)$ moves precisely so that $\mathcal{O}_{\text{bulk}}^{\text{dr}}(x)$ transforms as a boundary scalar:

$$\delta_\zeta \mathcal{O}_{\text{bulk}}^{\text{dr}}(x) = \zeta^i \partial_i \mathcal{O}_{\text{bulk}}^{\text{dr}}(x). \quad (2.2.13)$$

¹⁷Note that quantities built solely from boundary fields and derivatives tangent to the boundary are trivial as operators once Dirichlet boundary conditions are imposed.

¹⁸To check this, it is convenient to use the relation $\delta_\xi = \mathcal{L}_\xi - \bar{\delta}_\xi$, where $\mathcal{L}_\xi$ acts on both dynamical and non-dynamical fields, while $\bar{\delta}_\xi$ acts only on the non-dynamical ones. The action of $\mathcal{L}_\xi$ is trivial here, since the quantity in question transforms as a scalar on the boundary. Moreover, $\bar{\delta}_\xi$ acts only on Ω , so the explicit computation is rather straightforward.

¹⁹Since this is nothing but the canonical momentum, it can be obtained as the variation of the second-order derivatives scalar bulk action $S_{\text{bulk}}[\Phi]$ with respect to the boundary value $\phi := \Phi|_{\partial M}$,

$$n^I \partial_I \Phi(X(x)) \Big|_{\partial M} = \frac{\delta S_{\text{bulk}}[\Phi]}{\delta \phi(x)}. \quad (2.2.9)$$

From the fact that this variation is manifestly SGT invariant, one can also confirm the SGT invariance of this quantity.

²⁰As is well known, the BY tensor is constructed from the extrinsic curvature in, for example, Einstein–Hilbert gravity (see e.g. [39]). Although the extrinsic curvature itself is built from derivatives of the normal vector and the boundary data, one can explicitly verify by direct computation that it is invariant under SGT. Conversely, it is precisely this property that allows the extrinsic curvature to admit a clear geometric interpretation.

This is manifestly invariant under SGT and is therefore an observable.

As a concrete example, one may choose $F^A(x)$ to be the point reached by a geodesic fired from the boundary point x and evolved for a proper time τ :

$$F^A(x) := \mathcal{F}^A(x; \tau), \quad (2.2.14)$$

where $\mathcal{F}^A(x; \tau)$ satisfies the geodesic equation

$$\frac{d^2 \mathcal{F}^A}{d\tau^2} + \Gamma^A_{BC} \frac{d\mathcal{F}^B}{d\tau} \frac{d\mathcal{F}^C}{d\tau} = 0, \quad (2.2.15)$$

$$\mathcal{F}^A(0) = X^A(x)|_{\partial M}, \quad (2.2.16)$$

$$\left. \frac{d\mathcal{F}^A}{d\tau} \right|_{\tau=0} = n^A. \quad (2.2.17)$$

Indeed, one can verify that this $\mathcal{F}^A$ satisfies the condition (2.2.12). In this construction, the dressed operator depends on the metric along the geodesic and is therefore sensitive to quantum fluctuations of the gravitational field g . This type of construction is often referred to as a *gravitational line dressing* (of course, this is not the unique way to dressed operators [43]).

2.2.2 Holographic implication

In this subsection, we investigate to what extent “holographic” features can emerge purely from the kinematical constraints of a gravitational theory, without explicitly assuming holography. We discuss the Dirichlet boundary case as a more controllable setup, following Refs. [36, 37] and the references therein, which mainly focus on the AdS case.

Setups and quantization

We consider a $(D + 2)$ -dimensional Euclidean gravitational theory whose dynamical degrees of freedom are the metric field g_{IJ} and a scalar field Φ . We impose the following boundary conditions. Moreover, in this chapter we assume canonical quantization via the Euclidean path integral (see, for example, Ref. [45]).

First, the boundary geometry is taken to be the direct product of a unit-radius sphere S^D and a Euclidean time direction (which we will simply refer to as “time” in the following):

$$\mathcal{B} \simeq S^D \times \mathbb{R}_\tau. \quad (2.2.18)$$

On $\mathcal{B}$ we impose Dirichlet boundary conditions for the metric and fix it to

$$g_{ij} dx^i dx^j|_{\mathcal{B}} = d\tau^2 + \gamma_{\alpha\beta}(z) dz^\alpha dz^\beta, \quad (2.2.19)$$

where $\gamma_{\alpha\beta}$ is the round metric on the unit-radius sphere S^D . Similarly, we fix the scalar field by imposing the boundary condition $\Phi|_{\mathcal{B}} = 0$.

Writing the BY tensor as T_{ij} , which is defined as (A.3.6), we define the *Hamiltonian* as²¹ [39]

$$H(\tau) = \int_{\mathcal{C}_\tau} d^D z \sqrt{\gamma} T_{ij}(z, \tau) \zeta^i \zeta^j, \quad (2.2.20)$$

where ζ^i is a boundary time-like Killing vector :

$$\zeta^i = (\partial_\tau)^i, \quad (2.2.21)$$

and we denote the constant-time slices of the boundary by

$$\mathcal{C}_\tau \equiv \mathcal{B} \cap \{\tau = \text{const.}\}. \quad (2.2.22)$$

In what follows, we focus on the algebra generated by the Hamiltonian H and the extrapolate operators $\mathcal{O}(\tau, z)$. For simplicity, we will hereafter assume that $\mathcal{O}$ is a scalar. Of course, in general it is a tensor, but this will not affect the essence of the discussion. The commutator between the Hamiltonian H and an extrapolate observable $\mathcal{O}(\tau, z)$ is defined in the Euclidean path-integral formalism as (see, e.g. [45])

$$\langle 0 | \cdots [H, \mathcal{O}(\tau, z)] \cdots | 0 \rangle := \langle \cdots (H(\tau_1) \mathcal{O}(\tau, z) - H(\tau_2) \mathcal{O}(\tau, z)) \cdots \rangle. \quad (2.2.23)$$

Here we assume that all other operators have support away from the time interval $\tau \in [\tau_1, \tau_2]$, and the left-hand side is implicitly time-ordered. By definition, the commutator is thus represented as the difference between insertions of the Hamiltonian on the two time slices $\mathcal{C}_{\tau_1}$ and $\mathcal{C}_{\tau_2}$, which lie on either side of the insertion point of $\mathcal{O}(\tau, z)$. Moreover, these two time slices can be deformed so as to wrap around the operator $\mathcal{O}(\tau, z)$, and by using the relation (A.3.12), the right-hand side can be evaluated explicitly to give²²

$$[H, \mathcal{O}(x)] = \partial_\tau \mathcal{O}(x). \quad (2.2.24)$$

This is nothing but the Heisenberg equation of motion, thus, we can solve it and obtain :

$$\mathcal{O}(\tau, z) = e^{H\tau} \mathcal{O}(0, z) e^{-H\tau}. \quad (2.2.25)$$

²¹This can be understood as the fact that only the surface term appears in the Hamiltonian because we restrict attention to physical states obeying the bulk constraints and to Dirac observables, so that the bulk constraint term does not contribute.

²²In the appendix A.3, the signature used in the Lorentzian calculation is changed to the Euclidean one.

Assumptions and Statement

We now make the following two assumptions:

1. The spectrum of the boundary Hamiltonian H is bounded from below. In particular, by adding a suitable constant we normalize the vacuum energy to zero.
2. The Hilbert space is spanned by states obtained by acting *extrapolate observables* on the vacuum state $|0\rangle$.

In other words, suppressing the angular coordinate z , any ket can formally be written as a linear combination of²³

$$\mathcal{O}(-\tau_1) \cdots \mathcal{O}(-\tau_n)|0\rangle \quad (\forall \tau_i > 0). \quad (2.2.26)$$

Under these assumptions, we will establish the following two statements²⁴:

1. The Hilbert space can be constructed entirely from states obtained by acting extrapolate observables with support in the time band $\tau \in [0, \varepsilon]$ on the vacuum $|0\rangle$.

That is, even if we restrict the insertion times to the finite time band

$$\mathcal{O}(-\tau_1) \cdots \mathcal{O}(-\tau_n)|0\rangle, \quad (\tau_i \in [0, \varepsilon]), \quad (2.2.27)$$

we can still generate any state ket.

Equivalently, if we denote by $\mathcal{A}$ the set of all operators generated by polynomials in extrapolate observables supported in the time band $[-\varepsilon, 0]$, then any state can be written as

$$|X\rangle := X|0\rangle, \quad X \in \mathcal{A}. \quad (2.2.28)$$

2. Any observable on Hilbert space can be constructed solely from $\mathcal{A}$.

Proof of the first statement To show first statement, we first focus on “one-point states,” i.e. states obtained by acting on the vacuum with a single extrapolate observable.²⁵ By way of contradiction, suppose there exists a one-point state $|\Psi\rangle := \mathcal{O}'(-\tau')|0\rangle$ ($\tau' > 0$) such that

$$\langle \Psi | \mathcal{O}(-\tau) | 0 \rangle = 0 \quad (\tau \in [0, \varepsilon]). \quad (2.2.29)$$

²³At least in situations such as scalar field theories with non-dynamical gravity, this corresponds to regarding the scattering in-states as forming a complete basis of the Hilbert space, and such an assumption should be reasonable. Nonetheless, there remains room for debate as to how far this analogy can be justified. In fact, Ref. [46] criticizes this point, arguing that this assumption itself is already as strong as assuming holography.

²⁴The difference between these two statements may appear subtle. The qubit model discussed on page 43 of Ref. [37] may be helpful for understanding the distinction between them.

²⁵The structure of the argument is essentially the same as in the proof of the Reeh–Schlieder theorem.

where following standard Euclidean QFT quantization (see e.g. [45]), we define

$$\langle \Psi | = \langle 0 | \mathcal{O}'(\tau'). \quad (2.2.30)$$

In addition, we define $f(-\tau)$ as :

$$f(-\tau) := \langle \Psi | \mathcal{O}(-\tau) | 0 \rangle. \quad (2.2.31)$$

Using the Euclidean time evolution (see eq. (2.2.25))

$$\mathcal{O}(-\tau) = e^{-H\tau} \mathcal{O}(0) e^{H\tau}, \quad (2.2.32)$$

and inserting a complete set of energy eigenstates $\{|E\rangle\}$, we obtain

$$f(-\tau) = \int_0^\infty dE \langle \Psi | E \rangle \langle E | \mathcal{O}(0) | 0 \rangle e^{-E\tau}, \quad (2.2.33)$$

where we use the first assumption (2.2.29), the spectrum satisfies $E \geq 0$. Thus, $f(-\tau)$ is an analytic function for $\tau > 0$.

On the other hand, by assumption $f(-\tau)$ vanishes on the interval $\tau \in [0, \varepsilon]$. By uniqueness of analytic continuation, an analytic function that vanishes on a finite interval must vanish identically, so for any $\tau > 0$ we have

$$f(-\tau) = 0. \quad (2.2.34)$$

Equivalently,

$$\langle \Psi | \mathcal{O}(-\tau) | 0 \rangle = 0 \quad (\forall \tau > 0), \quad (2.2.35)$$

which means that $|\Psi\rangle$ is orthogonal to all one-point states $\mathcal{O}(-\tau) | 0 \rangle$ with $\tau > 0$.

However, by the second assumption, any one-point state can be generated from the vacuum by varying the insertion time τ . Hence any state orthogonal to all such states must be the zero vector, contradicting the assumption $|\Psi\rangle \neq 0$.

Thus, there is no nontrivial state orthogonal to the states generated by extrapolate observables supported in the time band $[0, \varepsilon]$. The same analyticity argument applies to higher-point correlation functions using edge of the wedge theorem [47], so the conclusion extends straightforwardly to states obtained by acting on the vacuum with polynomials in $\mathcal{O}(\tau)$. This proves the first statement.

Proof of the second statement In order to show second statement, for $X, Y \in \mathcal{A}$, consider the following operator :

$$|X\rangle \langle Y| = X | 0 \rangle \langle 0 | Y^\dagger. \quad (2.2.36)$$

where we use definition (2.2.28).

Here $|0\rangle\langle 0|$ is the projector onto the vacuum and picks out the zero-energy eigenspace of H . Formally, we may regard this projector as the operator $\delta[H]$, i.e. as a functional of H that can be approximated by polynomials in H . Since H itself is constructed from the BY tensor, which is an extrapolate observable, via (2.2.20), the operator $|0\rangle\langle 0|$ can also be regarded as built from extrapolate observables $\mathcal{A}$.

Physically, operators of the form (2.2.36) describe transitions from the state $|X\rangle$ to the state $|Y\rangle$, and by taking linear combinations of such transition operators we can represent arbitrary observables acting on our Hilbert space. Thus, this proves the second statement.

Summary

We prove the algebra $\mathcal{A}$ is already sufficient to give a complete description of the physics in this Hilbert space. This demonstrates that, purely from the kinematical properties of the gravitational theory, information contained in a limited time region on the boundary is enough to control the bulk states and observables, which is a clear indication of holography-like behavior at information level.

2.3 $U(1)$ gauge theory in Flat spacetime

In this section we derive the Ward identity of a $U(1)$ gauge theory in flat spacetime [48].

At first sight, however, there is a subtlety. The correlation functions (or scattering amplitudes) of the theory are defined by the path integral

$$\langle \mathcal{O} \rangle = \int \mathcal{D}A \, e^{iS_{\text{bulk}}[A]} \mathcal{O}, \quad (2.3.1)$$

where S_{bulk} is understood to include the vacuum wave functional prepared by a Euclidean path integral. The state created by the Euclidean path integral is invariant under the gauge symmetry of the action, so the bulk action is strictly $U(1)$ invariant. In particular, gauge invariance, including large gauge transformations, appears to be completely preserved, and one would then conclude that the Ward identity is trivial, i.e. that there is no nontrivial current. The purpose of this section is to resolve this apparent triviality and to derive a genuinely nontrivial Ward identity.

2.3.1 Cut-off Minkowski spacetime

We consider $(D+2)$ -dimensional Minkowski spacetime M in spherical coordinates (T, R, Z^α) , where T is the Minkowski time, $R \geq 0$ is the radial coordinate and Z^α ($\alpha = 1, \dots, D$) are coordinates

on the unit D -sphere, which metric denoted as $\gamma_{\alpha\beta}$ and tensor index of $\alpha, \beta, \dots$ raise/down by $\gamma_{\alpha\beta}$. We also introduce retarded and advanced time coordinates

$$U := T - R, \quad V := T + R. \quad (2.3.2)$$

Our goal is to describe massless scattering in Minkowski spacetime from the point of view of an observer living on the boundary. To this end we introduce a cut-off Minkowski region M^c obtained by restricting the entire spacetime to a finite domain bounded by²⁶

$$\partial M^c = \mathcal{I}^+ \cup \mathcal{I}^- \cup i^0 \cup i^+ \cup i^-. \quad (2.3.3)$$

The boundary ∂M^c consists of the following components:

- *Spacelike-infinity component i^0* : The spacelike-infinity boundary $i^\pm$ is defined by

$$i^0 = \{(T, R, Z^\alpha) \in M \mid R = R_c, -T_c < T < T_c\}, \quad (2.3.4)$$

where R_c is a radial cut-off, chosen to be much larger than any microscopic length scale in the problem. and T_c is time cut-off which is enough smaller than R_c

- *timelike-infinity component $i^\pm$* : The time-like infinity boundary $i^\pm$ is defined by

$$i^\pm = \{(T, R, Z^\alpha) \in M \mid T = \pm T_c, 0 < R < r_c\}, \quad (2.3.5)$$

where r_c is a radial cut-off, chosen to be enough large such as $O(R_c)$.

- *Null-infinity components $\mathcal{I}^\pm$* . The null-infinity boundary $\mathcal{I}^\pm$ ²⁷ is defined by

$$\mathcal{I}^\pm = \{(T, R, Z^\alpha) \in M \mid T = \pm T_c, r_c < R < R_c\}. \quad (2.3.6)$$

On $\mathcal{I}^+$ we use induced coordinates

$$u := U, \quad z^\alpha := Z^\alpha. \quad (2.3.7)$$

On $\mathcal{I}^-$ it is convenient to use antipodally inverted angular coordinates; we therefore choose

$$v := V, \quad z^\alpha := \text{Anti}(Z^\alpha), \quad (2.3.8)$$

where Anti denotes the antipodal map on the sphere.

²⁶A fully rigorous treatment would make use of the conformal compactification of Minkowski spacetime. For non-gravitational theories, however, we expect that the present cut-off description already reproduces the correct physics. The conformal compactification is technically subtle because the conformal factor is not smooth near spatial infinity [49]. In the gravitational case (section 2.4) we will indeed employ conformal compactification to handle the asymptotic boundary, although we will not treat the neighborhood of i^0 in full rigor and instead rely on analogies with the electromagnetic finite-boundary set-up.

²⁷Of course, this manifold is not a null manifold, that is, it is not a manifold equipped with a degenerate metric.

In addition, we define the codimension-two “corner” manifolds $\mathcal{I}_-^+$ and $\mathcal{I}_+^-$ as follows:

- $\mathcal{I}_-^+$ is the the past endpoint of the null-infinity boundary $\mathcal{I}^+$. Note that $\mathcal{I}_-^+$ coincides with the future boundary of i^0 , again with two disconnected components.
- $\mathcal{I}_+^-$ is the future endpoint of the null-infinity boundary $\mathcal{I}^-$. We have $\mathcal{I}_+^-$ coincides with the past boundary of i^0 . Although $\mathcal{I}_+^-$ and the past boundary of i^0 are geometrically the same set, the angular coordinates x^α on the two sides are related by the antipodal map due to our coordinate choice (2.3.8) .

In the following, we restrict our attention to the case $D = 2$, which corresponds to four-dimensional Minkowski spacetime with a two-dimensional celestial sphere. Although higher-dimensional generalizations involve certain technical subtleties, these have already been addressed in the existing literature using the canonical formalism [50, 51], and we therefore do not expect any fundamental obstruction to our path-integral formulation either.

We further focus on massless particles,²⁸ and assume that it is reasonable to neglect any fields in the neighborhood of timelike infinity. Under this assumption, we ignore contributions from timelike infinity.

2.3.2 Boundary conditions

We now impose boundary conditions on the gauge field. This is essential for deriving a non-trivial Ward identity, because, as mentioned above, if the action were just $S = S_{\text{bulk}}$, the resulting Ward identities would be purely trivial. As preparation, let us summarize the asymptotic behavior of the field strength, which will be used repeatedly below.

• Behavior at spatial infinity

We consider a localized charge distribution with finite total charge and no radiation at a fixed time t , so that the field is Coulombic. In this case, the components of the field strength behave as

$$F_{tR} = O(R_c^{-2}), \quad F_{t\alpha} = O(R_c^{-2}), \quad F_{R\alpha} = O(R_c^{-2}), \quad F_{\alpha\beta} = O(R_c^{-2}). \quad (2.3.9)$$

• Behavior at null infinity

For radiative configurations corresponding to a standard scattering setup, the field strength at future null-infinity $\mathcal{I}^+$ behaves as

$$F_{uR} = O(R_c^{-2}), \quad F_{u\alpha} = O(1), \quad F_{R\alpha} = O(R_c^{-2}), \quad F_{\alpha\beta} = O(1). \quad (2.3.10)$$

²⁸Formulations including massive particles have already been developed in the literature [52].

The behavior at past null infinity $\mathcal{I}^-$ is analogous. Here, the $O(1)$ parts of $F_{u\alpha}$ and $F_{\alpha\beta}$ encode the radiative data on $\mathcal{I}$.

In addition, for the field strength near spatial infinity i^0 , we can impose an antipodal matching condition, schematically written as [53] (see appendix A.4)

$$F_{IJ}^{(2)}|_{i^0}(-t, \text{Anti}(z)) = -F_{IJ}^{(2)}|_{i^0}(t, z), \quad (2.3.11)$$

where $F_{IJ}^{(2)}|_{i^0}$ denotes the $O(R_c^{-2})$ term in the large- R_c expansion. In what follows, we use the notation $C^{(n)}|_N$ to denote the $O(R_c^{-n})$ component of a quantity C restricted to a given boundary N such as i^0 or $\mathcal{I}$.

An important observation is that at the corners $\mathcal{I}_-^+$ and $\mathcal{I}_+^-$ one can impose the same kind of “no-radiation” condition as at spatial infinity i^0 . Concretely, we require

$$F_{\alpha\beta}^{(0)}|_{\mathcal{I}_\pm^\pm} = 0. \quad (2.3.12)$$

This implies that the codimension-two gauge fields

$$a^+ \equiv A_\alpha^{(0)}|_{\mathcal{I}_-^+}, \quad a^- \equiv A_\alpha^{(0)}|_{\mathcal{I}_+^-} \quad (2.3.13)$$

are pure gauge. To encode this boundary condition, we introduce background scalar fields $\bar{\phi}^\pm$ living on the corners, representing the $U(1)$ frame, and postulate the boundary actions

$$e^{iS_{\text{bdy}}^\pm[a^\pm]} = \delta[a^\pm + d\bar{\phi}^\pm]. \quad (2.3.14)$$

This merely specifies how to choose a frame on the corner, and it should be emphasized that this does not, of course, guarantee that the vacuum is preserved. It is interpreted as saying that the vacuum of the $U(1)$ gauge theory is degenerate, and that the set of vacua is labeled by the parameter $\bar{\phi}$ (see e.g. [17]).²⁹

Let us comment on the boundary action at spatial infinity. As we will see later, it does not play an essential role in the Ward identity. To make this explicit, we include an arbitrary boundary factor

$$e^{iS_{\text{bdy}}^0[A]|_{i^0}} \quad (2.3.15)$$

depending on the gauge field restricted to i^0 .

Finally, The full action then takes the form

$$S[A] = S_{\text{bulk}}[A] + S_{\text{bdy}}^+[a^+] + S_{\text{bdy}}^-[a^-] + S_{\text{bdy}}^0[A]|_{i^0}. \quad (2.3.16)$$

With this, the boundary conditions and boundary actions needed to derive the Ward identities associated with the soft theorem are fully specified.

²⁹As we have seen in section A.2, by promoting $\bar{\phi}$ to a dynamical variable one is naturally led to introduce edge modes. The fact that these edge modes are closely tied to the Ward identities in flat spacetime has already been suggested in the literature (see, for example, Ref. [40]), and our derivation below provides further support for the arguments presented in Ref. [40].

2.3.3 Ward identity

In this subsection we derive the Ward identity associated with the LGT introduced above. We consider the expectation value of a generic functional $\mathcal{O}$ in the gauge theory,

$$\langle \mathcal{O} \rangle = \int \mathcal{D}A e^{iS[A]} \mathcal{O}. \quad (2.3.17)$$

To obtain a useful identity, we restrict attention to a class of LGT parametrized by Λ satisfying the following condition, which we refer to as *antipodal matching*³⁰:

$$\Lambda^{(0)}|_{i^0}(-t, \text{Anti}(z)) = \Lambda^{(0)}|_{i^0}(t, z). \quad (2.3.18)$$

Moreover, by continuity³¹ we impose the corner condition

$$\Lambda^{(0)}|_{\mathcal{I}^+}(z) = \Lambda^{(0)}|_{\mathcal{I}^-}(z). \quad (2.3.19)$$

and we define $\bar{\Lambda}(z)$ as $\bar{\Lambda}(z) = \Lambda^{(0)}|_{\mathcal{I}^+}(z) = \Lambda^{(0)}|_{\mathcal{I}^-}(z)$.

With this set-up, we can now derive the Ward identity. The starting point is the invariance of the path integral under this class of gauge transformations:

$$\delta_\Lambda \langle \mathcal{O} \rangle = \int \mathcal{D}A \delta_\Lambda (e^{iS[A]} \mathcal{O}) = 0. \quad (2.3.20)$$

where we assume the gauge invariance of the measure. This can be rewritten as

$$0 = \langle \delta_\Lambda \mathcal{O} \rangle + i \left\langle \delta_\Lambda \left(S_{\text{bdy}}^0[A|_{i^0}] + \sum_{\sigma=\pm} S_{\text{bdy}}^\sigma[a^\sigma] \right) \mathcal{O} \right\rangle, \quad (2.3.21)$$

and we now evaluate each term on the right-hand side. The computation closely parallels the finite-boundary case discussed in the subsection 2.1.2.

It can be decomposed into the following two contributions.

- *Contribution from spatial infinity.*

We begin with

$$\langle \delta_\Lambda S_{\text{bdy}}^0[A|_{i^0}] \mathcal{O} \rangle = \left\langle \left(\int_{i^0} [d^3x] \partial_i \Lambda^{(0)}(x) \frac{\delta S_{\text{bdy}}^0[A|_{i^0}]}{\delta A_i^{(0)}(x)} \right) \mathcal{O} \right\rangle. \quad (2.3.22)$$

Strictly speaking, $F^{(2)Rj}(x)$ should be written as $F^{(2)Rj}|_{i^0}(x)$ and $\Lambda^{(0)}(x)$ as $\Lambda^{(0)}|_{i^0}(x)$, but whenever this causes no confusion we omit the explicit pullback notation $\cdot|_N$.

³⁰In fact, this condition can be relaxed. However, relaxing it merely introduces a somewhat peculiar flux term at spatial infinity, which is not particularly interesting for our purposes.

³¹Recall that the angular coordinates z^α on $\mathcal{I}^\pm$ are chosen so that they take the same values at antipodal points as Eq.(2.3.8).

Under our antipodal–matching assumptions we will see that the contribution (2.3.22) vanishes.³²

Now, we assume that $\mathcal{O}$ does not depend on $A|_{i^0}$, since we are ultimately interested in scattering amplitudes. Performing an integration by parts in field space and using the relation between the variation of the bulk action and the field–strength flux, we obtain from (2.3.22)³³

$$\begin{aligned}\langle \delta_\Lambda S_{\text{bdy}}^0[A|_{i^0}] \mathcal{O} \rangle &= - \left\langle \left(\int_{i^0} dt d^2 z \, R^2 \sqrt{\gamma} \partial_i \Lambda(x) \frac{\delta S_{\text{bulk}}[A]}{\delta A_i(x)} \right) \mathcal{O} \right\rangle \\ &= \left\langle \left(\int_{i^0} dt d^2 z \, \sqrt{\gamma} F^{(2)Rj}(x) \partial_j \Lambda(x) \right) \mathcal{O} \right\rangle.\end{aligned}\quad (2.3.23)$$

This contribution vanishes. Indeed, consider the change of integration variables $(t, z) \mapsto (-t, \text{Anti}(z))$. Under this transformation the integrand on the right–hand side is odd: each factor $dt d^2 z \sqrt{\gamma} n^i$, F^{ij} , and $\partial_i \Lambda$ picks up a minus sign due to the conditions (2.3.18) and (2.3.11). Hence the integral over i^0 is zero, and therefore the original term (2.3.22) vanishes (for essentially the same reason as argued in Ref. [53]).

- *Contribution from $\mathcal{I}^+$.*

Next, consider

$$\langle \delta_\Lambda S_{\text{bdy}}^+[a^+] \mathcal{O} \rangle = \left\langle \left(\int_{S^2} d^2 z \sqrt{\gamma} \partial_\alpha \bar{\Lambda}(z) \frac{\delta S_{\text{bdy}}^+[a^+]}{\delta a_\alpha^+(z)} \right) \mathcal{O} \right\rangle, \quad (2.3.24)$$

where $\bar{\Lambda}(z)$ denotes the $O(R_c^0)$ part of the gauge parameter pulled back to $\mathcal{I}$. After integrating by parts with respect to the functional derivative³⁴ we obtain

$$\langle \delta_\Lambda S_{\text{bdy}}^+[a^+] \mathcal{O} \rangle = - \left\langle \left(\int_{S^2} d^2 z \sqrt{\gamma} \partial_\alpha \bar{\Lambda}(z) \frac{\delta S_{\text{bulk}}[A]}{\delta a_\alpha^+(z)} \right) \mathcal{O} \right\rangle = -i \left\langle \left(\int_{S^2} d^2 z \sqrt{\gamma} \nabla_\alpha J^{+\alpha}(z) \bar{\Lambda}(z) \right) \mathcal{O} \right\rangle, \quad (2.3.25)$$

where, as in the subsection 2.1.2, we assume that $\mathcal{O}$ does not depend on a^+ . We have also defined the future soft current

$$J^{+\alpha}(z) := -i \frac{\delta S_{\text{bulk}}[A]}{\delta a_\alpha^+(z)} = -i \int_{\mathcal{I}^+} du F^{(0)R\alpha}|_{\mathcal{I}^+}(u, z). \quad (2.3.26)$$

³²If S_{bdy}^0 were gauge invariant, this term would vanish identically, so we explicitly assume that the boundary action is not gauge invariant.

³³More precisely, one should keep track of the restrictions $F^{(2)Rj} \rightarrow F^{(2)Rj}|_{i^0}$ and $\Lambda \rightarrow \Lambda|_{i^0}$, but this does not affect the argument.

³⁴Angular indices $\alpha, \beta, \dots$ are raised and lowered by the unit–sphere metric $\gamma_{\alpha\beta}$. In this notation, a factor of R_c^2 appears when converting between covariant and contravariant components.

An entirely analogous computation at $\mathcal{I}^-$ yields the past soft current and its contribution, so we omit the details. Combining the two, we obtain

$$\sum_{\sigma=\pm} \langle \delta_\Lambda S_{\text{bdy}}^\sigma[a^\sigma] \mathcal{O} \rangle = -i \left\langle \left(\int d^2z \sqrt{\gamma} \nabla_\alpha J^\alpha(z) \bar{\Lambda}(z) \right) \mathcal{O} \right\rangle, \quad (2.3.27)$$

where we have used the antipodal matching of the gauge parameter (2.3.19), and defined the total soft current

$$\begin{aligned} J^\alpha(z) &:= -i \frac{\delta S_{\text{bulk}}[A]}{\delta a_\alpha^+(z)} - i \frac{\delta S_{\text{bulk}}[A]}{\delta a_\alpha^-(z)} \\ &= -i \int_{\mathcal{I}^+} du F^{(0)R\alpha}|_{\mathcal{I}^+}(u, z) - i \int_{\mathcal{I}^-} dv F^{(0)R\alpha}|_{\mathcal{I}^-}(v, z). \end{aligned} \quad (2.3.28)$$

Substituting these results into (2.3.21), we arrive at

$$\left\langle \left(\int d^2z \sqrt{\gamma} \nabla_\alpha J^\alpha(z) \bar{\Lambda}(z) \right) \mathcal{O} \right\rangle = \langle \delta_\Lambda \mathcal{O} \rangle. \quad (2.3.29)$$

This is the most general Ward identity compatible with the boundary actions and gauge parameters specified above.

Now assume that $\mathcal{O}$ can be written as a product of “local” operators³⁵ on the celestial sphere,

$$\mathcal{O} = \prod_n \mathcal{O}_n(z_n). \quad (2.3.30)$$

In addition, we assume that each operator transforms linearly under the large gauge transformation,

$$\delta_\Lambda \mathcal{O}_n(z_n) \propto \bar{\Lambda}(z_n) \delta \mathcal{O}_n(z_n). \quad (2.3.31)$$

Then the Ward identity takes the local form at celestial sphere :

$$\left\langle \nabla_\alpha J^\alpha(z) \prod_n \mathcal{O}_n(z_n) \right\rangle = \frac{1}{\sqrt{\gamma}} \delta^{(D)}(z - z_m) \langle \delta \mathcal{O}_m(z_m) \prod_{n \neq m} \mathcal{O}_n(z_n) \rangle. \quad (2.3.32)$$

In particular, the Ward identity derived above implies the leading soft photon theorem [48]. A detailed derivation can be found in [48]; in this paper we only provide an intuitive sketch in the Appendix A.5.

Observables and additional boundary condition

In this section, we discuss other boundary conditions and the corresponding observables.³⁶

³⁵Each $\mathcal{O}_n(z_n)$ may be smeared in the u direction.

³⁶In what follows we focus on $\mathcal{I}^+$, but the same discussion applies equally to $\mathcal{I}^-$.

The setup considered here is slightly different from the Dirichlet boundary condition studied in the previous section. In the present case, LGT is not completely broken but only partially broken. More precisely, only the LGT at $\mathcal{I}_-^+$ is broken. Under this condition, only Wilson lines that end on $\mathcal{I}_-^+$ can be regarded as non-trivial observables carrying LGT charge.

On the other hand, if one wishes to insert a charged matter field Φ at a generic point on $\mathcal{I}^+$, such an insertion must be dressed by a Wilson line piercing $\mathcal{I}_-^+$. As a result, under our boundary conditions, the creation and annihilation operators that appear in the soft theorem must be dressed by Wilson lines so that they become overall gauge neutral.

If one would like to avoid such dressings, one needs to impose an additional boundary condition. Concretely, we add to the boundary action the term³⁷

$$e^{iS'_{\text{bdy}}} = \delta \left(A_u^{(0)} \right), \quad (2.3.33)$$

which imposes an extra constraint on A_u . Then the u -dependence of the LGT that preserves this boundary condition is removed, and in fact the LGT is completely broken. This additional boundary condition can always be imposed without spoiling the fall-off conditions of the field strength.

Consequently, the LGT that was originally a redundancy is promoted to a physical symmetry. As a result, under the new boundary condition, operators are allowed to carry arbitrary LGT charge, and all fields $\Phi(u, z)$ become observables without any Wilson-line dressing. Indeed, standard references [17], impose precisely this type of boundary condition—often formulated as a “fall-off condition” on the gauge field—and work within that framework.

Even if one considers more general LGT with u -dependent gauge parameter Λ , thereby further breaking this newly imposed boundary condition, most of the associated charges are known to diverge and hence are physically meaningless. The only nontrivial additional conserved current that remains is the one associated with an LGT whose parameter depends linearly on u , and this current is known to correspond to the subleading soft theorem [54, 55].

2.4 Gravity in AFS and Celestial holography

In this section we discuss the asymptotic symmetries of gravity in 4-dimensional asymptotic flat spacetime (AFS).³⁸

³⁷It is of course also possible to improve this to a gauge-invariant object by introducing edge modes. In that case, one has to dress it using the corresponding edge mode. This type of boundary condition is examined in Sec. 5.2 of Ref. [40], where the relation between edge modes and soft physics is studied within a canonical formalism, in contrast to our approach.

³⁸It is known that extending the analysis of asymptotic symmetries to higher dimensions is technically more involved than in four dimensions (See Ref. [56] and the references therein.)

Although several technical subtleties arise, the derivation of the Ward identities closely parallels the electromagnetic case. We begin by reviewing the asymptotic symmetries and the Bondi data, which encode the degrees of freedom at null infinity. We then derive the associated Ward identities. These results also motivated the idea that four-dimensional asymptotically flat spacetimes may admit a dual description in terms of a two-dimensional CFT, namely, *celestial holography*.

2.4.1 Definition of AFS

We briefly summarize the notion of an AFS (For a more detailed pedagogical discussion, see the lecture notes Ref. [17, 57]).

An AFS is a spacetime for which, at large r , there exists a coordinate system (u, r, z^α) in which the metric can be written as³⁹

$$g_{ij}dx^i dx^j = -du^2 - 2du dr + 2r^2 dz d\bar{z} + r C_{zz} dz^2 + r C_{\bar{z}\bar{z}} d\bar{z}^2 + \dots \quad (2.4.1)$$

Moreover, we assume that this coordinate system satisfies the *Bondi gauge conditions*

$$\begin{aligned} g_{rr} &= 0, & g_{r\alpha} &= 0, \\ \det(g_{\alpha\beta}) &= \det(r^2 \gamma_{\alpha\beta}). \end{aligned} \quad (2.4.2)$$

The tensor $C_{\alpha\beta}$ is called the *Bondi shear*, and can be regarded as the analogue of a gauge field living on $\mathcal{I}$ in a $U(1)$ gauge theory:⁴⁰

We further define the *Bondi news* by

$$N_{\alpha\beta}(u, z, \bar{z}) := \partial_u C_{\alpha\beta}(u, z, \bar{z}). \quad (2.4.3)$$

This quantity is the analogue of the radiative field strength in a $U(1)$ gauge theory.⁴¹

In general, following standard conventions, the coordinate system (u, r, z^α) introduced above is used to describe the metric in a neighborhood of future null infinity $\mathcal{I}^+$. Near past null infinity $\mathcal{I}^-$, it is often convenient to use a following coordinate system (v, r, z^α) , in which the metric takes the form⁴²

$$g_{ij}dx^i dx^j = -dv^2 + 2dv dr + 2r^2 dz d\bar{z} + r C_{zz} dz^2 + r C_{\bar{z}\bar{z}} d\bar{z}^2 + \dots \quad (2.4.4)$$

³⁹The definition of asymptotic flatness based on Penrose's conformal compactification is equivalent to the present definition in terms of the existence of a suitable class of coordinates. Indeed, we will see this type metric can be conformal compactification and null boundary have desired structure, which demanded by AFS condition. and Inverse statement is also true see e.g. [58].

⁴⁰Indeed, $C_{\alpha\beta}$ can be interpreted as parametrizing an affine connection on $\mathcal{I}$ [57].

⁴¹In fact, the relevant metric components can be rewritten in a form that makes the analogy with a field strength (more precisely, with components of the Weyl tensor) more explicit [57].

⁴²By the change of variables $v \rightarrow -u$, this coordinate system is mapped to the one introduced above (Eq.(2.4.1)), so that past null infinity is asymptotically flat in the same sense

We now turn to the geometry of the boundary. Since the original spacetime manifold has no boundary, we consider a compactified spacetime obtained by adjoining a boundary at $1/r = 0$. A natural question is what the induced metric and the induced null generator on this null boundary are. However, the naive induced structures diverge and cannot be defined, so the situation is not analogous to the case where the boundary is located at a finite position.

To address this issue, we first introduce a *defining function* Ω of the boundary:

$$\Omega = r^{-1} e^{\omega(r, z, \bar{z})}, \quad (2.4.5)$$

where e^ω is a smooth function that remains finite in the limit $r \rightarrow \infty$.

By multiplying the physical metric g by the conformal factor Ω^2 , we remove the divergent factor and then take the limit $r \rightarrow \infty$. In this way we obtain a finite three-dimensional degenerate metric

$$2\gamma_{z\bar{z}} dz d\bar{z} := 2 e^{2\omega(z, \bar{z})} dz d\bar{z} := \Omega^2 g_{ij} dx^i dx^j \Big|_{\mathcal{I}}, \quad (2.4.6)$$

where $\omega(z, \bar{z})$ is the $r \rightarrow \infty$ limit of $\omega(r, z, \bar{z})$. We regard this as the metric induced on $\mathcal{I}$. This procedure is, as is well known, referred to as conformal compactification [59]. Of course, since the result depends on the choice of e^ω only through its boundary value, the induced metric is defined only up to Weyl rescalings on $\mathcal{I}$.

Similarly, we define the induced null generator of the boundary using the defining function:

$$n^i := \Omega^{-2} g^{ij} (d\Omega)_j \Big|_{\mathcal{I}}. \quad (2.4.7)$$

In Bondi coordinates, this can be written as

$$n \propto e^{-\omega(z, \bar{z})} \partial_u. \quad (2.4.8)$$

Thus, up to the Weyl ambiguity encoded in ω , the pair (γ_{ij}, n^i) defines an induced structure on the boundary $\mathcal{I}$. Taking the quotient by Weyl rescalings, we obtain the equivalence class

$$[\gamma_{ij}, n^i], \quad (2.4.9)$$

which we call the *universal structure* and regard as capturing the true intrinsic geometry of $\mathcal{I}$. In what follows, we choose a representative of this universal structure by fixing $(\gamma_{z\bar{z}}, n^i) = (1, \delta_u^i)$, which corresponds $(2 dz d\bar{z}, \partial_u)$. In the next subsection 2.4.2, this choice will be implemented as one of our boundary conditions (2.4.24).

BMS transformation

We now characterize the LGT that preserve this universal structure. Such transformations form what is known as the BMS group [20, 21], and can be written as the following class of coordinate transformations on $\mathcal{I}$:⁴³

$$u \rightarrow u + f(z, \bar{z}) - \frac{1}{2} u \partial_\alpha Y^\alpha + \mathcal{O}(r^{-1}), \quad (2.4.10)$$

$$z^\alpha \rightarrow z^\alpha + Y^\alpha(z, \bar{z}) + \mathcal{O}(r^{-1}), \quad (2.4.11)$$

$$r \rightarrow r - \frac{1}{2} r \partial_\alpha Y^\alpha + \mathcal{O}(1), \quad (2.4.12)$$

where $f(z, \bar{z})$ is called a *supertranslation*, and $Y^\alpha(z, \bar{z})$ is a conformal Killing vector field on the two-sphere. These functions parametrize the BMS symmetry. Subleading terms in r is not universal term, so we suppress them.

From a more physical point of view, it is useful to regard BMS transformations as the analogue of isometries of the induced metric when the boundary is placed at a finite position. In both cases, they are LGTs that preserve the boundary condition (i.e., the boundary action). As we emphasize throughout this paper, such “boundary-action-preserving” LGTs give rise to Ward identities in which no boundary current appears (see Eq. (A.3.10)). In the next subsection 2.4.2, we will choose boundary conditions that explicitly break supertranslations, and we will see concretely that this choice generates a supertranslation current.

It is natural to consider extending the global conformal Killing vectors (CKVs) on the sphere to *local* CKVs, namely to allow Y^z ($Y^{\bar{z}}$) to be arbitrary meromorphic (anti-meromorphic) functions. This enlarged class of angular transformations is referred to as *superrotations*. Since their proposal in [23, 24], they have played a central role as the underlying symmetry of the framework known as celestial holography, leading to many interesting developments [29, 22, 17, 30, 31, 32]. However, it is important to stress that this extension is only defined locally. Because meromorphic functions have singularities on a measure-zero subset, the corresponding transformations are not genuine, globally defined diffeomorphisms on the sphere. Consequently, they do not preserve the universal structure, as we see concretely in next paragraph (see Eq.(2.4.19)). That said, the fact that they fail to preserve the universal structure is not, by itself, a serious problem. As we have seen, conserved currents are naturally associated with transformations that do *not* preserve the boundary conditions. In fact, we will confirm in the next subsection 2.4.2 that the contribution arising from the violation of the universal structure is indeed part of the superrotation current.

⁴³Note that a shift in r is equivalent to a redefinition of the defining function Ω or ω . This is needed when one wishes to keep a given representative fixed. If it is sufficient to preserve only the equivalence class, no transformation of r is required. Since we fixed a representative, we write this explicitly here.

Transformation law and gravitational vacua

Now, we summarize how BMS transformations act on physical quantities.⁴⁴

The action of a supertranslation with parameter $f(z, \bar{z})$ on the Bondi shear C_{zz} is

$$\delta_f C_{zz} = f \partial_u C_{zz} - 2 \partial_z^2 f. \quad (2.4.13)$$

In particular, even if we start from a vacuum configuration with $C_{zz} = 0$, a non-trivial supertranslation generates a non-zero Bondi shear. This gives rise to an infinite family of inequivalent *supertranslation gravitational vacua* [17].

Next, we consider a meromorphic superrotation generated by the vector field $Y^z(z) \partial_z$.⁴⁵ As a preliminary step, we decompose C_{zz} and N_{zz} as [60, 32]

$$C_{zz}(u, z, \bar{z}) = \mathcal{C}_{zz}(u, z, \bar{z}) - u \rho_{zz}(z, \bar{z}), \quad (2.4.14)$$

so that

$$N_{zz} = \partial_u C_{zz} = \mathcal{N}_{zz}(u, z, \bar{z}) - \rho_{zz}(z, \bar{z}), \quad (2.4.15)$$

where we have defined $\mathcal{N}_{zz} := \partial_u \mathcal{C}_{zz}$ ⁴⁶ and we required that $\mathcal{N}_{zz}(u, z, \bar{z})$ vanishes at the boundary $\partial\mathcal{I}$, which makes the decomposition (2.4.14) unique.

The action of a superrotation on the Bondi shear is [60, 32]

$$\delta_Y C_{zz} = Y^z \partial_z C_{zz} + 2 (\partial_z Y^z) C_{zz} + \frac{u}{2} (\partial_z Y^z) N_{zz} - u \partial_z^3 Y^z. \quad (2.4.16)$$

In terms of the decomposition above, this implies

$$\delta_Y \mathcal{C}_{zz} = Y^z \partial_z \mathcal{C}_{zz} + 2 (\partial_z Y^z) \mathcal{C}_{zz} + \frac{u}{2} (\partial_z Y^z) \mathcal{N}_{zz}, \quad (2.4.17)$$

$$\delta_Y \rho_{zz} = \partial_z^3 Y^z. \quad (2.4.18)$$

We can therefore interpret $\rho_{zz}(z, \bar{z})$ as the mode that captures the inhomogeneous part of the superrotation transformation and parametrizes the corresponding family of *superrotation gravitational vacua* [60, 32].

⁴⁴Since physical quantities such as the Bondi shear are defined in the Bondi gauge, it is necessary to introduce a compensating diffeomorphism that brings the configuration back to the Bondi gauge slice, using the omitted subleading terms (non-universal terms) of the BMS transformation.

⁴⁵Whether one regards Y^z as a meromorphic function or as a general diffeomorphism parameter $Y^z(z, \bar{z})$ (which is called generalized BMS [32]) makes no essential difference to the discussion below. In fact, the subsequent arguments and conclusions remain unchanged in either case.

⁴⁶Strictly speaking, $\mathcal{N}_{zz}$ is the genuine Bondi news tensor. In geometrical terms, ρ_{zz} is referred to as the Geroch tensor (see e.g. [60])

As discussed above, generic superrotations do not preserve the universal structure on $\mathcal{I}$. Under a meromorphic superrotation, the induced metric component $\gamma_{\bar{z}\bar{z}}$ transforms as [32]

$$\delta_Y \gamma_{\bar{z}\bar{z}} = 2 \partial_{\bar{z}} Y^z. \quad (2.4.19)$$

In our chosen representative, we initially have $\gamma_{\bar{z}\bar{z}} = 0$, but for a meromorphic Y^z the right-hand side can develop singular contributions localized at isolated points. For instance, for $Y^z = 1/z$ one finds $\partial_{\bar{z}} Y^z = 2\pi \delta^{(2)}(z)$, so that $\gamma_{\bar{z}\bar{z}}$ acquires a distributional $d\bar{z}^2$ component supported at $z = 0$. Note that the null generator n^i remains invariant under superrotations.

Subtle point for covariance

Let us comment on a slight subtlety in the present formulation. We are not restricting the domain of the path integral to configurations that themselves satisfy the gauge-fixing conditions. Rather, for any metric g , if there exists a diffeomorphism F such that its pullback F^*g lies on the gauge-fixing surface (2.4.2) and satisfies the AFS condition (2.4.1), then we regard such a metric g as describing an asymptotically flat geometry and include it in the domain of integration.

In this sense, there is no need to treat quantities such as the Bondi shear $C_{\alpha\beta}[g_{\text{Bondi}}]$ as being defined only for metrics that satisfy the Bondi gauge conditions and the AFS conditions. Instead, for an arbitrary metric g , we choose such an F and define

$$C_{\alpha\beta}[g] := C_{\alpha\beta}[F^*g], \quad (2.4.20)$$

thereby extending the definition of $C_{\alpha\beta}$ to the entire gauge orbit. In general, the choice of F is not unique, and there remains an ambiguity corresponding to residual diffeomorphisms that preserve the Bondi gauge and the asymptotic conditions. In the path integral, however, we do not integrate over this residual freedom; rather, once we choose F by some prescription, we treat it as a fixed map.⁴⁷

We adopt this somewhat indirect formulation because we wish to work with a concrete coordinate system, namely Bondi coordinates, while at the same time avoiding the additional complications that would arise from rewriting the path integral itself as an integral over the gauge-fixing surface and introducing the BRST formalism.⁴⁸ Such a reformulation would, in a way that is unnecessary for our purposes, obscure the fact that the discussion in this work is essentially kinematical in nature.

⁴⁷If the residual diffeomorphisms are not completely fixed by the boundary conditions and therefore remain as a freedom, $C_{\alpha\beta}[g]$ is not gauge invariant and hence is not an observable. This is why such an ambiguity is not physically problematic. Moreover, in the setup we adopt in the next subsection 2.4.2, we first fix a representative of the universal structure $[\gamma_{ij}, n^i]$ on the boundary and then choose a vacuum state. This procedure completely removes the residual freedom, and the map $g \mapsto C_{\alpha\beta}[g]$ becomes unique.

⁴⁸Of course, in a more formal treatment, it may be necessary to adopt such a prescription, but for the kinematical analysis carried out in this paper it is unlikely to lead to any essential modification.

2.4.2 Boundary conditions

We now derive the Ward identities associated with asymptotic symmetries in asymptotically flat spacetimes.

Schematically, we consider the gravitational path integral on AFS space-times as following :

$$\langle \mathcal{O} \rangle = \int \mathcal{D}g \, e^{iS[g]} \mathcal{O}, \quad (2.4.21)$$

where $\mathcal{O}$ denotes an observable.

On $\mathcal{I}^\pm$ we impose the following boundary conditions.

- **Choice of boundary structure** We restrict the path integral to geometries whose induced boundary data realize a fixed representative of the universal structure. Concretely, we integrate only over configurations for which the induced quantities ⁴⁹ $\gamma_{\alpha\beta}$ and n^i satisfy

$$\bar{\gamma}_{z\bar{z}} = 2 \, dz \, d\bar{z}, \quad (2.4.22)$$

$$\bar{n}^i = (\partial_u)^i. \quad (2.4.23)$$

Corresponding boundary action is :

$$e^{iS_{\text{univ}}^\pm[\gamma^\pm]} = \delta[\gamma_{\alpha\beta}^\pm - \bar{\gamma}_{\alpha\beta}]. \quad (2.4.24)$$

now we omit fixing term n^i because this is superrotation invariant, thus this is irrelevant. Thus, we implicitly impose this condition for n^i on path integral measure.

- **Choice of vacuum** On future null infinity $\mathcal{I}^+$ we single out an early-time portion, denoted $\mathcal{I}_-^+$, and assume that no gravitational radiation is present there. In this region the Bondi news tensor vanishes, and we can choose the shear $\mathcal{C}_{\alpha\beta}(u, z, \bar{z})$ to satisfy

$$c_{\alpha\beta}^+ := \mathcal{C}_{\alpha\beta}(u, z, \bar{z}) \Big|_{\mathcal{I}_-^+} = 0. \quad (2.4.25)$$

Thus, we implement this condition by a boundary action of the form

$$e^{iS_{\text{tran}}^\pm[c^\pm]} = \delta[c_{\alpha\beta}^\pm], \quad (2.4.26)$$

where $c_{\alpha\beta}^\pm$ denote the initial (or final) values of the shear on $\mathcal{I}_\mp^\pm$. Physically, this means that instead of summing over all vacua related by supertranslations, we prepare the in- and out-states in a definite super-translation vacuum, namely the one for which $c_{\alpha\beta} = 0$.

As like above we impose super-rotation vacuum condition as following :

$$e^{iS_{\text{rot}}^\pm[\rho^\pm]} = \delta[\rho^\pm] \quad (2.4.27)$$

⁴⁹The bar does not denote complex conjugation.

Now, we can surmalize boudary actions as following :

$$S_{\text{bdy}} = S_{\text{univ}}^{\pm}[\gamma^{\pm}] + S_{\text{tran}}^{\pm}[c^{\pm}] + S_{\text{rot}}^{\pm}[\rho^{\pm}] \quad (2.4.28)$$

From now on, we focus on the “diagonal” subgroup of asymptotic symmetries that act simultaneously on both $\mathcal{I}^-$ and $\mathcal{I}^+$. Concretely, we consider BMS vector fields ξ whose restrictions to future and past null infinity obey the antipodal matching condition

$$\xi|_{\mathcal{I}_+^+}(z, \bar{z}) = \xi|_{\mathcal{I}_+^-}(z, \bar{z}). \quad (2.4.29)$$

This is the gravitational counterpart of the “diagonal” LGT in gauge theory (see subsection 2.3.3). Within this analogy, this restriction is exactly the condition necessary to prevent the appearance of additional currents at spatial infinity (For a more precise and rigorous statement, see Ref.[61]). Accordingly, we neglect contributions from spatial infinity in the derivation of the Ward identities.

2.4.3 Ward identity

The Ward identities follow from the invariance of the path integral (2.4.21) under this class of diffeomorphisms. Performing an infinitesimal transformation generated by ξ , we obtain

$$\delta_{\xi} \langle \mathcal{O} \rangle = \int \mathcal{D}g \, \delta_{\xi} (e^{iS[g]} \mathcal{O}) = 0. \quad (2.4.30)$$

Using the diffeomorphism invariance of the bulk action and of the measure, the only non-trivial contributions come from the variation of the operator insertions and from the boundary actions $S_{\text{bdy}}^{\pm}$. We thus find

$$0 = \langle \delta_{\xi} \mathcal{O} \rangle + i \langle \delta_{\xi} S_{\text{bdy}} \mathcal{O} \rangle. \quad (2.4.31)$$

This is the general form of the Ward identity. The remaining steps proceed exactly as in the $U(1)$ case (see subsection 2.3.3): the infinitesimal diffeomorphism variation of the boundary action can be rewritten, by integrating by parts on the sphere, as a functional derivative of the bulk action.⁵⁰ Consequently, depending on which boundary action one chooses, one obtains the corresponding contributions [32, 17].⁵¹

- From S_{trans} (Eq. (2.4.25)),

$$\langle \delta_{\xi} S_{\text{trans}}[c] \mathcal{O} \rangle = \left\langle -\frac{i}{2} \left(\int_{S^2} d^2z \, \delta_{\xi} c_{zz}(z, \bar{z}) J^{zz}(z, \bar{z}) \right) \mathcal{O} \right\rangle, \quad (2.4.32)$$

⁵⁰The contribution from $\mathcal{I}^-$ is completely analogous, and hence in the following we consider only the contribution from $\mathcal{I}^+$ for simplicity.

⁵¹Here we use the relation between the on-shell variation of the action and the symplectic potential and charge; see, for example, [39].

where we define *holomorphic supertranslation current* as

$$J^{zz}(z, \bar{z}) := -i \frac{\delta S_{\text{bulk}}[g]}{\delta c_{zz}(z, \bar{z})} = -i \int_{-\infty}^{+\infty} du N^{zz}(u, z, \bar{z}). \quad (2.4.33)$$

- From S_{rot} (Eq. (2.4.27)),

$$\langle \delta_{\xi} S_{\text{rot}}[\rho] \mathcal{O} \rangle = - \left\langle \left(\int_{S^2} d^2 z \delta_{\xi} \rho_{zz}(z, \bar{z}) \mathcal{T}^{zz}(z, \bar{z}) \right) \mathcal{O} \right\rangle. \quad (2.4.34)$$

where⁵²

$$\mathcal{T}^{zz}(z, \bar{z}) := \frac{\delta S_{\text{bulk}}[g]}{\delta \rho_{zz}(z, \bar{z})} = \int du (C^{zz} - u N^{zz}). \quad (2.4.35)$$

- From S_{univ} (Eq. (2.4.24)),

$$\langle \delta_{\xi} S_{\text{univ}}[\gamma] \mathcal{O} \rangle = \left\langle \left(\int_{S^2} d^2 z \delta_{\xi} \gamma_{\bar{z}\bar{z}}(z, \bar{z}) \mathcal{T}^{\bar{z}\bar{z}}(z, \bar{z}) \right) \mathcal{O} \right\rangle, \quad (2.4.36)$$

where

$$\mathcal{T}^{\bar{z}\bar{z}}(z, \bar{z}) := \frac{\delta S_{\text{bulk}}[g]}{\delta \gamma_{\bar{z}\bar{z}}(z, \bar{z})} = - \int_{-\infty}^{+\infty} du \partial_z \partial_{\bar{z}} C^{\bar{z}\bar{z}}(u, z, \bar{z}), \quad (2.4.37)$$

Supertranslation Ward identity

We first consider the case where ξ generates a supertranslation [25]. The action of a holomorphic supertranslation on the shear is

$$\delta_f c_{zz} = -2 \partial_z^2 f(z), \quad (2.4.38)$$

which follows from (2.4.13) upon taking f to be holomorphic and starting from a configuration with vanishing shear.

Inserting (2.4.38) into (2.4.32) and using integration by parts on the sphere, we obtain⁵³

$$\langle \delta_{\xi} S_{\text{bdy}} \mathcal{O} \rangle = \left\langle - \frac{i}{2} \left(\int d^2 z \delta_f c_{zz} J^{zz} \right) \mathcal{O} \right\rangle = i \left\langle \left(\int d^2 z f(z, \bar{z}) \partial_z^2 J^{zz}(z, \bar{z}) \right) \mathcal{O} \right\rangle. \quad (2.4.39)$$

Substituting (2.4.39) into (2.4.31), we arrive at

$$\langle \delta_f \mathcal{O} \rangle - \left\langle \int d^2 z f \partial_z^2 J^{zz} \mathcal{O} \right\rangle = 0. \quad (2.4.40)$$

⁵²The divergent parts of $\mathcal{T}$ and $\mathcal{T}$ (we define below) are minimally subtracted by making use of the ambiguity associated with adding total derivatives [32, 62]

⁵³Supertranslations preserve the universal structure, i.e. $\delta_f \rho_{zz} = 0$ and $\delta_f \gamma_{\bar{z}\bar{z}} = 0$, so there is no contribution from Eqs. (2.4.36) and (2.4.34).

Furthermore, assuming that the observable is a product of local operator insertions,

$$\mathcal{O} = \prod_n \mathcal{O}_n(z_n, \bar{z}_n), \quad (2.4.41)$$

and that the supertranslation acts on each insertion as $\delta_f \mathcal{O}_n = f(z_n, \bar{z}_n) \delta \mathcal{O}_n$, we obtain the local form of the supertranslation Ward identity,

$$\langle \partial_z^2 J^{zz}(z, \bar{z}) \mathcal{O} \rangle = \sum_m \delta^{(2)}(z - z_m) \langle \delta \mathcal{O}_m \prod_{n \neq m} \mathcal{O}_n \rangle, \quad (2.4.42)$$

which, when expressed in momentum space, is equivalent to the leading soft graviton theorem [25].

Superrotation Ward identity and celestial holography

We now turn to the case where ξ generates a superrotation [28, 32]. The action of a meromorphic superrotation on $\rho_{zz}, \rho_{\bar{z}\bar{z}}$ is

$$\delta_Y \rho_{zz} = \partial_z^3 Y^z, \quad (2.4.43)$$

$$\delta_Y \rho_{\bar{z}\bar{z}} = -\partial_z \partial_{\bar{z}}^2 Y^z, \quad (2.4.44)$$

while the variation of the boundary metric is

$$\delta_Y \gamma_{\bar{z}\bar{z}} = 2 \partial_{\bar{z}} Y^z. \quad (2.4.45)$$

Inserting these variations into (2.4.34) and (2.4.36) and combining the two contributions, we obtain⁵⁴

$$\begin{aligned} \langle \delta_\xi S_{\text{bdy}} \mathcal{O} \rangle &= - \left\langle \left(\int_{S^2} d^2 z \left(\delta_Y \rho_{zz} \mathcal{T}^{zz} + \delta_Y \rho_{\bar{z}\bar{z}} \mathcal{T}^{\bar{z}\bar{z}} + \delta_Y \gamma_{\bar{z}\bar{z}} \mathcal{T}^{\bar{z}\bar{z}} \right) \right) \mathcal{O} \right\rangle \\ &= - \left\langle \left(\int du d^2 z 2 \left[\partial_z^3 Y^z (C_{\bar{z}\bar{z}} - u N_{\bar{z}\bar{z}}) + \partial_z \partial_{\bar{z}}^2 Y^z (C_{zz} + u N_{zz}) \right] \right) \mathcal{O} \right\rangle. \end{aligned} \quad (2.4.46)$$

We now improve this current by subtracting a total u -derivative, so as to simple form. Define

$$\partial_u \int d^2 z K = \partial_u \int d^2 z 2u \left[\partial_z^3 Y^z C_{\bar{z}\bar{z}} + \partial_{\bar{z}}^2 \partial_z Y^z C_{zz} \right]. \quad (2.4.47)$$

Subtracting this total derivative, we obtain

$$\langle \delta_\xi S_{\text{bdy}} \mathcal{O} \rangle = 4 \left\langle \left(\int du d^2 z \partial_z^3 Y^z u N_{\bar{z}\bar{z}} \right) \mathcal{O} \right\rangle. \quad (2.4.48)$$

⁵⁴Due to the decomposition (2.4.14), we have $\delta_Y c_{zz} = 0$, so there is no contribution from (2.4.32).

The current term can be rewritten as⁵⁵

$$\langle \delta_\xi S_{\text{bdy}} \mathcal{O} \rangle = i \left\langle \oint \frac{dz}{2\pi i} Y^z T_{zz}(z, \bar{z}) \right\rangle \mathcal{O}, \quad (2.4.50)$$

where the integration contour is taken to encircle the entire celestial sphere and we have defined the holomorphic *celestial stress tensor* [29] by

$$T_{zz}(z, \bar{z}) := 4i \int d^2w \int_{-\infty}^{+\infty} du \, u \frac{1}{z-w} \partial_w^3 N_{\bar{w}\bar{w}}(u, w, \bar{w}). \quad (2.4.51)$$

Substituting this back into (2.4.31), we obtain the superrotation Ward identity in integrated form

$$\left\langle \oint \frac{dz}{2\pi i} Y^z T_{zz}(z, \bar{z}) \right\rangle \mathcal{O} = \langle \delta_Y \mathcal{O} \rangle. \quad (2.4.52)$$

Now we would like to derive local form of this Ward identity. We assume that the observable factorizes as

$$\mathcal{O} = \prod_n \mathcal{O}_{h_n}(z_n, \bar{z}_n), \quad (2.4.53)$$

and that each local operator transforms as a two-dimensional primary with holomorphic weight h :

$$\delta_Y \mathcal{O}_h(z, \bar{z}) = (Y(z) \partial_z + h \partial_z Y(z)) \mathcal{O}_h(z, \bar{z}). \quad (2.4.54)$$

This transformation law is realized by expanding the spin s asymptotic creation and annihilation operators into a Mellin basis⁵⁶⁵⁷,

$$\mathcal{O}_h(z, \bar{z}) = \int_0^\infty d\omega \, \omega^{\Delta-1} a_\omega^{(s)}(z, \bar{z}), \quad (2.4.56)$$

where $h = (s + \Delta)/2$. This constitutes the basic *dictionary* of celestial holography [31, 30].

⁵⁵The right-hand side can equivalently be written as

$$\left\langle \left(\int d^2z \partial_{\bar{z}} Y^z T_{zz}(z, \bar{z}) \right) \mathcal{O} \right\rangle, \quad (2.4.49)$$

where Y^z is holomorphic except at the support of $\mathcal{O}$.

⁵⁶This provides the dictionary in the massless case. For the massive case, see Ref. [30]. In addition, there are ambiguity of definition of dictionary [63].

⁵⁷In mellin basis, celstial stress tensor can be written as

$$T_{zz}(z, \bar{z}) \sim \lim_{\Delta \rightarrow 0} \Delta \tilde{\mathcal{O}}_{zz}(z, \bar{z}) \quad (2.4.55)$$

where tilde is shadow transform, which is defined in [64, 30]

Finally, converting the integrated Ward identity to a local one, we obtain [29]⁵⁸

$$\langle T_{zz}(z, \bar{z}) \prod_{i=1}^n \mathcal{O}_{h_m}(z_m, \bar{z}_m) \rangle = \sum_m \left(\frac{h_i}{(z - z_m)^2} + \frac{1}{z - z_m} \partial_{z_m} \right) \langle \mathcal{O}_m \prod_{n \neq m} \mathcal{O}_n \rangle. \quad (2.4.57)$$

Thus, the operators $\mathcal{O}_n$ can be regarded as primary fields with conformal dimensions $(h_n, \bar{h}_n)$, and the celestial sphere inherits the structure of a two-dimensional conformal field theory. This is the central motivation for viewing four-dimensional asymptotically flat gravity as being dual to a two-dimensional CFT, namely, celestial holography. In the next chapter we will relate this celestial holographic picture to dS/CFT. We omit a detailed discussion of other boundary actions and observables, as it essentially parallels that in subsections (2.3.3, 2.2.1).

In conclusion, let us comment on some basic properties of this CFT. In this theory, the conformal dimension Δ takes complex values of the form $(1 + i\lambda)$ with $\lambda \in \mathbb{R}$, and in this sense the theory is non-unitary [30, 31, 35]. From the bulk point of view, however, this non-unitarity may rather be regarded as a natural consequence. In AdS/CFT, the dilatation generator of the boundary CFT is identified with the ADM energy in the bulk, and the latter is expected to be a positive operator whose spectrum is bounded from below. In contrast, in celestial CFT the dilatation is understood, from the bulk perspective, as the charge corresponding to a Lorentz boost in the $(z, \bar{z}) = (0, 0)$ direction contained in the superrotation symmetry, and it does not correspond to a timelike translation. Accordingly, there is no requirement that this charge have a spectrum bounded from below. A similar line of reasoning applies to dS/CFT as well, and in this sense the CFT appearing in dS/CFT [65] is likewise expected to be non-unitary from the bulk point of view. Moreover, in both cases it has been suggested that the corresponding theories may admit an imaginary central charge [35, 66], and this fact also strongly indicates that the dual CFT is non-unitary.

At least at this level, dS/CFT and celestial holography exhibit similar structures. In the next chapter, we will further investigate this similarity from a different perspective.

⁵⁸From the view from the momentum basis, this is equivalent to subleading soft theorem [28]

Chapter 3

Celestial holography meets dS/CFT

In this chapter, following the main paper [1], we construct a concrete correspondence between celestial amplitudes and cosmological correlators in $(D + 1)$ -dimensional de Sitter space.

Celestial holography conjectures that scattering amplitudes in $(D + 2)$ -dimensional Minkowski spacetime can be described as correlation functions of a theory living on a D -dimensional sphere. In the four-dimensional case ($D = 2$), this was already discussed in the previous chapter in Eq. (2.4.56). More generally, for a massless scalar field in $(D + 2)$ dimensions, the operators of the celestial CFT are defined via a Mellin transform [22] as

$$\mathcal{O}_\Delta(k) = \int_0^\infty d\omega \omega^{\Delta-1} a(\omega k), \quad (3.0.1)$$

where a is the annihilation operator of the massless scalar field. The operator $\mathcal{O}_\Delta$ can be interpreted as an operator that “lives” on the celestial sphere S^D .

In this thesis, by “cosmological correlators” we mean asymptotic limits of bulk correlation functions. More concretely, we consider the late-time (or early-time) behaviour of a bulk scalar field ψ in de Sitter space in the global patch:

$$\psi \sim e^{-D-\Delta t} V_{D-\Delta} + e^{-\Delta t} W_\Delta \quad (t \rightarrow \infty), \quad (3.0.2)$$

where $\Delta = D/2 + i\lambda$ with $\lambda \in \mathbb{R}$. We refer to the correlation functions of V_Δ and W_Δ as *cosmological correlators*. The extrapolated operators V_Δ and W_Δ play an important role in dS/CFT [65, 67] and cosmological bootstrap (see Ref. [68] and the references therein¹). As in the case of the celestial operators $\mathcal{O}_\Delta$, the operators V_Δ and W_Δ can also be regarded as living on S^D .

We will derive an explicit identification between $\mathcal{O}_\Delta$ and the operators V_Δ , W_Δ . The strategy is as follows. We start from a massless scalar field in flat Euclidean space $\mathbb{R}^{D+2}$. A Weyl transformation maps this theory to the cylindrical space $S^{D+1} \times \mathbb{R}$. After a Fourier transform along

¹In the context of the cosmological bootstrap, insertions are usually made only on the out states, whereas we differ slightly in that we also consider insertions on the in states.

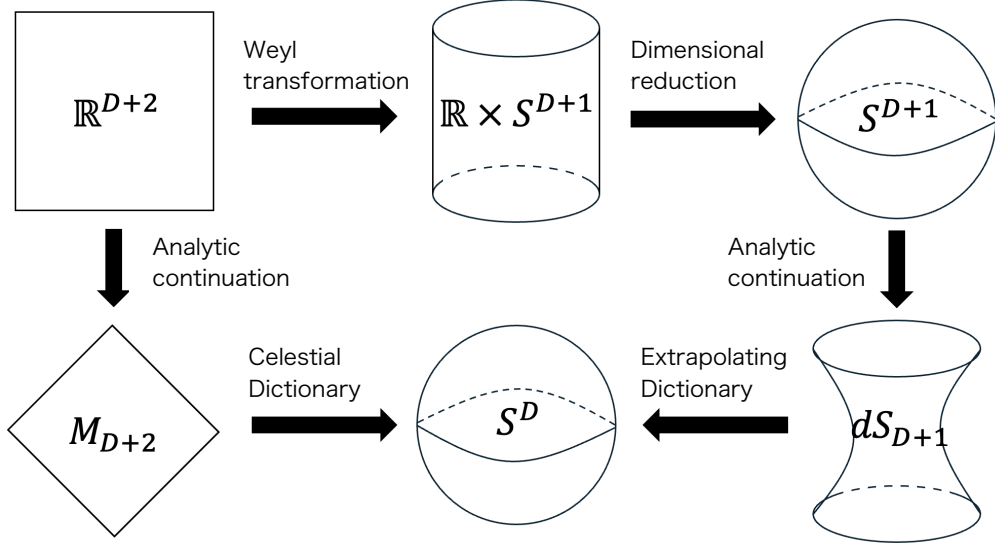


Figure 3.1: This figure provides a schematic overview of our strategy.

the $\mathbb{R}$ direction, we obtain a theory on the sphere S^{D+1} . This procedure is described in detail in section 3.1. We then analytically continue the theory on S^{D+1} to a theory in dS_{D+1} with the Euclidean vacuum, which is equivalent to Bunch–Davies vacuum [69, 70, 71, 72, 73, 74]. Under this analytic continuation, the original theory on $\mathbb{R}^{D+2}$ is mapped to a theory in Minkowski space M_{D+2} . In this way, we derive a relation between correlation functions in dS_{D+1} and in M_{D+2} in section 3.2.

In section 3.3, we apply the celestial holography dictionary developed in [63], and show that the extrapolated operators V_Δ, W_Δ can be expressed as linear combinations of the celestial operators $\mathcal{O}_\Delta$ and their shadow counterparts (the precise definition of the shadow operators will be given below). Our overall procedure is summarized schematically in Figure 3.1

Our work is largely inspired by [33, 34] (see also the recent related works [35, 75, 76, 77, 78, 79]), where it is proposed that M_{D+2} can be viewed, via a kind of “dimensional reduction”², as

$$M_{D+2} \sim dS_{D+1} + EAdS_{D+1}^+ + EAdS_{D+1}^-. \quad (3.0.3)$$

In our setup, we analytically continue a theory defined on a sphere. This viewpoint makes it particularly transparent why the Euclidean vacuum appears, since it is well known that dS correlation functions are obtained by analytic continuation from a Euclidean theory on the sphere. We

²This is not the standard Kaluza–Klein dimensional reduction, since we perform a mode expansion on a non-compact space (i.e., Fourier transformations).

confirm the connection with the earlier results of [33, 34] by appropriately deforming the contours of the path integral in section 3.2. In the same spirit as these works, we expect that our identification allows us to import the rich results of (A)dS/CFT into celestial holography, for instance the evaluation of the central charge in (A)dS₃/CFT₂ [65, 19, 18].

We also point out that Ref. [77] proposes a new celestial dictionary motivated by the extrapolate dictionary in (A)dS/CFT, and that Ref. [79] studies the relation between this new dictionary and the conventional celestial dictionary [22]. Ref. [22] likewise obtained a relation between the celestial operators $\mathcal{O}_\Delta$ and extrapolated operators in (A)dS. Although their approach and conclusions are closely related to ours, there are some differences in detail. We will explain the origin of these differences later in the paper.

Notation and useful formulas

Before proceeding, we collect our notation and some useful identities.

We define the conformal integration measure $D^D p$ following [64] by

$$\int D^D p f(p) \equiv \frac{2}{\text{Vol GL}(1, \mathbb{R})^+} \int_{p^0 > 0} d^{D+2} p \delta(p^2) f(p), \quad (3.0.4)$$

where $f(p)$ is required to have degree $-D$ under the rescaling $p \rightarrow ap$. In the “gauge” $p = (1, \mathbf{p})$, with $\mathbf{p}$ a unit vector in $\mathbb{R}^{D+1}$, this becomes

$$\int D^D p f(p) = \int_{S^D} d^D \mathbf{p} f(p)|_{p=(1, \mathbf{p})}. \quad (3.0.5)$$

The shadow transform $\widetilde{\mathcal{O}_\Delta}$ is defined by

$$\widetilde{\mathcal{O}_\Delta}(p) = \frac{1}{S_\Delta} \int d^D k (-2p \cdot k)^{\Delta-D} \mathcal{O}_\Delta(k). \quad (3.0.6)$$

The normalization factor S_Δ is chosen as

$$S_\Delta = 2^{2\Delta-D} \pi^h \frac{\Gamma(\Delta-h)}{\Gamma(D-\Delta)}, \quad h = \frac{D}{2}, \quad (3.0.7)$$

so that the transform is involutive:³

$$\widetilde{(\widetilde{\mathcal{O}_\Delta})} = \mathcal{O}_\Delta. \quad (3.0.8)$$

³The involutive condition (3.0.8) does not uniquely determine S_Δ . For example, one could choose $S_\Delta = \pi^h \frac{\Gamma(\Delta-h)}{\Gamma(D-\Delta)}$. We include the extra factor $2^{2\Delta-D}$ for later convenience.

Finally, we quote two integral identities, useful for computations involving shadow transforms [64]:

$$\int D^D k \frac{1}{(-2p \cdot k)^{D-\Delta}} \frac{1}{(-2X \cdot k)^\Delta} = \pi^h \frac{\Gamma(\Delta - h)}{\Gamma(\Delta)} \frac{(-X^2)^{h-\Delta}}{(-2p \cdot X)^{D-\Delta}}, \quad (3.0.9)$$

$$\int D^D k \frac{1}{(-2p \cdot k)^{h+i\lambda}} \frac{1}{(-2q \cdot k)^{h-i\lambda}} = \pi^D \frac{\Gamma(-i\lambda)\Gamma(i\lambda)}{\Gamma(h-i\lambda)\Gamma(h+i\lambda)} \delta^{(D)}(p-q), \quad (3.0.10)$$

where p and q are null vectors, X is an arbitrary non-null vector, and $\delta^{(D)}(p-q)$ denotes the delta function with respect to the measure $D^D p$.

We use following notation avoid notational complexity due to in/out index and make analyticity of p -complex plane manifest, namely, crossing symmetry. First, we define momentum operators as following :

$$a(\omega p) \equiv a_{\text{out}}(\omega \mathbf{p}), \quad (3.0.11)$$

$$a(-\omega p) \equiv a_{\text{in}}^\dagger(\omega \mathbf{p}). \quad (3.0.12)$$

We also define celestial operator as following by using above momentum operator.⁴

$$\mathcal{O}_\Delta(p) = \int_0^\infty \frac{d\omega}{\omega} \omega^\Delta a(\omega p). \quad (3.0.13)$$

3.1 QFT on S^{D+1} from QFT on $\mathbb{R}^{D+2}$

In this section, we show how to map a QFT defined on $\mathbb{R}^{D+2}$ to one on S^{D+1} by combining a Weyl transformation with a Fourier transform. This construction can be viewed as the Euclidean counterpart of the framework developed in [33, 34].

3.1.1 Free field

We begin with a free real scalar field on Euclidean space $\mathbb{R}^{D+2}$, with action

$$S = \int_{\mathbb{R}^{D+2}} [d^{D+2} X] \left(\frac{1}{2} G^{IJ} \nabla_I \Phi \nabla_J \Phi \right). \quad (3.1.1)$$

Here the metric is

$$ds^2 = G^{IJ} dX_I dX_J = (dX^1)^2 + \dots + (dX^{D+2})^2. \quad (3.1.2)$$

⁴Note that we now allow the embedding coordinate p of the operator $\mathcal{O}$ to be past-directed. Therefore, this is not the standard definition of an embedding-space CFT operator. Although this is merely a matter of convention, we can recover the usual celestial CFT operator by redefining $\mathcal{O}_\Delta(p)$ as $\mathcal{O}_\Delta^-(p)$ where p is past directed vector.

Introducing polar coordinates, we can rewrite this as

$$ds^2 = dr^2 + r^2 d\Omega_{D+1}^2 = e^{-2\xi} (d\xi^2 + d\Omega_{D+1}^2), \quad (3.1.3)$$

where $d\Omega_{D+1}^2$ is the line element on S^{D+1} and $r = e^{-\xi}$.

Next, we perform a Weyl transformation and redefine the metric and field as

$$g_{IJ} = e^{2\xi} G_{IJ}, \quad (3.1.4)$$

$$\phi = e^{-\frac{D}{2}\xi} \Phi. \quad (3.1.5)$$

Under this transformation, the theory originally defined on $\mathbb{R}^{D+2}$ is mapped to $S^{D+1} \times \mathbb{R}$, and the action becomes

$$S = \int_{S^{D+1}} [d^{D+1}x] \int d\xi \left(\frac{1}{2} g_S^{ij} \nabla_i \phi \nabla_j \phi + \frac{1}{2} \partial_\xi \phi \partial_\xi \phi + \frac{D^2}{8} \phi^2 \right), \quad (3.1.6)$$

where g_S is the metric on S^{D+1} .

We now Fourier transform along the ξ direction:

$$\begin{aligned} \psi_\lambda(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\xi e^{-i\lambda\xi} \phi(\xi, x), \\ \phi(\xi, x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\lambda e^{i\lambda\xi} \psi_\lambda(x), \end{aligned} \quad (3.1.7)$$

which yields an action for a continuum of fields ψ_λ on S^{D+1} :

$$S = \int_{S^{D+1}} [d^{D+1}x] \int_{-\infty}^{\infty} d\lambda \left(\frac{1}{2} g_S^{ij} \nabla_i \psi_{-\lambda} \nabla_j \psi_\lambda + \frac{1}{2} \left(\frac{D^2}{4} + \lambda^2 \right) \psi_{-\lambda} \psi_\lambda \right). \quad (3.1.8)$$

Note that, since the Fourier transform is taken along a non-compact direction, the Kaluza–Klein charge (KK charge) λ takes continuous values. Moreover, because the action is invariant under the global $U(1)$ transformation

$$\psi(\lambda) \rightarrow e^{i\alpha\lambda} \psi(\lambda), \quad (3.1.9)$$

λ can be interpreted as the conserved charge associated with this global $U(1)$ symmetry.

In summary, the original field Φ on $\mathbb{R}^{D+2}$ and the fields ψ_λ on S^{D+1} are related by

$$\psi_\lambda(x) = \frac{1}{\sqrt{2\pi}} \int d\xi e^{(D-\Delta)\xi} e^{-\epsilon r} \Phi(X(\xi, x)) = \frac{1}{\sqrt{2\pi}} \int \frac{dr}{r} r^{D-\Delta} e^{-\epsilon r} \Phi(X(r, x)), \quad (3.1.10)$$

with

$$\Delta = \frac{D}{2} + i\lambda, \quad (3.1.11)$$

where we have introduced an IR regulator $e^{-\epsilon r}$. Each mode ψ_λ has an effective mass

$$m_\lambda = \sqrt{\frac{D^2}{4} + \lambda^2}. \quad (3.1.12)$$

An explicit verification of the matching of two-point functions between the sphere and the plane is given in Appendix B.1.

3.1.2 Interactions

We consider a Φ^k interaction on $\mathbb{R}^{D+2}$ of the form

$$S_{\text{int}} = \int [d^{D+2}X] \frac{\alpha_k}{k!} \Phi^k, \quad (3.1.13)$$

where α_k is a coupling constant. After the Weyl transformation, this term becomes

$$S_{\text{int}} = \int [d^{D+1}x] \int d\xi \frac{\alpha_k}{k!} \exp \left[\left(D + 2 - \frac{kD}{2} \right) \xi \right] \phi^k. \quad (3.1.14)$$

Using the mode expansion (3.1.7), the interaction can be written as

$$S_{\text{int}} = \frac{\alpha_k}{k!} \int [d^{D+1}x] \int \prod_{i=1}^k \frac{d\lambda_i}{\sqrt{2\pi}} 2\pi \delta^G \left(- \sum_{i=1}^k \lambda_i + i \left(D + 2 - \frac{kD}{2} \right) \right) \prod_{i=1}^k \psi_{\lambda_i}, \quad (3.1.15)$$

where $\delta^G(z)$ is the generalized delta function defined by $\delta^G(z) = \frac{1}{2\pi} \int_0^\infty \frac{dr}{r} r^{-iz}$, for which an appropriate regularization is needed [32]. For real z it reduces to the usual delta function. In particular, for the Weyl-invariant interaction $k = 2(D+2)/D$, the generalized delta function reduces to the ordinary one. As a concrete example, we will later consider the case $k = 4$ for $D = 2$.

3.2 QFT in dS_{D+1} from QFT in M_{D+2}

From the correspondence developed in the previous section, we obtain the following relation between n -point functions on the sphere S^{D+1} and those on Euclidean space $\mathbb{R}^{D+2}$:

$$\langle \psi_{\lambda_1}(x_1) \cdots \psi_{\lambda_n}(x_n) \rangle_{S^{D+1}} = \int \left[\prod_{i=1}^n \frac{1}{\sqrt{2\pi}} \frac{dr_i}{r_i} r_i^{D-\Delta_i} e^{-er_i} \right] \langle \Phi(r_1, x_1) \cdots \Phi(r_n, x_n) \rangle_{\mathbb{R}^{D+2}}. \quad (3.2.1)$$

Introducing polar coordinates as⁵

$$x^{D+2} = -\cos \theta, \quad \mathbf{x} = (x^1, \dots, x^{D+1}) = \sin \theta \hat{n}, \quad (3.2.2)$$

we analytically continue (3.2.1) by continuing the polar angle θ_i in the spherical coordinates of the external vertices as

$$\theta_i = \frac{\pi}{2} + it_i. \quad (3.2.3)$$

⁵We choose $x^{D+2} = -\cos \theta$ instead of $x^{D+2} = +\cos \theta$ so that the direction of increasing θ is aligned with the x^{D+2} -axis. With this choice, the analytic continuation in Eq. (3.2.3) matches the standard convention for continuing between Lorentzian and Euclidean time (B.3.3).

Under this analytic continuation, the correlation functions in Euclidean space $\mathbb{R}^{D+2}$ can be identified with those in the standard Poincaré vacuum of Minkowski spacetime M_{D+2} (with the external points restricted to the spherical Rindler region $X^2 > 0$), while the correlation functions on S^{D+1} are interpreted as those in de Sitter spacetime dS_{D+1} in the Bunch–Davies (BD) vacuum (the Euclidean vacuum):

$$\langle \psi_{\lambda_1}(x_1) \cdots \psi_{\lambda_n}(x_n) \rangle_{dS^{D+1}} = \int \left[\prod_{i=1}^n \frac{1}{\sqrt{2\pi}} \frac{dr_i}{r_i} r_i^{D-\Delta_i} e^{-\epsilon r_i} \right] \langle \Phi(r_1, x_1) \cdots \Phi(r_n, x_n) \rangle_{M_{D+2}}. \quad (3.2.4)$$

At first sight, this relation between the Euclidean vacuum in dS and the Poincaré vacuum in Minkowski space may seem counterintuitive: using a foliation of Minkowski space (see Fig. 3.2), one typically expects a theory defined on $EAdS^+ + EAdS^- + dS$ [33, 34].

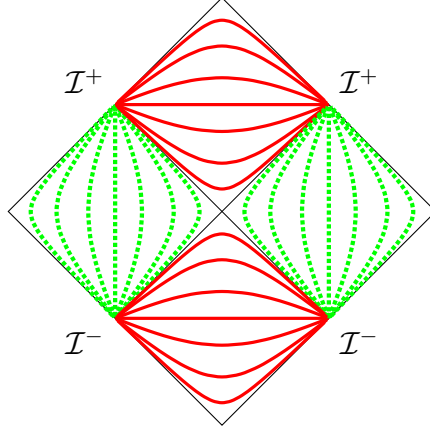


Figure 3.2: Penrose diagram of Minkowski space and the Milne slices. Red curves represent $EAdS$ slices, and green dotted curves represent dS slices. By performing the Fourier transformation, one obtains a picture of $EAdS^+ + dS + EAdS^-$.

Nevertheless, our construction is fully compatible with these earlier approaches. In practice, we analytically continue the external vertices on S^{D+1} and deform the integration contour for the polar angle $\theta \in [0, \pi]$ in the path integral to the contour shown below (and depicted in Fig. 3.3):⁶

$$\begin{aligned} A : \quad & \theta = -i\chi, \quad \chi : 0 \rightarrow \infty, \\ B : \quad & \theta = \frac{\pi}{2} + it, \quad t : -\infty \rightarrow \infty, \\ C : \quad & \theta = \pi - i\chi, \quad \chi : -\infty \rightarrow 0. \end{aligned} \quad (3.2.5)$$

⁶When deforming the contour into the paths $A + B + C$, the real part of the coordinate along each path must increase monotonically. Otherwise, the contour might cross a branch cut of the integrand, which is given by a product of propagators. For this reason, each path cannot be exactly vertical; it must be slightly tilted.

We assume that the integrand decays sufficiently rapidly at infinity so that this contour deformation is allowed. In this way, the path integral on the sphere is deformed to a path integral over $A+B+C$.

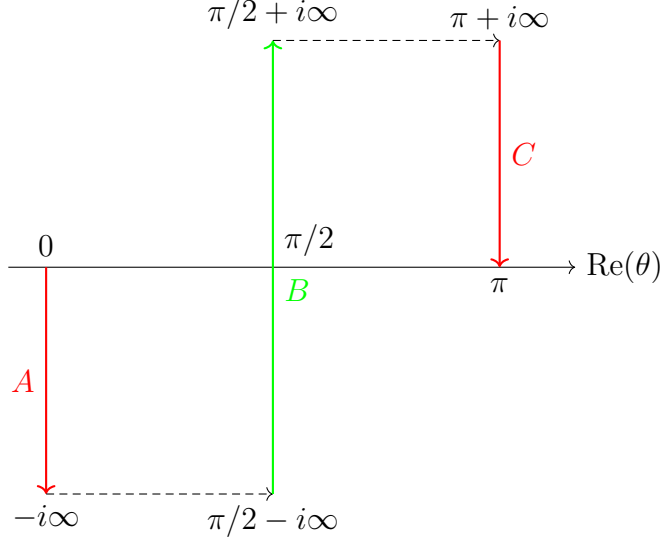


Figure 3.3: Deformation of the integration contour for the polar angle θ in the path integral. The original contour runs from 0 to π along the real axis. It is deformed into the contours A , B , and C (the dotted horizontal segments can be ignored).

One can see that path B corresponds to dS, path A to EAdS^- , and path C to EAdS^+ ,⁷ although we relegate a more detailed explanation to appendix B.2. The EAdS segments play the role of Euclidean vacuum for the initial and final states in dS.⁸ In this sense, our prescription is consistent with previous constructions.

Although there is no contradiction with the $\text{EAdS}^+ + \text{EAdS}^- + \text{dS}$ picture, interpreting the setup literally as a theory on this union of manifolds requires a somewhat unconventional quantum field theory with the following features:

- **Physical interpretation of the dS segment:** If one considers only the path integral on the dS segment with the standard Feynman $i\epsilon$ prescription, the resulting correlation functions do *not* coincide with those in Euclidean vacuum.⁹ In contrast, summing over the entire contour $\text{EAdS}^+ + \text{EAdS}^- + \text{dS}$ produces the correlation functions in the Euclidean vacuum of de Sitter space by definition.

⁷Further details in the Poincaré patch can be found in [80].

⁸In [81, 15], it is argued that the path integral on Euclidean AdS produces the wave functions of the Bunch–Davies vacuum for the initial states.

⁹See [82] for an explicit computation of such a two-point function, which differs from the in-in two-point function in the BD vacuum.

- **Interactions across EAdS and dS:** In an interacting theory, fields living in the EAdS region and the dS region must interact with each other in the sense that propagators connecting points in EAdS and dS contribute to correlation functions. However, since ordinary quantum field theory does not accommodate interactions between fields defined on disconnected manifolds, a substantial extension of the standard framework is required.
- **Propagators on EAdS:** Even for propagators with both endpoints in EAdS, one must use propagators obtained by analytic continuation from those on S^{D+1} . These differ in general from the propagators normally used in EAdS, which are typically defined with Dirichlet boundary conditions.

For these reasons, a direct interpretation in terms of a theory in de Sitter space itself seems particularly appropriate, especially in holographic applications, while the $\text{EAdS}^+ + \text{EAdS}^- + \text{dS}$ viewpoint remains useful as a complementary and powerful computational tool.

Remarks on the in-in contour and the GKPW dictionary

The contour introduced above may look unfamiliar at first sight. Nevertheless, it can be viewed as a natural extension of the standard contours commonly used in dS theory. In the latter part of this subsection, we also briefly comment on the GKPW dictionary. Some aspects lie beyond the scope of the present paper and are therefore left for future work.

We begin with a remark on the in-in contour. If we restrict ourselves to out insertions only, our cosmological correlator reduces to the usual in-in correlator, a situation that has been extensively discussed in the literature. In that case, one may adopt either of the following two contours¹⁰:

- One may prepare the vacuum on a hemisphere and then define correlators by subsequent dS time evolution. Concretely, this is implemented by the following contour:

$$\begin{aligned}
A : \quad \theta &= \Theta, & \Theta &: 0 \rightarrow \frac{\pi}{2}, \\
B : \quad \theta &= \frac{\pi}{2} + it, & t &: 0 \rightarrow \infty, \\
B' : \quad \theta &= \frac{\pi}{2} - it, & t &: -\infty \rightarrow 0, \\
A' : \quad \theta &= \Theta, & \Theta &: \frac{\pi}{2} \rightarrow \pi.
\end{aligned} \tag{3.2.6}$$

- Alternatively, one may construct the wavefunction in EAdS. Concretely, this is realized by the contour:

$$\begin{aligned}
A : \quad \theta &= i\chi, & \chi &: 0 \rightarrow \infty, \\
A' : \quad \theta &= \pi - i\chi, & \chi &: -\infty \rightarrow 0,
\end{aligned} \tag{3.2.7}$$

¹⁰We assume that the integrand decays sufficiently fast so that contributions from the far-away segments of the contour can be neglected.

The equivalence of these two descriptions, as well as their physical interpretation, was also pointed out in [15].

However, neither of these contours is directly applicable in our present context. From the flat-space perspective, it is well known that scattering amplitudes with only out insertions become trivial due to momentum conservation. Correspondingly, from the de Sitter perspective, correlators with only out insertions are also trivial in our Lagrangian. Indeed, for the in-in correlator, choosing either of the contours above yields the following effective vertex factor λ_{eff} of ϕ^k theory, which was computed in [76]:

$$\lambda_{\text{eff}} \sim \alpha \sin\left(\frac{D(k-2)}{4} + \frac{1}{2} \sum_{i=1}^k \left(\Delta_i - \frac{D}{2}\right)\right) \pi. \quad (3.2.8)$$

where α is coupling constant. However, in our theory the interaction includes a delta function imposing KK charge conservation on α (see Eq. (3.1.15)), which forces the sine phase to be an integer multiple of π , and hence implies $\lambda_{\text{eff}} = 0$. Therefore, to obtain a nontrivial correlator one must include both out and in insertions. Moreover, in order to correctly capture their contributions we employ the contour in Eq. (3.2.5).

That said, in the soft limit the distinction between out and in becomes ambiguous from crossing symmetry. From this viewpoint, restricting to the soft limit, it may still be meaningful to consider the in-in contour. Furthermore, in the extrapolate dictionary adopted in this work, the analytic continuation that exchanges out and in is manifest, suggesting that the gap between the two prescriptions may not be as large as it might appear. A more detailed investigation of this point is deferred to future work.

Finally, we comment on the GKPW dictionary. While the BDHM and GKPW dictionaries coincide in AdS/CFT, they generally differ in dS/CFT [15]. The reason is that the GKPW dictionary defines correlators via functional derivatives of the wavefunction, whereas the BDHM dictionary is based on the Born rule and involves the square of the wavefunction (more precisely, an appropriate combination involving complex conjugation). Nevertheless, their relation is not excessively complicated, and in the context of the cosmological bootstrap it can be made explicit through concrete examples (see e.g. [83]).

However, when both in and out insertions are allowed, no general and concrete relation is currently known. Consequently, even in dS theory the relation between the extrapolate and GKPW dictionaries remains unclear in that setting. Likewise, in celestial holography one is essentially led to an extrapolate-type dictionary, and a GKPW-type dictionary has not been constructed so far (see [84] for related work). Although it is an interesting problem to construct a GKPW-type dictionary in celestial holography and to clarify its relation to the global dS GKPW dictionary, in this paper we focus exclusively on the BDHM-type dictionary.

Subtleties in the Analytic Continuation

We now comment on a few subtleties that must be handled with care. Since Eq. (3.2.4) is obtained by analytically continuing both sides of Eq. (3.2.1), one should first perform the Mellin transform with respect to the variables r_i of the external vertices on the right-hand side of (3.2.1), and only then carry out the analytic continuation $\theta = \pi/2 + it$ together with the deformation of the path-integral contour.

Following this order of operations, the Mellin transform produces factors of the form of conformal primary waves, schematically

$$(-pr \cdot x + i\epsilon)^{-\Delta},$$

(where r denotes the radial coordinate of an internal vertex), which introduce nontrivial branch cuts in the complex r -plane. Since the contour for the internal vertices must subsequently be deformed, the following subtle issue arises.

If, in this situation, one tries to compute the Minkowski correlator from the Euclidean correlator using the standard QFT prescription—namely, rotating the integration contour of the internal vertex via $X^{d+2} = iX^0$, which in polar coordinates corresponds to a combined deformation of r and θ (see appendix B.3):

$$\begin{aligned} A : \theta &= -i\chi, \quad \chi : 0 \rightarrow \infty, \quad \text{and} \quad r = ir', \quad r' : 0 \rightarrow \infty, \\ B : \theta &= \frac{\pi}{2} + it, \quad t : -\infty \rightarrow \infty, \\ C : \theta &= \pi - i\chi, \quad \chi : -\infty \rightarrow 0, \quad \text{and} \quad r = ir', \quad r' : 0 \rightarrow \infty, \end{aligned} \tag{3.2.9}$$

then the contour in the complex r -plane inevitably crosses the branch cuts generated by the Mellin factors. One must then keep track of the associated discontinuity contributions, which makes the phase bookkeeping rather intricate.

For this reason, instead of using the full rotation $X^{d+2} = iX^0$, it is more natural and technically convenient to deform only the θ contour, as was done on the left-hand side of Eq. (3.2.4), namely deforming the original interval $[0, \pi]$ into the contour shown in Eq. (3.2.5).

In summary, it is preferable to adopt the prescription of deforming only the θ contour also on the right-hand side of Eq. (3.2.4), thereby avoiding any additional assumptions about the analytic structure in the r -plane.¹¹

¹¹Of course, in situations where the external legs are not Mellin transformed—so that the analytic structure in the complex r -plane is manifest, as in the usual Euclidean-to-Minkowski continuation of correlators—the two procedures are equivalent, as is explicitly demonstrated in Appendix B.3.

3.3 Connection between Celestial and Cosmological Correlators

The celestial dictionary relates the bulk field Φ in Minkowski spacetime M_{D+2} to the operators $\mathcal{O}_\Delta$ in the celestial CFT on S^D . On the other hand, we have related Φ in M_{D+2} to a family of bulk fields ψ_λ living in de Sitter spacetime dS_{D+1} . Combining these two relations, we can therefore connect the celestial operators $\mathcal{O}_\Delta$ on S^D with the extrapolated data (boundary operators) associated with ψ_λ in dS_{D+1} .

In what follows, we define the extrapolate operators associated with ψ_λ and identify them with the celestial operators. As a preparation, we first introduce a convenient parametrization.

We parametrize the coordinates on M_{D+2} as

$$X(t, r, \mathbf{p}) = (r \sinh t, r \cosh t \mathbf{p}), \quad (3.3.1)$$

where $\mathbf{p}$ is a unit vector on S^D . It is also useful to decompose X in terms of the null vectors

$$p_- = (1, -\mathbf{p}), \quad p = (1, +\mathbf{p}), \quad (3.3.2)$$

so that

$$X = \frac{1}{2} r e^t p - \frac{1}{2} r e^{-t} p_-. \quad (3.3.3)$$

The embedding coordinates of dS are parametrized as

$$x(t, \mathbf{p}) = (\sinh t, \cosh t \mathbf{p}). \quad (3.3.4)$$

These imply

$$X^I = r x^I, \quad (3.3.5)$$

$$x^I = \frac{1}{2} e^t p^I - \frac{1}{2} e^{-t} p_-^I. \quad (3.3.6)$$

The extrapolate operators in dS are defined as the leading coefficients in the late/early-time expansion of the dS bulk field¹²:

$$\psi_\lambda(x) \sim e^{-(D-\Delta)t} V_{D-\Delta}(p) + e^{-\Delta t} W_\Delta(p) \quad \text{as } t \rightarrow \infty, \quad (3.3.7)$$

$$\psi_\lambda(x) \sim e^{-(D-\Delta)t} V_{D-\Delta}(-p_-) + e^{-\Delta t} W_\Delta(-p_-) \quad \text{as } t \rightarrow -\infty. \quad (3.3.8)$$

¹²Following [85, 65], we use coordinates on the would-be dual space S^D such that each point on future infinity and its antipodal point on past infinity share the same angular coordinates.

Note that, since $\Delta = D/2 + i\lambda$, the two terms on the right-hand side contribute at the same order in t .

With these definitions in place, we proceed to identify the celestial operators with the extrapolate operators.

To this end, we focus on the behavior of Φ in the asymptotic region. In this region, we may use the free-field expansion of Φ to write¹³¹⁴

$$\begin{aligned}\psi_\lambda(x) &= \frac{1}{\sqrt{2\pi}} \int \frac{dr}{r} r^{D-\Delta} e^{-\epsilon r} \Phi(X) \\ &= \frac{1}{\sqrt{2\pi}} \int \frac{dr}{r} r^{D-\Delta} e^{-\epsilon r} \int \frac{\omega^{D-1} d\omega d^D k}{2(2\pi)^{D+1}} (a(\omega k) e^{i\omega k \cdot X - \epsilon \omega}).\end{aligned}\quad (3.3.9)$$

The UV cutoff factor $e^{-\epsilon \omega}$ is introduced by adding an infinitesimal imaginary part to the time component of X so as to project onto the vacuum.¹⁵

We now decompose the integral (3.3.9) into two contributions:

$$\psi_\lambda(x) := \psi_\lambda^{D-\Delta}(x) + \psi_\lambda^\Delta(x), \quad (3.3.10)$$

where $\psi_\lambda^{D-\Delta}$ denotes the contribution from the region $k \neq p$ (i.e. $\int_{k \neq p} d^D k \dots$), and ψ_λ^Δ denotes the contribution from a neighborhood of $k = p$ (i.e. $\int_{k \approx p} d^D k \dots$). As we will see, both terms contribute at the same order in the limit $t \rightarrow \infty$. The first (non-collinear) contribution yields the coefficient of the $e^{-(D-\Delta)t}$ term in (3.3.7), while the second (collinear) contribution provides the coefficient of the $e^{-\Delta t}$ term.

Finally, we note that a similar extrapolation was considered in [79]. More precisely, they study a quantity of the form

$$e^{(D-\Delta)t} \psi_\lambda(x) \sim V_{D-\Delta}(p) + e^{-2i\lambda} W_\Delta(p) \quad (t \rightarrow +\infty). \quad (3.3.11)$$

In this expression, the second term is highly oscillatory at late times, so one typically isolates only the first term. In contrast, our prescription systematically retains both contributions in (3.3.9), and the second term is precisely the collinear contribution described above.

Contribution from $k \neq p$

We first examine the contribution from the region $k \neq p$ in the integral (3.3.9). Using

$$\omega k \cdot X = \frac{1}{2} r \omega (e^t p \cdot k - e^{-t} p_- \cdot k), \quad (3.3.12)$$

¹³For simplicity, we focus on the $t \rightarrow \infty$ limit; the $t \rightarrow -\infty$ analysis proceeds in complete analogy.

¹⁴ $a_{\text{out}}^\dagger$ would contribute if we considered out-out two-point functions. Since we will not study such correlators in this paper, we omit this contribution here. Of course, it can be reinstated straightforwardly.

¹⁵For notational simplicity, we use the same symbol ϵ for the IR regulator of r and the UV regulator of ω , although they represent distinct positive infinitesimal parameters.

and the fact that $k \neq p$ implies $p \cdot k \neq 0$, we perform the change of variables

$$r = \frac{r'}{\omega e^t (-2p \cdot k)}. \quad (3.3.13)$$

Expanding Eq. (3.3.9) at large t and keeping only the leading term proportional to $e^{-D-\Delta t}$, we obtain

$$\begin{aligned} \psi_\lambda^{D-\Delta}(x) &\sim e^{-(D-\Delta)t} \frac{1}{2(2\pi)^{D+3/2}} \int \frac{dr' d^D k d\omega}{r'} r'^{D-\Delta} e^{-\epsilon r'} \omega^{\Delta-1} (-2p \cdot k)^{\Delta-D} \left(a(\omega k) e^{-\frac{i}{4} r'} \right) \\ &= e^{-(D-\Delta)t} \frac{2^{2D-\Delta} \Gamma(D-\Delta)}{2(2\pi)^{D+3/2}} \int d^D k d\omega \omega^{\Delta-1} (-2p \cdot k)^{\Delta-D} \left(i^{-(D-\Delta)} a(\omega k) \right). \end{aligned} \quad (3.3.14)$$

Thus, using the celestial dictionary (3.0.13), we find

$$\begin{aligned} \psi_\lambda^{D-\Delta}(x) &\sim e^{-(D-\Delta)t} S_\Delta \frac{2^{-2i\lambda} \Gamma(D-\Delta)}{2^{5/2} \pi^{D+3/2}} i^{-(D-\Delta)} \tilde{\mathcal{O}}_{D-\Delta}(p) \\ &\sim e^{-(D-\Delta)t} N_{-\lambda} i^{-(D-\Delta)} \tilde{\mathcal{O}}_{D-\Delta}(p), \end{aligned} \quad (3.3.15)$$

with

$$N_\lambda = \frac{\Gamma(-i\lambda)}{2^{5/2} \pi^{D/2+3/2}}. \quad (3.3.16)$$

Comparing this with (3.3.7), we identify

$$V_\Delta(p) = N_\lambda i^{-\Delta} \tilde{\mathcal{O}}_\Delta(p). \quad (3.3.17)$$

Note that $V_{D-\Delta}$ carries KK charge λ , and hence V_Δ carries KK charge $-\lambda$.

Contribution from the neighborhood of $k = p$

We now turn to the collinear contribution from the neighborhood of $k = p$ in the integral (3.3.9). Assuming that k is close to p (so that $p_- \cdot k \neq 0$), we introduce the change of variables

$$r = \frac{r'}{\omega e^{-t} (-p_- \cdot k/2)}. \quad (3.3.18)$$

Then the integral (3.3.9) becomes

$$\begin{aligned} \psi_\lambda^\Delta(x) &\sim e^{-(D-\Delta)t} \frac{1}{2(2\pi)^{D+3/2}} \int \frac{dr' d^D k d\omega}{r'} r'^{D-\Delta} e^{-\epsilon r'} \omega^{\Delta-1} (-p_- \cdot k/2)^{-(D-\Delta)} \\ &\quad \times \left[a(\omega k) \exp \left(-ir' e^{2t} \frac{p \cdot k}{p_- \cdot k} + ir' \right) \right]. \end{aligned} \quad (3.3.19)$$

We further introduce the variable

$$z^2 = -e^{2t} \frac{p \cdot k}{p_- \cdot k}. \quad (3.3.20)$$

The variable z can be written in terms of the angle ϕ between $\mathbf{k}$ and $\mathbf{p}$ as

$$z = e^t \tan \frac{\phi}{2}. \quad (3.3.21)$$

Under this change of variables, the measure $d^D k$ becomes

$$d^D k = \sin^{D-1} \phi d\phi d\Omega_{D-1} = 2^D e^{-Dt} z^{D-1} (1 + e^{-2t} z^2)^{-D} dz d\Omega_{D-1}, \quad (3.3.22)$$

and (3.3.19) can be rewritten as

$$\begin{aligned} \psi_\lambda^\Delta(x) \sim e^{-\Delta t} \frac{2^D}{2(2\pi)^{D+3/2}} \int \frac{dr' dz d\omega d\Omega_{D-1}}{r'} r'^{D-\Delta} e^{-\epsilon r'} \omega^{\Delta-1} z^{D-1} (1 + e^{-2t} z^2)^{-\Delta} \\ \times [a(\omega k) \exp(-ir' z^2 + ir')] , \end{aligned} \quad (3.3.23)$$

where k is now a function of z and the remaining angular variables.

Expanding in powers of $e^{-2t} z^2 \ll 1$ and keeping the leading term in which k localizes to $\phi = 0$, we obtain

$$\psi_\lambda^\Delta(x) \sim e^{-\Delta t} \frac{2^D \Omega_{D-1}}{2(2\pi)^{D+3/2}} \int \frac{dr' dz d\omega}{r'} r'^{D-\Delta} e^{-\epsilon r'} \omega^{\Delta-1} z^{D-1} [a(\omega p) \exp(-ir' z^2 + ir')] , \quad (3.3.24)$$

where Ω_{D-1} is the volume of the $(D-1)$ -dimensional unit sphere.¹⁶

Finally, performing the integrals over r' and z and using the celestial dictionary (3.0.13), we find

$$\psi_\lambda^\Delta(x) \sim e^{-\Delta t} N_\lambda i^{-\Delta} \mathcal{O}_\Delta(p). \quad (3.3.25)$$

Comparing this with (3.3.7), we obtain the identification

$$W_\Delta(p) = N_\lambda i^{-\Delta} \mathcal{O}_\Delta(p). \quad (3.3.26)$$

Note that W_Δ carries KK charge λ , and hence $W_{D-\Delta}$ carries KK charge $-\lambda$.

The $t \rightarrow -\infty$ relations can be derived in exactly the same way. An explicit computation shows that the final expressions coincide with those obtained above; this equality reflects the fact that our definition of the cosmological operators (3.3.7) is compatible with the crossing symmetry of flat-space scattering.

3.4 Consistency Check for Correlation Functions

We have established an identification rule between the extrapolated operators V_Δ, W_Δ and the celestial operators $\mathcal{O}_\Delta$. In this section, we independently compute the late/early cosmological correlators in dS and the celestial amplitudes, and verify that they agree, thereby providing a consistency check of our identification rule.

¹⁶We define $\Omega_0 = 2$ for $D = 1$.

3.4.1 Two-Point Functions

In this subsection, we evaluate the two-point functions in general dimensions.

Example: $\langle W_1 V_2 \rangle$

We first verify that

$$\langle W_1 V_2 \rangle = i^{-\Delta_1} i^{-\Delta_2} N_{\lambda_1} N_{\lambda_2} \langle \mathcal{O}_1 \tilde{\mathcal{O}}_2 \rangle. \quad (3.4.1)$$

Here, O_n is a shorthand for $O_{\Delta_n}(p_n)$.

From the celestial holography dictionary, the following two-point function is obtained:

$$\langle \mathcal{O}_{\Delta_1} \mathcal{O}_{D-\Delta_2} \rangle = 2(2\pi)^{D+2} \delta(\lambda_1 - \lambda_2) \delta^{(D)}(p_1 + p_2). \quad (3.4.2)$$

Applying the shadow transform and including all relevant normalization factors, the right-hand side of (3.4.1) becomes

$$\begin{aligned} & i^{-\Delta_1} i^{-\Delta_2} N_{\lambda_1} N_{\lambda_2} \langle \mathcal{O}_1 \tilde{\mathcal{O}}_2 \rangle \\ &= i^{-2\Delta_1} \frac{\Gamma(\Delta_1) \Gamma(-i\lambda_1)}{4\pi^{D/2+1}} \delta(\lambda_1 - \lambda_2) (2p_1 \cdot p_2)^{-\Delta_1}. \end{aligned} \quad (3.4.3)$$

This agrees with the late/early two-point function $\langle W_1 V_2 \rangle$ in dS, computed in Appendix B.4 [see (B.4.14)].

Example: $\langle W_1 W_2 \rangle$

Next, we examine the correlator

$$\langle W_1 W_2 \rangle = i^{-\Delta_1} i^{-\Delta_2} N_{\lambda_1} N_{\lambda_2} \langle \mathcal{O}_1 \tilde{\mathcal{O}}_2 \rangle. \quad (3.4.4)$$

Using (3.4.2), the right-hand side evaluates to

$$\begin{aligned} & i^{-\Delta_1} i^{-\Delta_2} N_{\lambda_1} N_{\lambda_2} \langle \mathcal{O}_1 \tilde{\mathcal{O}}_2 \rangle \\ &= 2^D i^{-D} \frac{\Gamma(i\lambda_1) \Gamma(-i\lambda_1)}{4\pi} \delta(\lambda_1 + \lambda_2) \delta^{(D)}(p_1 + p_2). \end{aligned} \quad (3.4.5)$$

This expression again matches the corresponding two-point function in dS [see (B.4.14)]. Although the appearance of a pure delta-function (a contact term) may seem unusual at first sight, it is known [86, 87, 88] that such contact terms arise in late-late two-point functions. In Appendix B.4, we also verify that similar contact terms show up in the late-early correlators.

3.4.2 Four-Point Function ($D = 2$, ϕ^4 interaction)

In this subsection, we study the tree-level four-point function in $D = 2$ for a theory with a ϕ^4 interaction.

Example: $\langle W_1 W_2 W_3 W_4 \rangle$

We aim to show that

$$\langle W_1 W_2 W_3 W_4 \rangle = i^{-\sum \Delta_i} N_{\lambda_1} N_{\lambda_2} N_{\lambda_3} N_{\lambda_4} \langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4 \rangle. \quad (3.4.6)$$

Left-hand side The left-hand side (LHS) is computed from the QFT with interaction (3.1.15) on S^{D+1} , using the analytic continuation described in section 3.2. The external bulk-to-boundary legs in dS are obtained from those on the sphere by the continuation (3.2.3). Moreover, by deforming the path-integral contour, the interaction vertex is pushed into the region $\text{EAdS}^+ + \text{dS} + \text{EAdS}^-$ as in (3.2.5), and we can apply the deformation formula (B.2.9) when carrying out this contour deformation. Performing these steps, we obtain¹⁷

$$\begin{aligned} \langle W_1 W_2 W_3 W_4 \rangle = & -i\alpha_4 2\pi \delta\left(\sum_i \lambda_i\right) \left(\int_{\text{EAdS}^+} [d^3 y] + \int_{\text{EAdS}^-} [d^3 y] + \int_{\text{dS}} [d^3 y] \right) \\ & \times \prod_{i=1}^4 \frac{1}{\sqrt{2\pi}} \frac{2^{\Delta_i} \Gamma(1 - \Delta_i) \Gamma(\Delta_i)}{4\pi^2} [-2p_i \cdot x + i\epsilon]^{-\Delta_i}, \end{aligned} \quad (3.4.7)$$

where $x := -iy$ if $y \in \text{EAdS}$ and $x := y$ if $y \in \text{dS}$, and α_4 is the coupling constant defined in (3.1.15) and the overall factor of λ follows from the conservation of the KK charge.

As already emphasized in Sec. 3.2, the contribution from EAdS appears solely as a technical device to prepare the Euclidean vacuum. It does *not* mean that we are working with a disconnected spacetime obtained by gluing EAdS and dS. In our setup, EAdS arises only as the region into which the integration contour for internal vertices is deformed. The EAdS integrals are therefore to be regarded as a calculational artifact of the Euclidean construction in dS, and all physical statements should be understood as referring to dS spacetime itself.

Right-hand side The corresponding Lorentzian four-point celestial amplitude is obtained by applying a Mellin transform to the tree-level amplitude in four-dimensional Minkowski spacetime, which itself is reached by analytic continuation of the Euclidean correlator. In performing the angular integration over the internal vertex, we can again make use of the contour-deformation formula (B.2.9) appearing in the relevant equations.

¹⁷The prefactor i^{-D-2} multiplying $\int_{\text{EAdS}}$ equals 1 in the present case $D = 2$.

$$\begin{aligned}
\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4 \rangle &= -i\alpha_4 \int dr r^3 \left(\int_{\text{EAdS}^+} [d^3 y] + \int_{\text{EAdS}^-} [d^3 y] + \int_{\text{dS}} [d^3 y] \right) \\
&\quad \times \int \left(\prod_{i=1}^4 d\omega_i \omega_i^{\Delta_i-1} e^{-\epsilon \omega_i} e^{-i\omega_i r p_i \cdot x} \right) \\
&= -i\alpha_4 2\pi \delta \left(\sum_i \lambda_i \right) \left(\int_{\text{EAdS}^+} [d^3 y] + \int_{\text{EAdS}^-} [d^3 y] + \int_{\text{dS}} [d^3 y] \right) \\
&\quad \times \prod_{i=1}^4 (2i)^{\Delta_i} \Gamma(\Delta_i) [-2p_i \cdot x + i\epsilon]^{-\Delta_i}.
\end{aligned} \tag{3.4.8}$$

Here, to obtain the second line, we performed the change of variables $\omega_i \rightarrow \omega_i/r$.

Matching of both sides Multiplying (3.4.8) by the overall prefactors, we find

$$\begin{aligned}
i^{-\sum \Delta_i} N_{\lambda_1} N_{\lambda_2} N_{\lambda_3} N_{\lambda_4} \langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4 \rangle &= -i\alpha_4 2\pi \delta \left(\sum_i \lambda_i \right) \left(\int_{\text{EAdS}^+} [d^3 y] + \int_{\text{EAdS}^-} [d^3 y] + \int_{\text{dS}} [d^3 y] \right) \\
&\quad \times \prod_{i=1}^4 2^{\Delta_i} N_{\lambda_i} \Gamma(\Delta_i) [-2p_i \cdot x + i\epsilon]^{-\Delta_i}.
\end{aligned} \tag{3.4.9}$$

Using

$$2^{\Delta_i} N_{\lambda_i} \Gamma(\Delta_i) = \frac{1}{\sqrt{2\pi}} \frac{2^{\Delta_i} \Gamma(1 - \Delta_i) \Gamma(\Delta_i)}{4\pi^2}, \tag{3.4.10}$$

we see that the right-hand side indeed reproduces the LHS in (3.4.7), and thus (3.4.6) is verified.

Example: $\langle W_1 V_2 W_3 W_4 \rangle$

In the previous example (3.4.6), no contribution from the shadow conformal primary wave function appeared. We now turn to a configuration for which such a contribution is present. More precisely, we replace the second operator in the previous example and consider

$$\langle W_1 V_2 W_3 W_4 \rangle = i^{-\sum \Delta_i} N_{\lambda_1} N_{\lambda_2} N_{\lambda_3} N_{\lambda_4} \left\langle \mathcal{O}_1 \tilde{\mathcal{O}}_2 \mathcal{O}_3 \mathcal{O}_4 \right\rangle. \tag{3.4.11}$$

First, the expression on the LHS of (3.4.11) is

$$\begin{aligned}
\langle W_1 V_2 W_3 W_4 \rangle &= -i\alpha_4 2\pi \delta(\lambda_1 - \lambda_2 + \lambda_3 + \lambda_4) \left(\int_{\text{EAdS}^+} [d^3 y] + \int_{\text{EAdS}^-} [d^3 y] + \int_{\text{dS}} [d^3 y] \right) \\
&\quad \times \prod_{i=1}^4 \frac{1}{\sqrt{2\pi}} \frac{2^{\Delta_i} \Gamma(1 - \Delta_i) \Gamma(\Delta_i)}{4\pi^2} [-2p_i \cdot x + i\epsilon]^{-\Delta_i}.
\end{aligned} \tag{3.4.12}$$

Next, we evaluate the RHS of (3.4.11). Applying the shadow transform with respect to p_2 , and using (3.0.9) on both sides of (3.4.8), we obtain the desired expression.

$$\begin{aligned}
& \langle \mathcal{O}_1 \tilde{\mathcal{O}}_2 \mathcal{O}_3 \mathcal{O}_4 \rangle \\
&= -i\alpha_4 2\pi\delta(\lambda_1 - \lambda_2 + \lambda_3 + \lambda_4) \left(\int_{\text{EAdS}^+} [d^3y] + \int_{\text{EAdS}^-} [d^3y] + \int_{\text{dS}} [d^3y] \right) \\
&\times (2i)^{D-\Delta_2} \Gamma(D - \Delta_2) \frac{1}{S_{D-\Delta_2}} \frac{\pi^h \Gamma(-i\lambda_2)}{\Gamma(D - \Delta_2)} (-1 + i\epsilon)^{i\lambda_2} [-2p_2 \cdot x + i\epsilon]^{-\Delta_2} \prod_{i=1,3,4} (2i)^{\Delta_i} \Gamma(\Delta_i) [-2p_i \cdot x + i\epsilon]^{-\Delta_i} \\
&= -i\alpha_4 2\pi\delta(\lambda_1 - \lambda_2 + \lambda_3 + \lambda_4) \left(\int_{\text{EAdS}^+} [d^3y] + \int_{\text{EAdS}^-} [d^3y] + \int_{\text{dS}} [d^3y] \right) \\
&\times \prod_{i=1}^4 (2i)^{\Delta_i} \Gamma(\Delta_i) [-2p_i \cdot x + i\epsilon]^{-\Delta_i}. \tag{3.4.13}
\end{aligned}$$

Comparing this with the dS expression above, we see that our correspondence continues to hold in this case as well.

At this stage, one might be tempted to simply replace an insertion of $\mathcal{O}_{D-\Delta}$ by an insertion of $\tilde{\mathcal{O}}_{D-\Delta}$. However, an important difference is that such a replacement flips the sign of the argument of the delta function from $-\lambda$ to λ . This provides an alternative way to understand why the operator corresponding to $W_{D-\Delta}$ must be identified with $\mathcal{O}_{D-\Delta}$, while the operator corresponding to $V_{D-\Delta}$ must instead be $\tilde{\mathcal{O}}_{D-\Delta}$.

3.5 Discussion

In this section, we summarize the implications suggested by the framework developed in this paper.

3.5.1 celestial stress tensor vs. Geroch tensor

The purpose of this subsection is to clarify the relation between the (boundary) stress tensor in de Sitter (dS) spacetime and the celestial stress tensor / Geroch tensor in celestial CFT. In the conventional slicing-based construction of celestial holography, essentially only the quantity known as the Geroch tensor, ρ_{zz} , tends to become manifest. As a result, it has been difficult to treat the celestial stress tensor as an *intrinsic* observable of the dS theory defined via slicing [33, 89]. However, within the formalism developed in this paper, we will see that the (formally) massless mode responsible for the celestial stress tensor does exist in the dimensionally reduced theory, and moreover can be defined and identified intrinsically.

Below, we first explain why, from the (A)dS perspective, the Geroch tensor ρ_{zz} alone is easily (mis)interpreted as the only viable candidate for a boundary stress tensor, and consequently why a naive zero-mode truncation that discards the ξ dependence necessarily breaks down. We then show

that, in our formalism, two “massless-like” spin-2 sectors—the $\mathfrak{h}$ -sector and the h -sector—coexist in the dS theory, and in particular that the latter corresponds to the celestial stress tensor generating the correct Ward identities. Finally, we formulate a new question that arises naturally in our framework, and point out critical NMG as a useful toy model for understanding the resulting behavior.

Setup (4D Euclidean gravity in radial gauge)

We begin with four-dimensional Euclidean Einstein gravity,

$$S = \frac{1}{2} \int [d^4 X] \mathcal{R}[G], \quad (3.5.1)$$

where $\mathcal{R}[G]$ is the 4D Ricci scalar.

We introduce the radial-gauge metric along the ξ direction :

$$G_{IJ} dX^I dX^J = e^{2\xi} \left(d\xi^2 + g_{ij}(\xi, x) dx^i dx^j \right). \quad (3.5.2)$$

Using the standard ADM decomposition, the action can be written as

$$S = \frac{1}{2} \int d\xi e^{2\xi} \int [d^3 x] \sqrt{g} \left(R[g] - 2 + K^2 - K_{ij} K^{ij} \right), \quad (3.5.3)$$

where $R[g]$ is the 3D Ricci scalar and

$$K_{ij} = \frac{1}{2} \partial_\xi g_{ij}, \quad K = g^{ij} K_{ij} \quad (3.5.4)$$

is the extrinsic curvature of constant- ξ slices. Note that while ξ is a radial coordinate from the 4D viewpoint, from the 3D viewpoint (in particular, after analytic continuation to dS) it should be interpreted as a mode label.

Confusion caused by the presence of the Geroch tensor ρ_{zz}

Here we explain where the naive slicing construction can be misleading. The discussion below is a rewrite, in our notation, of similar arguments in the existing literature [33, 89]. The underlying issue has already been recognized in [60]; our goal here is to make it more explicit.

We decompose the 4D metric as

$$G_{IJ} dX^I dX^J = \eta_{IJ} dX^I dX^J + H_{IJ} dX^I dX^J, \quad (3.5.5)$$

where η_{IJ} is the Euclidean flat metric and H_{IJ} is a perturbation satisfying radial gauge. In polar coordinates, the metric takes the form

$$G_{IJ} dX^I dX^J = e^{2\xi} \left(d\xi^2 + d\theta^2 + 2 \cos^2 \theta \gamma_{z\bar{z}} dz d\bar{z} + e^{-2\xi} H_{ij}(\xi, t, z, \bar{z}) dx^i dx^j \right), \quad (3.5.6)$$

where $\gamma_{z\bar{z}}$ is the S^2 metric in complex coordinates used e.g. in [17].

Next, analytically continuing $\theta = \frac{\pi}{2} + it$ yields the Minkowski metric in the Milne patch. We then compare this with the AFS metric in Bondi gauge (see eq. (2.4.1)) rewritten in radial gauge^{18,19}

$$G_{IJ}dX^I dX^J = -e^{2\xi}(d\xi^2 - dt^2 + 2\cosh^2 t \gamma_{z\bar{z}} dz d\bar{z} + e^{-\xi} \cosh t C_{zz} dz^2 + e^{-\xi} \cosh t C_{\bar{z}\bar{z}} d\bar{z}^2) + \mathcal{O}(R^0), \quad (3.5.7)$$

where we perform the coordinate change

$$r_{\text{Bondi}} \rightarrow e^\xi \cosh t, \quad u \rightarrow -e^\xi e^{-t}. \quad (3.5.8)$$

From this comparison, we obtain the relation between H_{zz} and the Bondi shear C_{zz} ,

$$H_{zz}(\xi, t, z, \bar{z}) = e^\xi \cosh t C_{zz}(u, z, \bar{z}) + \mathcal{O}(R^0). \quad (3.5.9)$$

Expanding in u , and extracting leading term of u , we find

$$H_{zz}^{(1)}(\xi, t, z, \bar{z}) = e^{2\xi} e^{-t} \cosh t \rho_{zz}(z, \bar{z}) + \mathcal{O}(R^0), \quad (3.5.10)$$

where we used the decomposition (2.4.14). In the limit $t \rightarrow \infty$, this becomes

$$H_{zz}^{(1)}(\xi, t, z, \bar{z}) = \frac{1}{2} e^{2\xi} \rho_{zz}(z, \bar{z}) + \mathcal{O}(R^0). \quad (3.5.11)$$

Thus, in radial gauge the metric behaves asymptotically in $t \rightarrow \infty$ as

$$G_{IJ}dX^I dX^J \sim e^{2\xi} \left(d\xi^2 - dt^2 + 2e^{2t} \gamma_{z\bar{z}} dz d\bar{z} + \frac{1}{2} \rho_{zz} dz^2 + \frac{1}{2} \rho_{\bar{z}\bar{z}} d\bar{z}^2 \right), \quad (3.5.12)$$

so that the part one may naively interpret as a dS₃ metric is

$$g_{ij}dx^i dx^j \sim -dt^2 + 2e^{2t} \gamma_{z\bar{z}} dz d\bar{z} + \frac{1}{2} \rho_{zz} dz^2 + \frac{1}{2} \rho_{\bar{z}\bar{z}} d\bar{z}^2. \quad (3.5.13)$$

At this stage, it is tempting to identify the Geroch tensor ρ_{zz} with the (traceless part of the) boundary stress tensor in dS, following the dS/CFT dictionary [65].

A natural next step along the naive “ ρ as a stress tensor” intuition is to discard the ξ dependence and keep only the zero mode [33],

$$g_{ij}(\xi, x) \rightarrow g_{ij}(x). \quad (3.5.14)$$

¹⁸Strictly speaking, radial gauge is not preserved at subleading order, but this does not affect the present discussion, so we omit the compensating diffeomorphism that would restore radial gauge exactly.

¹⁹Here, in order to be consistent with the conventions adopted in this chapter, we have generalized the boundary metric from the flat metric to a general one.

Thus, the resulting toy-model 3D reduced action from (3.5.3)

$$S = \frac{a}{2} \int d^3x \sqrt{g} (R[g] - 2), \quad a = \int_{\xi'_c}^{\xi_c} d\xi e^{2\xi} \quad (3.5.15)$$

depends strongly on the IR cutoff through a , and typically diverges. Estimating the central charge via $c \sim \frac{3\ell}{2G_3}$ (with $\ell = i$ and $G_3 \sim G_4/a$) then leads to a divergent central charge. This is incompatible with the features expected for the celestial stress tensor from double soft limit [90, 91].

The crucial point is that this truncation retains only the geometry involving ρ . As we saw in section 2.4.2, ρ should be interpreted, in celestial CFT, not as the stress tensor itself but rather as a *source* or *Goldstone mode*²⁰ that couples to the stress tensor. Consequently, this truncation systematically discards the degrees of freedom responsible for the celestial Ward identities, and therefore fails to capture the celestial stress tensor as a conserved quantity at the boundary. This makes explicit issue in the conventional slicing construction.

Two massless-like spin-2 sectors

In our framework, the “massless” (more precisely, massless-like) modes that appear in the dS theory after dimensional reduction are not unique. Concretely, while the r^2 (i.e. $e^{2\xi}$) component that asymptotes to the boundary metric provides a “dS-natural” massless-like mode $\mathfrak{h}_{ij}$ ²¹, an independent r^0 (i.e. $e^{0\xi}$) component can also survive as another massless-like mode h_{ij} . It is the latter that asymptotes to the “celestial stress tensor” that generates the correct Ward identities in celestial holography.

We define these two modes as Mellin projections of $H_{ij}(r, x)$:²²

$$\mathfrak{h}_{ij}(x) := \lim_{\Delta \rightarrow 2} (\Delta - 2) \int_0^\infty \frac{dr}{r} r^{-\Delta} H_{ij}(r, x), \quad (3.5.16)$$

$$h_{ij}(x) := \lim_{\Delta \rightarrow 0} \Delta \int_0^\infty \frac{dr}{r} r^{-\Delta} H_{ij}(r, x). \quad (3.5.17)$$

Below we verify, at least at the level of fall-offs, that these behave as massless-like modes.

Using

$$H_{ij} = \frac{\partial X^I}{\partial x^i} \frac{\partial X^J}{\partial x^j} H_{IJ} \quad (3.5.18)$$

²⁰By introducing edge modes, the degrees of freedom fixed by Dirichlet boundary conditions can be relaxed, allowing one to construct gauge-invariant Goldstone modes as observables (see appendix B.5).

²¹In the wedge-holography construction [35], it may be interpreted as a graviton localized on the end-of-the-world (EOW) brane inserted deep in the bulk.

²²Strictly speaking, these integrals should be defined by computing in a domain of convergence and then analytically continuing. Here we use a compact notation to denote the corresponding Mellin projections.

together with $X^I = rx^I$, we have in the limit $t \rightarrow \infty$ that

$$\frac{\partial x^I}{\partial x^i} \longrightarrow e^t \frac{\partial p^I}{\partial x^i} \quad (t \rightarrow \infty), \quad (3.5.19)$$

where we use the parametrization (3.3.6). Therefore $\mathfrak{h}_{ij}$ and h_{ij} can be written as following :

$$\mathfrak{h}_{ij}(x) \sim e^{2t} \left(\frac{\partial p^I}{\partial x^i} \frac{\partial p^J}{\partial x^j} \right) \lim_{\Delta \rightarrow 2} (\Delta - 2) \int_0^\infty \frac{dr}{r} r^{2-\Delta} H_{IJ}(r, x), \quad (3.5.20)$$

$$h_{ij}(x) \sim e^{2t} \left(\frac{\partial p^I}{\partial x^i} \frac{\partial p^J}{\partial x^j} \right) \lim_{\Delta \rightarrow 0} \Delta \int_0^\infty \frac{dr}{r} r^{2-\Delta} H_{IJ}(r, x). \quad (3.5.21)$$

From the discussion in section 3.3, these have asymptotic components proportional to e^{2t} and to $e^{0 \cdot t}$. Thus, at the level of fall-offs, both can be regarded as massless spin-2 modes. In particular, the $e^{0 \cdot t}$ fall-off is naturally compatible with an interpretation in terms of a stress-tensor operator from the viewpoint of conformal dimension.²³

Moreover, one can identify the corresponding celestial operators by the same logic as in section 3.3. Focusing in particular on the $e^{0 \cdot t}$ component that can correspond to a stress tensor, one finds schematically

$$\mathfrak{h}_{zz} \sim \lim_{\Delta \rightarrow 2} (\Delta - 2) \mathcal{O}_{zz, \Delta}, \quad (3.5.22)$$

$$h_{zz} \sim \lim_{\Delta \rightarrow 0} \Delta \tilde{\mathcal{O}}_{zz, \Delta}. \quad (3.5.23)$$

Here the fall-off component $\lim_{\Delta \rightarrow 0} \Delta \mathcal{O}_{zz, \Delta}$ of $\mathfrak{h}_{zz}$ corresponds to the Geroch tensor (superrotation goldstone mode) ρ [92], in agreement with the naive discussion above. The important point is that the asymptotic mode $\lim_{\Delta \rightarrow 0} \Delta \tilde{\mathcal{O}}_{zz, \Delta}$ associated with h is precisely the celestial stress tensor T that generates the correct Ward identities.²⁴ Thus, the bulk mode h_{ij} corresponding to the *physical* stress tensor (i.e. the one responsible for the correct Ward identities and the central term) is in fact present in the dimensionally reduced theory, although it tends to be missed in the conventional discussion.

Intuitively, this parallels the scalar example where a “massless mode” arises from two analytic continuations corresponding to $\lambda = \pm iD/2$ (see (3.1.8)). Of course, as long as the λ integral runs along the real axis, these values do not appear in the spectrum. However, if one can justify rotating the contour into the imaginary direction by analytic continuation, the procedure itself is not necessarily unreasonable. That said, substituting $\lambda = \pm iD/2$ into the action leads, after diagonalization, to a wrong sign of the kinetic term, indicating the appearance of ghosts. How the consistency of the full theory is maintained is left for future work.

²³Here “massless” is an operational terminology based on the dS_3 fall-offs and the structure of the Ward identities.

²⁴This expression follows by rewriting the definition of the celestial stress tensor in (2.4.51) in Mellin space; see e.g., [92] for details.

For essentially the same reason,²⁵ it is natural to expect that in gravity as well two independent massless-like sectors can arise. The intuition suggested here is that, in addition to the possibility of analytic continuation enabled by the presence of a continuous spectrum, the induced mass shift by $D^2/4$ by weyl transformation plays a key role: together these effects allow two “massless” components to emerge, even though a naive dimensional reduction would yield only one.

Our claim: an apparent tension and a new question

Let us summarize the claim. Since $\mathfrak{h}_{ij}$ cancels the prefactor $e^{-2\xi}$ in (3.5.6), it is easy to think of it as the unique massless mode in the dS metric fluctuations:

$$g_{ij}^{(\text{dS})} = \bar{g}_{ij}^{(\text{dS})} + \mathfrak{h}_{ij} \quad (\text{schematically}). \quad (3.5.24)$$

This makes it tempting to regard $\mathfrak{h}$ as the bulk mode corresponding to the boundary stress tensor of the dS theory.

On the other hand, the correct Ward identities are tied to h_{ij} . At first sight, one might therefore feel a tension:

1. the massless mode $\mathfrak{h}_{ij}$ appears massless but does not correspond to the boundary stress tensor, whereas
2. h_{ij} yields the “physical stress tensor” but is not manifest on the dS side.

In our viewpoint, however, this tension arises only from the assumption that $\mathfrak{h}$ is the *only* massless spin-2 mode in the dimensionally reduced dS theory. In reality, the reduced theory contains not only $\mathfrak{h}_{ij}$ but also another massless-like spin-2 mode h_{ij} .

The new question is then: *why, among the two massless-like spin-2 modes, does only one (the h -sector) correspond to the boundary stress tensor (i.e. the one that generates the correct Ward identities)?* A tentative answer is that the “masslessness” of $\mathfrak{h}$ is sensitive to the IR cutoff (or to the choice of boundary conditions), and for a finite cutoff the exact symmetry may be lost so that $\mathfrak{h}$ ceases to be truly massless (see related work [93]). The picture in which both modes appear simultaneously massless-like should be regarded as an approximation valid for a sufficiently large IR cutoff, while in practice a tiny mass generation may prevent $\mathfrak{h}$ from being associated with a conserved boundary charge.

Critical NMG as a toy-model

Finally, we comment that in (A)dS₃ there exist explicit models in which two massless-like spin-2 modes are present while only a single boundary stress tensor appears. A candidate that closely

²⁵This analogy is, of course, not perfect, since in three dimensions massless spin-2 fields do not carry propagating degrees of freedom.

matches the above qualitative requirement—namely, two massless-like spin-2 modes whose boundary stress tensor (carrying the central term) effectively combines into a single object—is New Massive Gravity (NMG) [94] at its critical point. In the auxiliary-field formulation,

$$S_{\text{NMG}}[g, f] = \frac{1}{16\pi G_3} \int d^3x \sqrt{g} \left[\sigma \left(R + \frac{2}{\ell^2} \right) + f^{ij} G_{ij}(g) - \frac{m^2}{4} (f_{ij} f^{ij} - f^2) \right]. \quad (3.5.25)$$

For AdS_3 , the dual central charge is

$$c = \frac{3\ell}{2G_3} \left(\sigma + \frac{1}{2m^2\ell^2} \right). \quad (3.5.26)$$

Generically, away from the critical point, a massless sector corresponding to the boundary graviton coexists with a massive spin-2 mode propagating in the bulk. At the critical point ($c=0$), however, the linearized spectrum can be described as exhibiting “two massless (massless-like) spin-2 modes” (equivalently, a degeneracy).²⁶

The dual CFT is expected to become a log CFT at the critical point (see e.g. [95]). Since celestial holography suggests a vanishing central charge, such a critical NMG-like structure appears consistent with our expectations. It is also intriguing that this is compatible with indications from purely celestial analyses (without invoking slicing) that the boundary theory may be a log CFT [96].

3.5.2 Cosmological correlators from celestial amplitudes

Up to this point, we have seen that there is a relatively simple correspondence between cosmological correlators and celestial correlators. In this subsection, we discuss the possibility of treating cosmological correlators as scattering amplitudes in Minkowski spacetime by exploiting this correspondence.

In general, even apart from holography, celestial correlators are often easier to handle than cosmological correlators, since one only needs to compute scattering amplitudes and then perform a Mellin transform. Moreover, the theoretical framework of scattering amplitudes is already highly developed: in flat spacetime, hidden structures that are not manifest at the Lagrangian level—such as BCFW recursion and the double copy—have been uncovered (see Ref. [97] and references therein). If such structures could be transplanted into the de Sitter context, they might also provide new insights for the cosmological bootstrap.

This naturally raises the question of whether cosmological correlators themselves can be treated as scattering amplitudes in Minkowski spacetime. In our previous discussion, we started from an action in Minkowski spacetime and derived the de Sitter action by a “dimensional reduction.” From this perspective, it is tempting to reverse the logic: start from the de Sitter action, uplift it to an action in Minkowski spacetime, and compute cosmological correlators as scattering amplitudes.

²⁶More precisely, this degeneracy arises from a double root of the linearized equations of motion, and is accompanied by the appearance of a logarithmic mode (a log partner).

However, this approach does not work in full generality. The reason is that only a very limited class of de Sitter theories can be interpreted as bona fide Minkowski theories with the required symmetries such as Poincaré invariance, and such theories are expected to be essentially restricted to those with a continuous mass spectrum, as in the examples studied in this paper.

What we would like to emphasize here is that, even without literally reversing the construction, it may still be possible to start from a flat-space theory and compute nontrivial cosmological correlators. As an example, we consider Yang–Mills (YM) theory on dS_3 , and explain a procedure for computing the tree-level cosmological correlators of YM theory within the scattering-amplitude framework.

First, four-dimensional Euclidean YM theory can be written as

$$S = \frac{1}{2} \int_{\mathbb{R}^4} [d^4 X] \operatorname{tr}(F_{IJ} F^{IJ}). \quad (3.5.27)$$

Applying to this action the same manipulations used in the previous section, and further organizing the result following Ref. [33] (and [98]), we arrive at the following three-dimensional action:

$$S = S_0 + S_{\text{int}}. \quad (3.5.28)$$

The free part S_0 is given by

$$S_0 = \int_{S^3} [d^3 x] \int_{-\infty}^{\infty} d\lambda \operatorname{tr} \left[-\frac{1}{2} f_{ik, -\lambda}^{(0)} f_{\lambda}^{(0), ik} + \lambda^2 a_{i, -\lambda} a_{\lambda}^i \right], \quad (3.5.29)$$

where $f_{ij}^{(0)} = \partial_i a_j - \partial_j a_i$ is the linearized field strength. We work in the radial gauge $A_r = 0$.²⁷ The interaction part reads

$$S_{\text{int}} = \int_{S^3} [d^3 x] \int [d\lambda] \left[-\delta \left(\sum_{i=1}^3 \lambda_i \right) f^{abc} \partial_i A_{j, \lambda_1}^a A_{\lambda_2}^{b, i} A_{\lambda_3}^{c, j} \right. \\ \left. - \frac{1}{4} \delta \left(\sum_{i=1}^4 \lambda_i \right) f^{abe} f^{cde} A_{\mu, \lambda_1}^a A_{\nu, \lambda_2}^b A_{\lambda_3}^{c\mu} A_{\lambda_4}^{d\nu} \right], \quad (3.5.30)$$

where λ is a global KK charge (a $U(1)$ charge). By analytic continuation, these theories give rise to a theory in four-dimensional Minkowski spacetime and a three-dimensional theory defined on dS_3 .

The resulting three-dimensional theory on de Sitter space contains, in addition to the YM field, an infinite tower of massive colored complex vector fields. Formally, this theory is equivalent to

²⁷If the r -direction were compact, this gauge choice would not be justified for the zero mode due to a nontrivial holonomy. Even in our non-compact setup, this gauge fixing may fail to be justified once one takes into account that the asymptotic symmetry group can act as a physical symmetry. In that case, an adjoint scalar corresponding to A^r would have to be included in the three-dimensional theory.

YM theory on Minkowski spacetime, but in this form it does not exhibit an especially transparent physical structure.

The crucial point is that if we restrict the external legs of the cosmological operators to $\lambda = 0$, namely to the modes corresponding to the YM field, then the massive colored vector fields do not contribute to internal lines due to $U(1)$ charge conservation.²⁸ Therefore, at tree level, cosmological correlators can be computed within a closed sector consisting solely of the YM field, without ever including the massive colored vectors.

Equivalently, if one wishes to compute the cosmological correlators defined by YM theory on three-dimensional de Sitter space,

$$S_0 = \frac{1}{2} \int_{S^3} [d^3x] \operatorname{tr} \left[-\frac{1}{2} f_{ik} f^{ik} \right], \quad (3.5.31)$$

one may proceed as follows: introduce a large number of auxiliary massive vector fields, uplift the theory to four-dimensional Minkowski spacetime, compute the corresponding celestial amplitudes, and then take the $\lambda = 0$ limit of the external legs (equivalently, the $\Delta = 1$ limit). In the context of celestial holography, this limit corresponds to a multi-conformal soft limit. Moreover, in this limit the distinction between in and out becomes ambiguous. Thus, schematically, one is led to the correspondence

$$\text{multi-conformal soft limit of YM celestial amplitudes} \sim (\text{in-in}) \text{ YM cosmological correlator}. \quad (3.5.32)$$

However, these conformal multi-soft limits are generally non-commutative [99]. Understanding what this non-commutativity corresponds to in the intrinsic dS_3 description is an interesting open problem, and we leave it to future work.

In this way, the existence of a closed sector enforced by the $U(1)$ charge conservation in λ suggests that certain de Sitter theories can be efficiently analyzed using scattering amplitudes, once one supplements them with additional particles with a continuous mass spectrum. From a conceptual standpoint, an important implication is that structures known from gravity/YM in Minkowski spacetime—such as BCFW recursion and the double copy—may also be realized, in some form, in gravity/YM on dS_3 . It is also interesting to understand how this viewpoint is related to the BCFW/double-copy structures developed independently in (A)dS contexts (see, e.g., [100, 101]).

In particular, explicit computations show that the $\lambda = 0$ sector of four-dimensional gravity corresponds, in de Sitter space, to a massive graviton with the special mass value of a so-called

²⁸Of course, this reasoning breaks down at loop level. In this sense, a similar limitation is already present in the usual scattering-amplitude story: for instance, in non-supersymmetric theories, loop effects are often difficult to control, and the simplifications at tree level need not persist.

partially massless graviton [98, 102]. Since it is well known that four-dimensional flat-space gravity and four-dimensional flat-space YM amplitudes are related by the double copy (BCJ relations), this suggests that the amplitudes of YM on dS_3 and those of partially massless gravitons on dS_3 might also be related by a sort of double-copy construction.

Chapter 4

Conclusion

In chapter 2, we reviewed how gravity already possesses a holographic character at a purely kinematical level, before entering dynamics, thereby providing motivation for considering holographic frameworks that are not restricted to AdS. We also organized the symmetry structure of gravity in asymptotically flat spacetimes (AFS) and presented an overview of celestial holography. To make these points transparent, we adopted a stepwise strategy. We first analyzed a $U(1)$ gauge theory with a boundary at finite location and then generalized this setup to gravity. We subsequently turned to a $U(1)$ theory in flat spacetime and, finally, extended the discussion to gravity in AFS and its celestial holographic description. Although the chapter is formally organized as a review of existing work, many results were rederived in the language of path integrals, and in several places we provided modest generalizations of known results.

In chapter 3, we began with a massless scalar field on flat space $\mathbb{R}^{D+2}$. By performing a Weyl transformation followed by a Fourier transform along the “radial” direction, we mapped this theory to a theory of massive scalar fields on the sphere S^{D+1} . Upon analytic continuation, the theory on S^{D+1} is mapped to a theory on dS_{D+1} in the Euclidean vacuum, while the original theory on $\mathbb{R}^{D+2}$ is mapped to a theory on Minkowski spacetime M_{D+2} . In this way, we obtained a correspondence between correlation functions in Minkowski spacetime and those in de Sitter space.

This correspondence provides a natural bridge between celestial holography and the dS/CFT correspondence. To explore this connection, we assumed the existence of a celestial dictionary and studied how the celestial operators $\mathcal{O}_\Delta$ on the sphere are related to the extrapolated operators V_Δ and W_Δ in de Sitter space. We found that these operators are related as

$$V_\Delta(p) = N_\lambda i^{-\Delta} \tilde{\mathcal{O}}_\Delta(p), \quad (4.0.1)$$

$$W_\Delta(p) = N_\lambda i^{-\Delta} \mathcal{O}_\Delta(p). \quad (4.0.2)$$

Furthermore, by independently computing celestial amplitudes and cosmological correlators, we explicitly verified that these relations hold. These identification rules thus yield a simple correspon-

dence between scattering amplitudes in Minkowski spacetime M_{D+2} and cosmological correlators in dS_{D+1} .

In celestial holography, the putative theory on S^D is assumed to be a conformal field theory whose primary operators are the celestial operators $\mathcal{O}_\Delta$. Our analysis suggests that this same CFT can simultaneously capture cosmological correlators in dS_{D+1} . In particular, the fact that V_Δ can be expressed in terms of the nonlocal shadow operator $\tilde{\mathcal{O}}_\Delta$ indicates a potential tension between celestial holography and the naive BDHM-type dS/CFT dictionary.¹

In addition, we discussed two further implications of our construction. First, our analysis suggests that there may be a subtle but conceptually important difference between the celestial stress tensor and the naive stress tensor appearing in dS/CFT. Second, our results indicate that cosmological correlators may encode deep mathematical structures—namely, nontrivial patterns already present in flat-space scattering amplitudes but obscured at the level of the Lagrangian description.

Independently of our approach, there have also been attempts to relate flat holography and AdS/CFT by taking the flat-space limit of AdS/CFT [103, 104, 105]. In addition, related connections have been explored by considering the flat-space limit of end-of-the-world branes in AdS/BCFT [106]. Moreover, recent developments [107, 108, 80, 109] have uncovered several bridges between de Sitter and anti-de Sitter space. Motivated by (A)dS/CFT, extrapolate-type dictionaries for celestial holography have also been proposed [76, 77, 79, 78].

By combining our results with these developments, it may eventually become possible to treat (A)dS/CFT and flat holography on comparable footing and to move toward a unified framework for holographic descriptions of quantum gravity, or, more broadly, toward a unified understanding of the mathematical structures underlying Yang–Mills and gravitational correlators/amplitudes in (A)dS and flat spacetimes.

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¹On the other hand, a more direct connection between our framework and more conventional dS/CFT dictionaries—such as the GKPW-type dictionary—requires further investigation.

Appendix A

Appendix to Chapter 2

In this chapter, we present the appendices to Chapter 2, which are intended to aid the reader.

A.1 Elitzur's theorem with boundary action

Elitzur's theorem [110] states that local gauge symmetries cannot be spontaneously broken in a gauge theory. In particular, the expectation value of any local operator that is not gauge invariant must vanish:

$$\langle \mathcal{O}[A] \rangle = 0 \quad \text{if } \mathcal{O}[A] \text{ is not gauge invariant.} \quad (\text{A.1.1})$$

Setup. Consider a pure $U(1)$ gauge theory with the path integral

$$\langle \mathcal{O}[A] \rangle = \int \mathcal{D}A e^{-S[A]} \mathcal{O}[A], \quad (\text{A.1.2})$$

where the action and the measure are invariant under gauge transformations

$$A^\Lambda = A + d\Lambda. \quad (\text{A.1.3})$$

Proof. Under a gauge transformation, we can rewrite the expectation value as

$$\langle \mathcal{O}[A] \rangle = \frac{1}{Z} \int \mathcal{D}A e^{-S[A]} \mathcal{O}[A] = \frac{1}{Z} \int \mathcal{D}A e^{-S[A]} \mathcal{O}[A^\Lambda], \quad (\text{A.1.4})$$

since both $S[A]$ and $\mathcal{D}A$ are gauge invariant. Note that if $S[A]$ contain non LGT invariant boundary action, above computation only hold for SGT. We can now average over all local gauge transformations $\alpha(x)$ with a normalized Haar measure:

$$\langle \mathcal{O}[A] \rangle = \int \mathcal{D}A e^{-S[A]} \left[\int \mathcal{D}\Lambda \mathcal{O}[A^\Lambda] \right]. \quad (\text{A.1.5})$$

If $\mathcal{O}[A]$ is not gauge invariant, it transforms nontrivially under $\Lambda(x)$, so the average over $\Lambda(x)$ vanishes by Riemann–Lebesgue lemma:

$$\int \mathcal{D}\alpha \mathcal{O}[A^\alpha] = 0. \quad (\text{A.1.6})$$

Therefore,

$$\langle \mathcal{O}[A] \rangle = 0. \quad (\text{A.1.7})$$

Example: Wilson line. Consider the Wilson line along a curve Γ ,

$$W_\Gamma[A] = \exp \left(i \int_\Gamma A \right). \quad (\text{A.1.8})$$

Under a $U(1)$ gauge transformation,

$$A \rightarrow A + d\Lambda, \quad W_\Gamma[A] \rightarrow e^{i(\Lambda(x_f) - \Lambda(x_i))} W_\Gamma[A], \quad (\text{A.1.9})$$

where x_i and x_f are the endpoints of the path. For an open Wilson line, the phase factor $e^{i(\Lambda(x_f) - \Lambda(x_i))}$ depends on the local gauge parameter. Averaging over $\alpha(x)$ gives by Riemann–Lebesgue lemma :

$$\langle W_\Gamma[A] \rangle = 0. \quad (\text{A.1.10})$$

However, for a closed Wilson loop ($x_i = x_f$), the phase cancels, and

$$\langle W_{\text{loop}}[A] \rangle \neq 0, \quad (\text{A.1.11})$$

which is consistent with gauge invariance.

Elitzur’s theorem implies that spontaneous breaking of local gauge symmetries is not meaningful in gauge theories. Only gauge-invariant quantities such as the field strength F_{IJ} or closed Wilson loops can acquire nonzero expectation values.

A.2 Edge modes for $U(1)$ gauge theory

In this section, we surmalize symmetry aspect of edge modes.

A.2.1 Connection to edge modes

The boundary condition used in the subsection 2.1.2 is somewhat artificial in the sense that a boundary observer chooses a particular $U(1)$ frame and thereby treats the boundary gauge field

itself as an observable. In practice, it might be preferred to think that what we should really impose is only the condition $F_{ij}|_{\partial M} = 0$, rather than a condition directly on the gauge field. Indeed, by promoting the fixed $U(1)$ frame $\bar{\phi}$ to a dynamical frame ϕ , one can impose $F_{ij}|_{\partial M} = 0$ while keeping gauge invariance intact. The additional dynamical degree of freedom ϕ introduced in this way to restore gauge invariance is called an *edge mode* (see e.g. [38, 111, 40]).¹ In this edge-mode description, the Ward identity derived above no longer follows in the same way, but in fact non-trivial Ward identities still arise even after introducing edge modes.

The Dirichlet boundary condition can be “improved” to a gauge-invariant form by introducing the edge mode ϕ as

$$e^{iS_{\text{bdy}}[a+d\phi]} = \int \mathcal{D}\phi \delta(a + d\phi). \quad (\text{A.2.1})$$

We also assign the following gauge transformation rule to ϕ :

$$\delta_{\Lambda}\phi = -\Lambda|_{\partial M}. \quad (\text{A.2.2})$$

With this prescription, all pure-gauge boundary configurations of the form $a = d\phi$ appear with equal weight (equal probability), and large gauge invariance is restored at the boundary.

Let us first reconsider observables. As explained above, LGT is no longer broken in this description, so Elitzur’s theorem can be applied to all GT non-invariant operators. Thus, any operator that transforms non-trivially under GT is not an observable. At first sight, one may then conclude that observables should be constructed only from Wilson loops and the field strength. However, by using the additional field ϕ , one can construct GT-invariant operators such as

$$\widetilde{W}_{\Gamma(x)} = \exp(iq\phi(x)) \exp\left(iq \int_{\Gamma(x)} A\right), \quad (\text{A.2.3})$$

where $\Gamma(x)$ is a path whose endpoint lies at the boundary point x .² Therefore, by appropriately dressing Dirichlet observables with edge modes, one can reconstruct the observables corresponding to the Dirichlet setup in a fully GT-invariant manner

Another interesting observable is the following. We first Hodge-decompose the boundary gauge field a as

$$a = a^{\text{dr}} - d\varphi. \quad (\text{A.2.4})$$

¹Various definitions of edge modes coexist in the literature. The differences among these definitions and the necessity of distinguishing them are discussed in detail in Ref. [40]. The field ϕ introduced in this paper corresponds to what is called an *extrinsic edge mode* in Ref. [40].

²The dressing factor $\exp(iq\phi)$ cancels the phase rotation along the gauge orbit. This is the essential reason why the usual “averaging over the gauge orbit” argument in Elitzur’s theorem can be evaded.

We then regard a^{dr} as the gauge-invariant part, and assign to φ the gauge transformation rule ³

$$\delta_\Lambda \varphi = -\Lambda. \quad (\text{A.2.5})$$

In this setup, the combination

$$\phi_G := \varphi - \phi \quad (\text{A.2.6})$$

is invariant under gauge transformations and can be treated as a genuine physical observable. In Ref. [40], this ϕ_G is referred to as a Goldstone mode. It carries soft radiation and is observable as a memory effect in IR physics context.

We now turn to the derivation of the Ward identity. We define the correlator by

$$\left\langle \prod_k \mathcal{O}_k(x_k) \right\rangle = \int \mathcal{D}\phi \mathcal{D}A e^{iS_{\text{bulk}}[A]} \delta[a + d\phi] \mathcal{O}. \quad (\text{A.2.7})$$

In this formulation, both the action and the operators are gauge invariant, so applying δ_Λ directly yields no non-trivial Ward identity. To obtain a non-trivial identity, we proceed as follows.

To avoid this triviality, we introduce the transformation $\bar{\delta}_\Lambda$ defined by

$$\bar{\delta}_\Lambda A = 0, \quad (\text{A.2.8})$$

$$\bar{\delta}_\Lambda \phi = -\bar{\Lambda}, \quad (\text{A.2.9})$$

which we call a *boundary gauge transformation* (BGT) [38].⁴ Using BGT, we then form the combination GT–BGT, namely, $\delta_\Lambda - \bar{\delta}_\Lambda$:

$$(\delta_\Lambda - \bar{\delta}_\Lambda) A = d\Lambda, \quad (\text{A.2.10})$$

$$(\delta_\Lambda - \bar{\delta}_\Lambda) \phi = 0. \quad (\text{A.2.11})$$

To derive the Ward identity, we consider

$$(\delta_\Lambda - \bar{\delta}_\Lambda) \left\langle \prod_k \mathcal{O}_k(x_k) \right\rangle = (\delta_\Lambda - \bar{\delta}_\Lambda) \int \mathcal{D}\phi \mathcal{D}A e^{iS_{\text{bulk}}[A]} \delta[a + d\phi] \mathcal{O} = 0. \quad (\text{A.2.12})$$

What we are doing here is nothing but fixing the boundary $U(1)$ frame ϕ . In other words, this corresponds to performing a gauge transformation that fixes the boundary frame, thereby reproducing the Dirichlet setup. From this point onward, the computational steps are exactly the same as in the Dirichlet case, and as a result we obtain, once again, a Ward identity of the form (2.1.10):

$$\left\langle \left(\int_{\partial M} [d^D x] \nabla_i J^i \Lambda \right) \mathcal{O} \right\rangle = -i \left\langle \left(-\bar{\delta}_\Lambda - \int_{\partial M} [d^D x] \nabla_i \Lambda(x) \frac{\delta}{\delta a(x)} \right) \mathcal{O} \right\rangle. \quad (\text{A.2.13})$$

³which is called the intrinsic edge mode in Ref. [40])

⁴Ref. [38] call this transformation as surface symmetry and Ref. [40] call this as frame reorientation.

Here we have used the fact that $\delta_\Lambda \mathcal{O} = 0$, i.e. that the operator $\mathcal{O}$ is invariant under GT.

What is physically important is that even without introducing a Dirichlet-type boundary term that explicitly breaks LGT, the same form of Ward identity arises naturally once we make use of BGT. In other words, once we impose the condition that the gauge-invariant boundary electromagnetic field is set to zero, any boundary action consistent with this condition will automatically lead to a Ward identity of the form discussed above; in this sense, the Ward identity, of course, is not merely an artifact of having added a boundary term that breaks gauge symmetry. Rephrased from the edge-mode perspective, the essential point is that changing only the bulk gauge frame while keeping the boundary gauge frame (i.e. the edge mode) fixed, or conversely moving only the boundary gauge frame, does in fact affect the physics, and it is precisely this fact that underlies the emergence of a non-trivial Ward identity.

Analogy from toy-model Although this may look somewhat involved, In fact, the structure becomes quite natural once we consider the following analogy with translational symmetry in a setup where both the detector and the particle are treated as dynamical degrees of freedom, and the position of the particle is measured by the detector.

As a starting point, it is natural to adopt the following correspondence:

- The position of the detector corresponds to the edge mode. Since the edge mode is nothing but the boundary observer's $U(1)$ frame promoted to a dynamical degree of freedom, this identification is natural.
- The position of the particle corresponds to the bulk gauge-field configuration.

In this language, the three types of transformations discussed above can be interpreted as follows:

- **GT, δ_Λ :** This corresponds to translating both the detector and the particle simultaneously. In this case, the position of the particle relative to the detector does not change. This is analogous to the fact that a gauge transformation does not change the boundary action and hence does not change the state.
- **BGT, $\bar{\delta}_\Lambda$, and GT–BGT, $(\delta_\Lambda - \bar{\delta}_\Lambda)$:** BGT corresponds to moving only the detector, whereas GT–BGT corresponds to moving the particle while keeping the detector fixed. In either case, the position of the particle relative to the detector changes, and therefore the results of observation change. This provides an analogy for the fact that BGT and GT–BGT act non-trivially on the boundary action (i.e. on the state) and on observables.

From this viewpoint, the overall structure becomes more transparent. The only transformations we regard as *meaningful symmetries* are those that change the relative configuration between the measuring apparatus and the physical system [38, 112]. Accordingly, the reason why the BGT and

GT–BGT transformations impose non-trivial constraints on the system is that they precisely correspond to transformations that alter this relative configuration—in other words, transformations that qualify as genuine physical symmetries. Conversely, the Dirichlet-type boundary conditions employed in the previous section can be interpreted as describing the limiting situation in which the “detector” is no longer treated as a physical degree of freedom.⁵

Edge modes from a subregion perspective

Let us add one more comment on the physical meaning of ϕ .

Suppose that the theory equipped with the boundary action for the edge mode is realized as the theory on a subregion M embedded in a larger manifold N , on which the same gauge theory is defined and on whose boundary we impose Dirichlet boundary conditions. In other words, we consider a situation with $M \subset N$, and a single gauge theory living on all of N .

In this setup, the degree of freedom $\phi(x)$ can be reconstructed from the fields in the complement $N \setminus M$ (the region obtained by removing M from N). Concretely, one may consider a Wilson line that pierces both ∂N and ∂M . For example, we may define

$$e^{i\phi(x)} = \exp\left(i \int_{\Gamma(x, y(x))} A\right), \quad (\text{A.2.14})$$

where $y(x)$ is a point on ∂N which depend on x , x is a point on ∂M , and $\Gamma(x, y(x))$ is some fixed path connecting these two points from x to $y(x)$. With this definition, the variable $\phi(x)$ inherits precisely the transformation property under LGT on M that we require for the edge mode (A.2.2).

Identifying this degree of freedom with the edge mode, a Wilson line on M that is dressed by the edge mode coincides with a Wilson line in the theory on N that pierces ∂N . Thus, in this construction, the edge mode can be interpreted as a proxy for (a subset of) the degrees of freedom in the exterior region $N \setminus M$. By using the edge-mode degrees of freedom, one is effectively able to access part of the information in $N \setminus M$ while still working with a theory defined purely on M .

From this perspective, the BGT transformation can be regarded as a LGT supported on ∂N . Such a transformation leaves the bulk configuration inside M invariant, while it changes $\phi(x)$, which appears in the Wilson line defined above, through its support on ∂N . This precisely satisfies the transformation law defined in (A.2.9).

The physical meaning of edge modes from this subregion viewpoint, and their realization in a canonical formalism, has been systematically analyzed and organized in the literature [40].

⁵In Ref. [40], the authors summarize both the differences and the equivalence between the Dirichlet description (which they refer to as boundary gauge fixing) and the formulation based on edge modes from different perspective.

A.3 Ward identity for gravitational theory

In this section, we derive Ward identity for gravitational theory using same method in subsection 2.1.2.

As a concrete example, let us next consider the Dirichlet boundary condition for gravity, although almost all of the following manipulations do not rely on the detailed form of the boundary action. In the gravitational case, the boundary action is chosen so as to implement Dirichlet boundary conditions for the induced metric γ_{ij} on the boundary ∂M :

$$e^{iS_{\text{bdy}}[\gamma]} = \delta[\gamma_{ij} - \bar{\gamma}_{ij}], \quad (\text{A.3.1})$$

where $\bar{\gamma}$ is non-dynamical metric. Equivalently, one could restrict the integration domain of the path integral to metrics satisfying

$$\gamma_{ij}(x) = \bar{\gamma}_{ij}(x). \quad (\text{A.3.2})$$

In other words, we demand that the induced boundary metric is fixed to a prescribed background metric $\bar{\gamma}_{ij}$. Thus, at the boundary, LGT invariant break and only isometry is symmetry.

The correlator is then written as

$$\langle \mathcal{O} \rangle = \int Dg e^{iS_{\text{bulk}}[g]} \delta[\gamma_{ij} - \bar{\gamma}_{ij}] \mathcal{O}. \quad (\text{A.3.3})$$

Using the LGT invariance of the bulk path integral, we obtain the Ward identity

$$\delta_\xi \langle \mathcal{O} \rangle = \int Dg \delta_\xi \left(e^{iS_{\text{bulk}}[g]} e^{iS_{\text{bdy}}[\gamma]} \mathcal{O} \right) = 0, \quad (\text{A.3.4})$$

where ξ^I is the infinitesimal LGT parameter and $\xi^i(x)$ denotes its tangential component on the boundary. Since $\delta_\xi S_{\text{bulk}} = 0$, the Leibniz rule yields two terms: one where δ_ξ acts on the functional delta function, and one where it acts on the operator insertions $\prod_k \mathcal{O}_k$:

$$0 = \int Dg e^{iS_{\text{bulk}}[g]} \left[\left(\int_{\partial M} d^D x (2\nabla_{(i} \xi_{j)}(x)) \frac{\delta}{\delta \gamma_{ij}(x)} e^{iS_{\text{bdy}}[\gamma]} \right) \mathcal{O} \right] + \langle \delta_\xi \mathcal{O} \rangle, \quad (\text{A.3.5})$$

where ∇_i is the covariant derivative with respect to the induced boundary metric. Focusing on the first term, we integrate by parts with respect to the boundary variable γ_{ij} . The functional derivative is then transferred to $e^{iS_{\text{bulk}}}$ and to the operators, which produces the following term :

$$\int_{\partial M} d^D x (2\nabla_{(i} \xi_{j)}(x)) \frac{\delta S_{\text{bulk}}}{\delta \gamma_{ij}(x)} =: \int_{\partial M} d^D x \sqrt{\gamma} T^{ij} \nabla_i \xi_j = - \int_{\partial M} d^D x \sqrt{\gamma} (\nabla_i T^i_j) \xi^j, \quad (\text{A.3.6})$$

where we integrated by parts on the boundary and we define T^{ij} as *Brown York tensor* (BY tensor). As a result, we obtain the gravitational analogue of (2.1.10):

$$\left\langle \left(\int_{\partial M} d^D x \sqrt{\gamma} \nabla_i T^i_j \xi^j \right) \mathcal{O} \right\rangle = -i \left\langle \left(\delta_\xi - \int_{\partial M} d^D x (2\nabla_{(i} \xi_{j)}) \frac{\delta}{\delta \gamma_{ij}(x)} \right) \mathcal{O} \right\rangle. \quad (\text{A.3.7})$$

Although (A.3.7) is the general form of the Ward identity for LGT in gravity, its form is rather unfamiliar. To recover the familiar form of Ward identity, we impose two additional assumptions :

- $\mathcal{O}$ does not depend functionally on γ_{ij} . This assumption allows us to drop the term proportional to $\int d^D x (2\nabla_{(i}\xi_{j)}) \delta/\delta\gamma_{ij}$ in (A.3.7). In the Dirichlet problem we always assume this, because the correlator (A.3.3) has support only on $\gamma = \bar{\gamma}$, so that any γ -dependence of $\mathcal{O}$ can be traded for a dependence on the fixed background $\bar{\gamma}$.⁶
- $\mathcal{O}$ can be written as a product of boundary “local” operators $\mathcal{O}_n(x_n)$, each labelled by a boundary point x_n , and each transforming linearly under the boundary diffeomorphism with parameter $\xi^i(x)$, which is defined by restricting bulk diffeomorphism at boundary :

$$\delta_\xi \mathcal{O}_n(x_n) \propto \mathcal{L}_\xi \mathcal{O}_n(x_n), \quad (\text{A.3.8})$$

where $\mathcal{L}$ is Lie derivative on boundary manifold. GT-invariant operators are included as the special case where the proportionality constant vanishes.

Under these assumptions, (A.3.7) reduces to the standard D dimensional Ward identity for the stress tensor. For a product of boudary local operators we obtain

$$\left\langle \left(\int_{\partial M} d^D x \sqrt{\bar{\gamma}} \nabla_i T^i_j(x) \xi^j(x) \right) \prod_n \mathcal{O}_n(x_n) \right\rangle = -i \sum_m \left\langle \mathcal{L}_\xi \mathcal{O}_m(x_m) \prod_{n \neq m} \mathcal{O}_n(x_n) \right\rangle \quad (\text{A.3.9})$$

which is precisely the usual Ward identity expressing the conservation of the non-gravitational QFT stress tensor in the presence of operator insertions (see e.g. [45]).

Note that in the case that ξ is isometry of boundary, we can get following identity :

$$\sum_m \left\langle \mathcal{L}_\xi \mathcal{O}_m(x_m) \prod_{n \neq m} \mathcal{O}_n(x_n) \right\rangle = 0, \quad (\text{A.3.10})$$

where we use killing equation $\nabla_{(i}\xi_{j)} = 0$. This is called global Ward identity.

In addition substituting $\xi = \alpha(x) \zeta$, where ζ is a Killing vector and $\alpha(x)$ is scalar function, into first line of the expression (A.3.9), we obtain

$$\left\langle \left(\int_{\partial M} d^D x \sqrt{\bar{\gamma}} (\nabla_i \alpha) P^i(x) \right) \prod_n \mathcal{O}_n(x_n) \right\rangle = -i \sum_m \left\langle \mathcal{L}_\xi \mathcal{O}_m(x_m) \prod_{n \neq m} \mathcal{O}_n(x_n) \right\rangle \quad (\text{A.3.11})$$

where we define momentum $P^i = T^i_j \zeta^j$. Here, if we choose $\alpha(x)$ to be a step-function with support equal to one only on a small region B containing only x_m , we further obtain

$$\left\langle P_{\partial B} \prod_n \mathcal{O}_n(x_n) \right\rangle = -i \left\langle \mathcal{L}_\xi \mathcal{O}_m(x_m) \prod_{n \neq m} \mathcal{O}_n(x_n) \right\rangle \quad (\text{A.3.12})$$

⁶As in the $U(1)$ example, one can say that the term with the functional derivative is required to compensate the difference between evaluating δ_ξ *after* substituting $g^{(0)} = \eta$ into $\mathcal{O}$ and evaluating it *before* the substitution.

where we define $P_{\partial B}$ as

$$P_{\partial B} = \int_{\partial B} d^{D-1}z \sqrt{\gamma} n_i P^i(z) \quad (\text{A.3.13})$$

Therefore, $P_{\partial B}$ is an operator that acts as the Lie derivative $\mathcal{L}_\xi$ on operators supported in the region contained in B .

A.4 Anti-podal matching condition for spacial infinity

In this section we verify, following Ref. [53], that the boundary conditions in Eqs. (2.3.11) are physically reasonable. We consider scattering processes in which, in the distant past, each charged particle approaches the interaction region with an asymptotically constant velocity. If we compute the gauge field using the retarded Green's function, then, by causality, the electromagnetic field at the points i^0 , $\mathcal{I}_+^+$, and $\mathcal{I}_+^-$ depends only on the behavior of the charges in the far past. In particular, near i^0 the field coincides with the Liénard–Wiechert field of a charge moving with constant velocity.

For concreteness, let us consider a point particle of charge e moving with constant three-velocity $\vec{\beta}$ and, for simplicity, assume that its trajectory passes through the origin.⁷ As noted above, the leading $1/r$ behavior is insensitive to a constant spatial shift of the worldline, so this assumption entails no loss of generality. The corresponding Liénard–Wiechert potential is known to be

$$A_\mu(t, \vec{r}) = \frac{e}{4\pi} \frac{(1, -\vec{\beta})}{\sqrt{(\vec{r} - \vec{\beta}t)^2 + (\vec{r} \cdot \vec{\beta})^2 - \beta^2 r^2}}. \quad (\text{A.4.1})$$

What matters at i^0 is the behavior of this configuration under inversion of the spacetime point,

$$(t, \vec{r}) \longrightarrow (-t, -\vec{r}). \quad (\text{A.4.2})$$

From (A.4.1) we see that, in the neighborhood of i^0 , the potential is an even function under this map,

$$A_\mu(-t, -\vec{r}) = A_\mu(t, \vec{r}) \quad (\text{at } i^0). \quad (\text{A.4.3})$$

Since the field strength is constructed from derivatives of A_μ , it is therefore odd:

$$F_{\mu\nu}(-t, -\vec{r}) = -F_{\mu\nu}(t, \vec{r}) \quad (\text{at } i^0). \quad (\text{A.4.4})$$

⁷Here we are only concerned with the finiteness of the total energy at large r . The divergence of the energy density of the Coulomb field near a point charge is an unavoidable feature of unrenormalized classical electromagnetism and is irrelevant for the present discussion.

A.5 Soft theorem from Ward identity

In this section, we review derivation of soft photon theorem from Ward identity (2.3.32) roughly following [48].

We take the scattering sphere to be very large and assume that the scattering region is localized near a point on this sphere for simplicity. In this limit we can neglect the curvature of the scattering sphere and approximate the boundary metric by the flat metric on the complex plane,

$$\gamma_{\alpha\beta} dz^\alpha dz^\beta = 2dz d\bar{z}, \quad (\text{A.5.1})$$

with complex coordinate $(z, \bar{z})$. To make contact with the standard momentum–space soft theorem, we parametrize the null momentum of the soft photon by

$$q^\mu(z, \bar{z}) = \left(1 + z\bar{z}, z + \bar{z}, -i(z - \bar{z}), 1 - z\bar{z}\right), \quad (\text{A.5.2})$$

and choose the polarization vector as

$$\varepsilon_z^\mu(z, \bar{z}) = \partial_z q^\mu(z, \bar{z}). \quad (\text{A.5.3})$$

In this setup, the Ward identity (2.3.32) reads

$$\langle (\partial\bar{J} + \bar{\partial}J)(z, \bar{z}) \prod_n O_n(z_n) \rangle \propto \sum_m \delta^{(2)}(z - z_m, \bar{z} - \bar{z}_m) \langle \delta O_m(z_m) \prod_{n \neq m} O_n(z_n) \rangle, \quad (\text{A.5.4})$$

where $\partial = \partial_z$, $\bar{\partial} = \partial_{\bar{z}}$, $J \equiv J_z$ and $\bar{J} \equiv J_{\bar{z}}$. We now take the operators O_n to be the annihilation operators a_n of asymptotic charged particles. Under a global $U(1)$ transformation we have

$$\delta a_m = iQ_m a_m, \quad (\text{A.5.5})$$

where Q_m is the $U(1)$ charge carried by a_m . The holomorphic part of Ward identity (A.5.4) then implies that the holomorphic current has simple poles at the insertion points:

$$\langle J(z, \bar{z}) \prod_n a_n \rangle \propto \sum_m \frac{Q_m}{z - z_m} \langle \prod_n a_n \rangle, \quad (\text{A.5.6})$$

Next we relate the current J to the soft photon operator. At future null infinity $\mathcal{I}^+$, the current is given by the u -zero mode of F_{uz} ,⁸

$$J(z, \bar{z}) \sim \int_{-\infty}^{+\infty} du F_{uz}(u, z, \bar{z}) \sim \lim_{\omega \rightarrow 0} \omega a_z(\omega, z, \bar{z}). \quad (\text{A.5.7})$$

where we use additional boundary condition (2.3.33) and $a_z(\omega, z, \bar{z}) = \partial_z q^I a_I$ is the annihilation operator for an outgoing positive–helicity photon of frequency ω . The last step follows from viewing

⁸Note that lowering the index of R gives u .

the u -integral as the $\omega \rightarrow 0$ limit of its Fourier transform, $\int du e^{i\omega u}$, so that the zero mode in u precisely picks out the soft $\omega \rightarrow 0$ component of the photon.

For a massless external particle with momentum $p_m^\mu = \omega_m q^\mu(z_m, \bar{z}_m)$, one then has the identity

$$\frac{1}{z - z_m} = \frac{\varepsilon_z(z, \bar{z}) \cdot p_m}{q(z, \bar{z}) \cdot p_m}. \quad (\text{A.5.8})$$

Using (A.5.7) to replace J by the soft photon operator in (A.5.6) and then employing (A.5.8), we obtain

$$\lim_{\omega \rightarrow 0} \omega \langle a_z(\omega, z, \bar{z}) \prod_n a_n \rangle \propto \sum_m Q_m \frac{\varepsilon_z(z, \bar{z}) \cdot p_m}{q(z, \bar{z}) \cdot p_m} \langle \prod_n a_n \rangle. \quad (\text{A.5.9})$$

This is precisely the leading soft photon theorem.

Appendix B

Appendix to Chapter 3

B.1 Two-point Functions on S^2 from $\mathbb{R}^3$

For $D = 1$, the relation (3.1.10) implies that the two-point functions on $\mathbb{R}^3$ and S^2 are connected via

$$\begin{aligned}\langle \psi_{\lambda_1}(x_1) \psi_{\lambda_2}(x_2) \rangle &= \frac{1}{2\pi} \int d\xi_1 d\xi_2 e^{(1/2-i\lambda_1)\xi_1} e^{(1/2-i\lambda_2)\xi_2} \langle \Phi(X_1) \Phi(X_2) \rangle \\ &= \frac{1}{8\pi^2} \int d\xi_1 d\xi_2 e^{(1/2-i\lambda_1)\xi_1} e^{(1/2-i\lambda_2)\xi_2} \frac{1}{\sqrt{e^{2\xi_1} + e^{2\xi_2} - 2e^{\xi_1+\xi_2} \cos \gamma}},\end{aligned}\quad (\text{B.1.1})$$

where γ denotes the angle between the two points X_1 and X_2 .

We now verify that the right-hand side reproduces the two-point function corresponding to the Lagrangian (3.1.8) on S^2 . Introducing new variables $u = (\xi_1 - \xi_2)/2$ and $v = (\xi_1 + \xi_2)/2$, we can rewrite the expression as

$$\begin{aligned}\langle \psi_{\lambda_1}(x_1) \psi_{\lambda_2}(x_2) \rangle &= \frac{1}{4\pi^2} \int du dv e^{iu(\lambda_1+\lambda_2)} e^{iv(\lambda_1+\lambda_2)} \frac{1}{\sqrt{e^{2u} + e^{-2u} - 2 \cos \gamma}} \\ &= \frac{1}{2\pi} \delta(\lambda_1 + \lambda_2) \int du e^{2iu\lambda_1} \frac{1}{\sqrt{e^{2u} + e^{-2u} - 2 \cos \gamma}}.\end{aligned}\quad (\text{B.1.2})$$

We next partially expand the integrand using the standard multipole (Legendre) expansion in

three dimensions:¹

$$\begin{aligned}
\langle \psi_{\lambda_1}(x_1) \psi_{\lambda_2}(x_2) \rangle &= \frac{1}{2\pi} \delta(\lambda_1 + \lambda_2) \int du e^{2iu\lambda_1} \sum_{l=0}^{\infty} e^{-(2l+1)|u|} P_l(\cos \gamma) \\
&= \frac{1}{\pi} \delta(\lambda_1 + \lambda_2) \sum_{l=0}^{\infty} \frac{(2l+1) P_l(\cos \gamma)}{(2l+1)^2 + 4\lambda_1^2} \\
&= \delta(\lambda_1 + \lambda_2) \frac{1}{4\pi} \sum_{l=0}^{\infty} \frac{(2l+1) P_l(\cos \gamma)}{l(l+1) + m_{\lambda_1}^2} \\
&= \delta(\lambda_1 + \lambda_2) G(x_1, x_2),
\end{aligned} \tag{B.1.4}$$

where $G(x_1, x_2)$ is the two-point function of a massive scalar on S^2 , as reviewed below. Thus, the final expression agrees with the two-point function obtained from the Lagrangian (3.1.8).

We now recall that the Green's function on S^2 can be written as

$$G(\gamma) = \frac{1}{4\pi} \sum_{l=0}^{\infty} \frac{2l+1}{l(l+1) + m^2} P_l(\cos \gamma), \tag{B.1.5}$$

where γ is the geodesic distance between the two points.

Consider a massive scalar field with mass m on the two-dimensional sphere S^2 . The Green's function $G(x_1, x_2)$ satisfies

$$-(\Delta - m^2)G(x_1, x_2) = \frac{1}{\sqrt{g_S}} \delta^{(2)}(x_1 - x_2), \tag{B.1.6}$$

where Δ is the Laplacian on S^2 and g_S is the determinant of its metric.

The spherical harmonics $Y_{lm}(x)$ obey

$$\Delta Y_{lm} = -l(l+1)Y_{lm}, \tag{B.1.7}$$

so we may expand the Green's function as

$$G(x_1, x_2) = \sum_{l=0}^{\infty} \sum_{m'=-l}^l \frac{Y_{lm'}(x_1) Y_{lm'}^*(x_2)}{l(l+1) + m^2}. \tag{B.1.8}$$

Using the addition theorem for spherical harmonics,

$$\sum_{m=-l}^l Y_{lm}(x_1) Y_{lm}^*(x_2) = \frac{2l+1}{4\pi} P_l(\cos \gamma), \tag{B.1.9}$$

¹Recall

$$\frac{1}{\sqrt{r^2 + r'^2 - 2rr' \cos \theta}} = \sum_{l=0}^{\infty} \frac{r_{<}^l}{r_{>}^{l+1}} P_l(\cos \theta), \tag{B.1.3}$$

where $r_{<} = \min(r, r')$ and $r_{>} = \max(r, r')$.

with $\cos \gamma = x_1 \cdot x_2$, we recover (B.1.5).

The Green's function G can also be expressed in a hypergeometric form:²

$$G(x_1, x_2) = \frac{\Gamma(\Delta) \Gamma(D - \Delta)}{4\pi} {}_2F_1 \left(\Delta, D - \Delta; 1; \frac{1 + \cos \gamma}{2} \right), \quad (\text{B.1.11})$$

with

$$\Delta = \frac{1}{2} + i \sqrt{m^2 - \frac{1}{4}}.$$

B.2 Evaluating Integrals on S^{D+1} by Contour Deformation of the Polar Angle

In this appendix we work in an $(D+2)$ -dimensional Euclidean space with coordinates

$$(X^1, \dots, X^{D+1}, X^{D+2}) = rx, \quad (\text{B.2.1})$$

where x denotes embedding coordinates on S^{D+1} .

We consider the integral

$$I = \int_{S^{D+1}} [d^{D+1}x] f(x), \quad (\text{B.2.2})$$

for a function f that decays sufficiently rapidly at infinity, and we deform the contour of the polar angle of S^{D+1} as in Fig. 3.3 and Eq. (3.2.5).

Introduce the polar angle θ by

$$x^{D+2} = -\cos \theta, \quad \mathbf{x} = (x^1, \dots, x^{D+1}) = \sin \theta \hat{n}, \quad \hat{n} \in S^D. \quad (\text{B.2.3})$$

The Euclidean volume element is then $[d^{D+1}x] = (\sin \theta)^D d\theta d^D \hat{n}$.

In section 3.2 we analytically continued θ as $\theta = \pi/2 + it$. Under this continuation,

$$x^{D+2} \rightarrow i \sinh t =: ix^0, \quad \mathbf{x} \rightarrow \cosh t \hat{n}, \quad (\text{B.2.4})$$

and the embedding coordinates $(x^0, \mathbf{x})$ obey $-(x^0)^2 + |\mathbf{x}|^2 = 1$, which is the global embedding of dS_{D+1} .

²One way to show this is to write

$$G(x_1, x_2) = \frac{1}{8\pi} \sum_{l=-\infty}^{\infty} \frac{2l+1}{l(l+1)+m^2} (-1)^l P_l(-\cos \gamma) = \frac{1}{16\pi i} \oint_C dz \frac{f(z)}{\sin(\pi z)}, \quad (\text{B.1.10})$$

where C is a contour encircling all integer poles of $1/\sin(\pi z)$ and $f(z) := \frac{2z+1}{z(z+1)+m^2} P_z(-\cos \gamma)$. By picking up the poles $z = -\Delta_{\pm}$ of $f(z)$, one obtains (B.1.11).

For the continuation $\theta = -i\chi$, we obtain

$$x^{D+2} \rightarrow -\cosh \chi =: -iy^{D+2} = y^0, \quad \mathbf{x} \rightarrow -i \sinh \chi \hat{n} =: -i\mathbf{y}, \quad (\text{B.2.5})$$

so that $(\mathbf{y}, y^0)$ with $-(y^0)^2 + |\mathbf{y}|^2 = -1$ gives the embedding of global coordinates on EAdS_{D+1}^- (the unit hyperbolic space H^-).

Similarly, for $\theta = \pi - i\chi$,

$$x^{D+2} \rightarrow \cosh \chi =: -iy^{D+2} = y^0, \quad \mathbf{x} \rightarrow i \sinh \chi \hat{n} =: -i\mathbf{y}, \quad (\text{B.2.6})$$

which corresponds to global coordinates on EAdS_{D+1}^+ (hyperbolic space H^+).

For a sufficiently rapidly decaying test function $f(x^1, \dots, x^{D+2})$, we start from

$$\begin{aligned} I &= \int_{S^{D+1}} [d^{D+1}x] f(\sin \theta \hat{n}, -\cos \theta) \\ &= \int_{S^D} [d^D \hat{n}] \int_0^\pi d\theta \sin^D \theta f(\sin \theta \hat{n}, -\cos \theta). \end{aligned} \quad (\text{B.2.7})$$

The original contour $\theta \in [0, \pi]$ can be deformed as in Eq. (3.2.5) and Fig. 3.3 because the additional horizontal segments at $\text{Im } \theta = \pm\infty$ do not contribute, assuming suitable decay of f and analyticity in the θ -plane. We can therefore write $I = I_A + I_B + I_C$, with

$$I_A = i^{-D-1} \int_{S^D} [d^D \hat{n}] \int_0^\infty d\chi \sinh^D \chi f(-i \sinh \chi \hat{n}, -\cosh \chi), \quad (\text{B.2.8a})$$

$$I_B = i \int_{S^D} [d^D \hat{n}] \int_{-\infty}^\infty dt \cosh^D t f(\cosh t \hat{n}, i \sinh t), \quad (\text{B.2.8b})$$

$$I_C = i^{-D-1} \int_{S^D} [d^D \hat{n}] \int_0^\infty d\chi \sinh^D \chi f(-i \sinh \chi \hat{n}, \cosh \chi). \quad (\text{B.2.8c})$$

Thus we obtain

$$\begin{aligned} I &= I_A + I_B + I_C \\ &= i \left(i^{-D-2} \int_{\text{EAdS}^-} [d^{D+1}y] + \int_{\text{dS}} [d^{D+1}y] + i^{-D-2} \int_{\text{EAdS}^+} [d^{D+1}y] \right) f(x), \end{aligned} \quad (\text{B.2.9})$$

where

$$x = \begin{cases} -iy & (y \in \text{EAdS}), \\ y & (y \in \text{dS}). \end{cases} \quad (\text{B.2.10})$$

B.3 Minkowski Slicing and Contour Deformation

We now consider the integral

$$J = \int_{\mathbb{R}^{D+2}} [d^{D+2}X] f(X). \quad (\text{B.3.1})$$

In polar coordinates this becomes

$$J = \int_0^\infty dr r^{D+1} \int_{S^{D+1}} [d^{D+1}x] f(rx). \quad (\text{B.3.2})$$

We evaluate (B.3.1) in two different ways (as discussed in Appendices B.3.1 and B.3.2) and show that, provided f is analytic in r , the two results agree.

B.3.1 Minkowski Slicing

We first perform the Wick rotation

$$X^{D+2} = iX^0. \quad (\text{B.3.3})$$

After analytic continuation to Minkowski spacetime, we consider the same integral and partition the integration region into three parts A, B, C . Region B is defined by $(X)^2 > 0$, region A by $(X)^2 < 0$ with $X^0 < 0$, and region C by $(X)^2 < 0$ with $X^0 > 0$. Each region can be foliated by either EAdS or dS slices, as illustrated in Fig. 3.2.

Spherical Rindler region B . The contribution from region B is

$$\begin{aligned} J_B &= i \int_{M_{D+2}, (X)^2 > 0} [d^{D+2}X] f(\mathbf{X}, iX^0) \\ &= i \int_0^\infty dr r^{D+1} \int_{\text{dS}} [d^{D+1}x] f(r \cosh t \hat{n}, ir \sinh t), \end{aligned} \quad (\text{B.3.4})$$

where we use the parametrization $X^0 = r \sinh t$, $\mathbf{X} = r \cosh t \hat{n}$.

Milne regions A and C . The contributions from regions A and C are

$$\begin{aligned} J_A &= i \int_{M_{D+2}, (X)^2 < 0, X^0 < 0} [d^{D+2}X] f(\mathbf{X}, iX^0) \\ &= i \int_0^\infty dr r^{D+1} \int_{\text{EAdS}^-} [d^{D+1}x] f(r \sinh \chi \hat{n}, -ir \cosh \chi), \end{aligned} \quad (\text{B.3.5})$$

and

$$\begin{aligned} J_C &= i \int_{M_{D+2}, (X)^2 < 0, X^0 > 0} [d^{D+2}X] f(\mathbf{X}, iX^0) \\ &= i \int_0^\infty dr r^{D+1} \int_{\text{EAdS}^+} [d^{D+1}x] f(r \sinh \chi \hat{n}, ir \cosh \chi). \end{aligned} \quad (\text{B.3.6})$$

Region A is covered by Milne coordinates $X^0 = -r \cosh \chi$, $X^i = r \sinh \chi \hat{n}^i$, and region C by $X^0 = r \cosh \chi$, $X^i = r \sinh \chi \hat{n}^i$, with the same ranges for $r, \chi, \hat{n}$. Putting everything together,

$$\begin{aligned} J &= J_A + J_B + J_C \\ &= i \left[\int_0^\infty dr r^{D+1} \left(\int_{\text{EAdS}^-} [d^{D+1}x] + \int_{\text{dS}} [d^{D+1}x] + \int_{\text{EAdS}^+} [d^{D+1}x] \right) \right] f(rx). \end{aligned} \quad (\text{B.3.7})$$

B.3.2 Contour Deformation

Alternatively, we may start from (B.3.2) and deform the θ -contour as in Appendix B.2, which splits the angular integral into contributions from $\text{EAdS}^\pm$ and dS as in (B.2.9). Concretely,

$$J = i \int_0^\infty dr r^{D+1} \left(i^{-D-2} \int_{\text{EAdS}^-} [d^{D+1}y] + \int_{\text{dS}} [d^{D+1}y] + i^{-D-2} \int_{\text{EAdS}^+} [d^{D+1}y] \right) f(rx). \quad (\text{B.3.8})$$

From here, the A and C contributions can be written as

$$J_A = i^{-D-1} \int_0^\infty dr r^{D+1} \int_{\text{EAdS}^-} [d^{D+1}y] f(-iry), \quad (\text{B.3.9a})$$

$$J_C = i^{-D-1} \int_0^\infty dr r^{D+1} \int_{\text{EAdS}^+} [d^{D+1}y] f(-iry), \quad (\text{B.3.9b})$$

and after a Wick rotation $r \rightarrow ir$ in both terms we obtain

$$J_A = i \int_0^\infty dr r^{D+1} \int_{\text{EAdS}^-} [d^{D+1}y] f(ry), \quad (\text{B.3.10a})$$

$$J_C = i \int_0^\infty dr r^{D+1} \int_{\text{EAdS}^+} [d^{D+1}y] f(ry). \quad (\text{B.3.10b})$$

Thus (B.3.8) reduces to the same expression as (B.3.7). The evaluation based on contour deformation in (θ, r) therefore agrees with the Minkowski slicing approach, provided f is analytic in r . In our applications, however, the integrand typically contains factors such as $(-2rp \cdot x)^{-\Delta}$, which introduce branch cuts in the complex r -plane. In such situations, the two procedures are not guaranteed to be equivalent.

B.4 Early/Late Correlators in dS

We now turn to early/late-time limits of correlators in de Sitter space.

We start from the Euclidean Green's function of a massive scalar field on S^{D+1} (see, e.g., [72]):

$$G(x_1, x_2) = 2^{\frac{1-D}{2}} \frac{\Gamma(\Delta)\Gamma(D-\Delta)}{2(2\pi)^{\frac{D+1}{2}} \Gamma(\frac{D+1}{2})} {}_2F_1 \left(\Delta, D-\Delta; \frac{D+1}{2}; \frac{1+x_1 \cdot x_2}{2} \right). \quad (\text{B.4.1})$$

By analytically continuing the polar angle as $\theta_i = \frac{\pi}{2} + it_i$, we obtain the two-point function in de Sitter space with the Euclidean vacuum [69, 70, 71]:

$$G(x_1, x_2) = \frac{\Gamma(\Delta)\Gamma(D-\Delta)}{(4\pi)^{\frac{D+1}{2}} \Gamma(\frac{D+1}{2})} {}_2F_1 \left(\Delta, D-\Delta; \frac{D+1}{2}; \frac{1+x_1 \cdot x_2 - i\epsilon}{2} \right), \quad (\text{B.4.2})$$

where the $i\epsilon$ prescription selects the Feynman branch.

From the free action (3.1.8), the two-point function of ψ_λ in dS must take the form

$$\langle \psi_{\lambda_1}(x_1) \psi_{\lambda_2}(x_2) \rangle = \delta(\lambda_1 + \lambda_2) G(x_1, x_2). \quad (\text{B.4.3})$$

Late-time asymptotics. In section 3.3, we parametrize the embedding coordinates of dS as $x(t, \mathbf{p}) = (\sinh t, \cosh t \mathbf{p})$ and introduce $p = (1, \mathbf{p})$, $p_- = (1, -\mathbf{p})$. Setting $x_1 \simeq \frac{1}{2}e^{t_1}p_1$, we find for $t_1 \rightarrow +\infty$,

$$G(x_1, x_2) \sim \frac{1}{4\pi^{h+1}} [e^{-\Delta t_1} 2^\Delta \Gamma(h - \Delta) \Gamma(\Delta) (-2p_1 \cdot x_2 + i\epsilon)^{-\Delta} + (\Delta \leftrightarrow D - \Delta)], \quad (\text{B.4.4})$$

where $h = D/2$ and we have denoted the two contributions compactly as Δ and $D - \Delta$ (equivalently $\Delta_\pm$).

Integral symmetries. Define

$$M_\lambda = 2^{-2i\lambda} \frac{\Gamma(h - i\lambda)}{\Gamma(-i\lambda)} i^{2i\lambda} \pi^{-h}. \quad (\text{B.4.5})$$

Then (B.4.4) is invariant under the simultaneous transformations

$$F_+(p_1) \longrightarrow M_\lambda \int D^d p' (-2p_1 \cdot p')^{-\Delta_-} F_+(p') = F_-(p_1), \quad e^{-\Delta_+ t_1} \longrightarrow e^{-\Delta_- t_1}, \quad (\text{B.4.6})$$

$$F_-(p_1) \longrightarrow M_{-\lambda} \int D^d p' (-2p_1 \cdot p')^{-\Delta_+} F_-(p') = F_+(p_1), \quad e^{-\Delta_- t_1} \longrightarrow e^{-\Delta_+ t_1}, \quad (\text{B.4.7})$$

where $\Delta_\pm = D/2 \pm i\lambda$, and $F_\pm$ denote the coefficients of the $e^{-\Delta_\pm t_1}$ terms. For instance, in the present case,

$$F_\pm(p_1) = \frac{2^{\Delta_\pm}}{4\pi^{h+1}} \Gamma(h - \Delta_\pm) \Gamma(\Delta_\pm) (-2p_1 \cdot x_2 + i\epsilon)^{-\Delta_\pm}. \quad (\text{B.4.8})$$

Late-early correlator and contact terms. Now consider the mixed limit $t_1 \rightarrow +\infty$ and $t_2 \rightarrow -\infty$, with $x_2 \simeq -\frac{1}{2}e^{-t_2}p_{2-}$. A naive evaluation gives

$$G_{\text{naive}}(x_1, x_2) \sim e^{-\Delta_+ t_1 + \Delta_+ t_2} C_{\Delta_+} i^{-2\Delta_+} (-2p_1 \cdot p_{2-})^{-\Delta_+} + (\Delta_+ \leftrightarrow \Delta_-), \quad (\text{B.4.9})$$

with

$$C_\Delta = \frac{2^{2\Delta}}{4\pi^{h+1}} \Gamma(h - \Delta) \Gamma(\Delta), \quad (\text{B.4.10})$$

where we used $(-x + i\epsilon)^a = i^{2a} x^a$ for $x > 0$. However, the expression (B.4.9) does not respect the symmetry relations (B.4.6)–(B.4.7), because we implicitly assumed $x_1 \cdot x_2 \neq 0$.

As in section 3.3, there are additional contributions from the kinematic region where $x_1 \cdot x_2 \sim 0$, corresponding to the collinear $p \sim k$ region. Instead of reconstructing these from scratch, we can determine the missing contribution by enforcing the symmetry. The symmetry uniquely fixes the need for contact terms, leading to

$$\begin{aligned} G(x_1, x_2) \sim & \sum_{\sigma=\pm} e^{-\Delta_\sigma(t_1+t_2)} C_{\Delta_\sigma} i^{-2\Delta_\sigma} (-2p_1 \cdot p_{2-})^{-\Delta_\sigma} \\ & + \sum_{\sigma=\pm} e^{-\Delta_\sigma t_1 - \Delta_{-\sigma} t_2} D_\lambda i^{-D} \delta^{(D)}(p_1 - p_{2-}), \end{aligned} \quad (\text{B.4.11})$$

where

$$D_\lambda = \frac{2^D}{4\pi} \Gamma(i\lambda) \Gamma(-i\lambda). \quad (\text{B.4.12})$$

Similar delta-function-like terms in late-time two-point functions have been observed in [88, 87].

Boundary two-point functions. Finally, multiplying (B.4.11) by $\delta(\lambda_1 - \lambda_2)$ as in (B.4.3), we obtain the late/early boundary correlators of the complex scalar field ψ_λ with continuous label λ appearing in (3.1.8):

$$\langle V_1 W_2 \rangle = \delta(\lambda_1 - \lambda_2) i^{-2\Delta_1} C_{\Delta_1} (2p_1 \cdot p_2)^{-\Delta_1}, \quad (\text{B.4.13})$$

$$\langle W_1 W_2 \rangle = \delta(\lambda_1 + \lambda_2) i^{-D} D_{\lambda_1} \delta^{(D)}(p_1 + p_2), \quad (\text{B.4.14})$$

which are the boundary two-point functions used in the main text.

B.5 Super-rotation edge-mode/Goldstone mode

In this section, we introduce the description of edge modes corresponding superrotation at the linearized level. For simplicity, we choose the supertranslation vacuum $c_{zz} = 0$ and only consider around $\rho_{zz} \sim 0$, $\rho_{\bar{z}\bar{z}} \sim 0$ and $\gamma_{\bar{z}\bar{z}} \sim 0$. The construction presented here heavily relies on appendix A.2 in particular around Eq.(A.2.6). Non-linear analysis can be seen in [62].

We begin by introducing an edge mode ϕ^z as a new boundary degree of freedom, which we define to transform under a superrotation as

$$\delta_Y \phi^z = -Y^z. \quad (\text{B.5.1})$$

Next, since ρ_{zz} and $\gamma_{\bar{z}\bar{z}}$ are regarded as pure-gauge fields in our setup, they can be parametrized in terms of φ^z as

$$\rho_{zz} = -\partial_z^3 \varphi^z, \quad (\text{B.5.2})$$

$$\rho_{\bar{z}\bar{z}} = \partial_z \partial_{\bar{z}}^2 \varphi^z, \quad (\text{B.5.3})$$

$$\gamma_{\bar{z}\bar{z}} = -2 \partial_{\bar{z}} \varphi^z. \quad (\text{B.5.4})$$

With this definition, the transformation law of φ^z becomes

$$\delta_Y \varphi^z = -Y^z. \quad (\text{B.5.5})$$

where we use (2.4.44),(2.4.45).

In this parametrization, we can construct a gauge-invariant Goldstone mode as the combination of the edge mode ϕ^z and the pure-gauge mode φ^z :

$$\phi_G^z = \varphi^z - \phi^z. \quad (\text{B.5.6})$$

This Goldstone mode corresponds to subleading soft radiation and is expected to represent the observable associated with the superrotation memory effect [62].

Finally, to match the notation of Ref. [62], we introduce the following celestial Liouville field Φ :

$$\Phi = -\partial_z \phi_G^z \tag{B.5.7}$$

With this definition, dressed vacua $\tilde{\rho} := -\partial_z^3 \phi_G^z = -\partial_z^2 \Phi$ naturally appears to play the role of a linearized stress tensor field of Liouville theory.

Bibliography

- [1] H. Furugori, N. Ogawa, S. Sugishita, and T. Waki, *Celestial Holography meets dS/CFT*, [arXiv:2507.17558](#).
- [2] J. D. Bekenstein, *Black holes and entropy*, *Phys. Rev. D* **7** (Apr, 1973) 2333–2346.
- [3] J. M. Bardeen, B. Carter, and S. W. Hawking, *The Four laws of black hole mechanics*, *Commun. Math. Phys.* **31** (1973) 161–170.
- [4] S. W. Hawking, *Particle Creation by Black Holes*, *Commun. Math. Phys.* **43** (1975) 199–220. [Erratum: *Commun.Math.Phys.* 46, 206 (1976)].
- [5] J. D. Bekenstein, *Black holes and entropy*, *Phys. Rev. D* **7** (1973) 2333.
- [6] J. D. Bekenstein, *Universal upper bound on the entropy-to-energy ratio for bounded systems*, *Phys. Rev. D* **23** (Jan, 1981) 287–298.
- [7] J. D. Bekenstein, *Entropy bounds and black hole remnants*, *Phys. Rev. D* **49** (1994) 1912–1921, [[gr-qc/9307035](#)].
- [8] G. 't Hooft, *Dimensional reduction in quantum gravity*, *Conf. Proc. C* **930308** (1993) 284–296, [[gr-qc/9310026](#)].
- [9] L. Susskind, *The World as a hologram*, *J. Math. Phys.* **36** (1995) 6377–6396, [[hep-th/9409089](#)].
- [10] R. Bousso, *The Holographic principle*, *Rev. Mod. Phys.* **74** (2002) 825–874, [[hep-th/0203101](#)].
- [11] J. M. Maldacena, *The Large N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231–252, [[hep-th/9711200](#)].
- [12] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, *Gauge theory correlators from noncritical string theory*, *Phys. Lett. B* **428** (1998) 105–114, [[hep-th/9802109](#)].

- [13] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253–291, [[hep-th/9802150](#)].
- [14] T. Banks, M. R. Douglas, G. T. Horowitz, and E. J. Martinec, *AdS dynamics from conformal field theory*, [hep-th/9808016](#).
- [15] D. Harlow and D. Stanford, *Operator Dictionaries and Wave Functions in AdS/CFT and dS/CFT*, [arXiv:1104.2621](#).
- [16] S. B. Giddings, *The Boundary S matrix and the AdS to CFT dictionary*, *Phys. Rev. Lett.* **83** (1999) 2707–2710, [[hep-th/9903048](#)].
- [17] A. Strominger, *Lectures on the Infrared Structure of Gravity and Gauge Theory*, [arXiv:1703.05448](#).
- [18] J. D. Brown and M. Henneaux, *Central Charges in the Canonical Realization of Asymptotic Symmetries: An Example from Three-Dimensional Gravity*, *Commun. Math. Phys.* **104** (1986) 207–226.
- [19] A. Strominger, *Black hole entropy from near horizon microstates*, *JHEP* **02** (1998) 009, [[hep-th/9712251](#)].
- [20] H. Bondi, M. G. J. van der Burg, and A. W. K. Metzner, *Gravitational waves in general relativity. 7. Waves from axisymmetric isolated systems*, *Proc. Roy. Soc. Lond. A* **269** (1962) 21–52.
- [21] R. Sachs, *Asymptotic symmetries in gravitational theory*, *Phys. Rev.* **128** (1962) 2851–2864.
- [22] S. Pasterski, S.-H. Shao, and A. Strominger, *Flat Space Amplitudes and Conformal Symmetry of the Celestial Sphere*, *Phys. Rev. D* **96** (2017), no. 6 065026, [[arXiv:1701.00049](#)].
- [23] G. Barnich and C. Troessaert, *Supertranslations call for superrotations*, *PoS CNCFG2010* (2010) 010, [[arXiv:1102.4632](#)].
- [24] G. Barnich and C. Troessaert, *BMS charge algebra*, *JHEP* **12** (2011) 105, [[arXiv:1106.0213](#)].
- [25] A. Strominger, *On BMS Invariance of Gravitational Scattering*, *JHEP* **07** (2014) 152, [[arXiv:1312.2229](#)].
- [26] D. Kapec, V. Lysov, S. Pasterski, and A. Strominger, *Semiclassical Virasoro symmetry of the quantum gravity $\mathcal{S}$ -matrix*, *JHEP* **08** (2014) 058, [[arXiv:1406.3312](#)].

- [27] S. Weinberg, *Infrared Photons and Gravitons*, *Phys. Rev.* **140** (1965) B516–B524.
- [28] F. Cachazo and A. Strominger, *Evidence for a New Soft Graviton Theorem*, [arXiv:1404.4091](#).
- [29] D. Kapec, P. Mitra, A.-M. Raclariu, and A. Strominger, *2D Stress Tensor for 4D Gravity*, *Phys. Rev. Lett.* **119** (2017), no. 12 121601, [[arXiv:1609.00282](#)].
- [30] S. Pasterski and S.-H. Shao, *Conformal basis for flat space amplitudes*, *Phys. Rev. D* **96** (2017), no. 6 065022, [[arXiv:1705.01027](#)].
- [31] S. Pasterski, S.-H. Shao, and A. Strominger, *Gluon Amplitudes as 2d Conformal Correlators*, *Phys. Rev. D* **96** (2017), no. 8 085006, [[arXiv:1706.03917](#)].
- [32] L. Donnay, S. Pasterski, and A. Puhm, *Asymptotic Symmetries and Celestial CFT*, *JHEP* **09** (2020) 176, [[arXiv:2005.08990](#)].
- [33] C. Cheung, A. de la Fuente, and R. Sundrum, *4D scattering amplitudes and asymptotic symmetries from 2D CFT*, *JHEP* **01** (2017) 112, [[arXiv:1609.00732](#)].
- [34] J. de Boer and S. N. Solodukhin, *A Holographic reduction of Minkowski space-time*, *Nucl. Phys. B* **665** (2003) 545–593, [[hep-th/0303006](#)].
- [35] N. Ogawa, T. Takayanagi, T. Tsuda, and T. Waki, *Wedge holography in flat space and celestial holography*, *Phys. Rev. D* **107** (2023), no. 2 026001, [[arXiv:2207.06735](#)].
- [36] D. Marolf, *Holography without strings?*, *Class. Quant. Grav.* **31** (2014) 015008, [[arXiv:1308.1977](#)].
- [37] S. Raju, *Lessons from the information paradox*, *Phys. Rept.* **943** (2022) 1–80, [[arXiv:2012.05770](#)].
- [38] W. Donnelly and L. Freidel, *Local subsystems in gauge theory and gravity*, *JHEP* **09** (2016) 102, [[arXiv:1601.04744](#)].
- [39] D. Harlow and J.-Q. Wu, *Covariant phase space with boundaries*, *JHEP* **10** (2020) 146, [[arXiv:1906.08616](#)].
- [40] G. Araujo-Regado, P. A. Hoehn, F. Sartini, and B. Tomova, *Soft edges: the many links between soft and edge modes*, *JHEP* **07** (2025) 180, [[arXiv:2412.14548](#)].
- [41] C. Rovelli, *Why Gauge?*, *Found. Phys.* **44** (2014), no. 1 91–104, [[arXiv:1308.5599](#)].

- [42] R. Loll, G. Fabiano, D. Frattulillo, and F. Wagner, *Quantum Gravity in 30 Questions*, *PoS CORFU2021* (2022) 316, [[arXiv:2206.06762](#)].
- [43] S. B. Giddings, *Quantum gravity observables: observation, algebras, and mathematical structure**, *J. Phys. A* **58** (2025), no. 41 415401, [[arXiv:2505.22708](#)].
- [44] C. Rovelli, *Partial observables*, *Phys. Rev. D* **65** (2002) 124013, [[gr-qc/0110035](#)].
- [45] D. Simmons-Duffin, *The Conformal Bootstrap*, in *Theoretical Advanced Study Institute in Elementary Particle Physics: New Frontiers in Fields and Strings*, pp. 1–74, 2017. [arXiv:1602.07982](#).
- [46] S. B. Giddings and A. Kinsella, *Gauge-invariant observables, gravitational dressings, and holography in AdS*, *JHEP* **11** (2018) 074, [[arXiv:1802.01602](#)].
- [47] R. F. Streater and A. S. Wightman, *PCT, spin and statistics, and all that*. 1989.
- [48] T. He, P. Mitra, A. P. Porfyriadis, and A. Strominger, *New Symmetries of Massless QED*, *JHEP* **10** (2014) 112, [[arXiv:1407.3789](#)].
- [49] A. Ashtekar and R. O. Hansen, *A unified treatment of null and spatial infinity in general relativity. I - Universal structure, asymptotic symmetries, and conserved quantities at spatial infinity*, *J. Math. Phys.* **19** (1978) 1542–1566.
- [50] D. Kapec, V. Lysov, and A. Strominger, *Asymptotic Symmetries of Massless QED in Even Dimensions*, *Adv. Theor. Math. Phys.* **21** (2017) 1747–1767, [[arXiv:1412.2763](#)].
- [51] T. He and P. Mitra, *Asymptotic symmetries and Weinberg’s soft photon theorem in $Mink_{d+2}$* , *JHEP* **10** (2019) 213, [[arXiv:1903.02608](#)].
- [52] M. Campiglia and A. Laddha, *Asymptotic symmetries and subleading soft graviton theorem*, *Phys. Rev. D* **90** (2014), no. 12 124028, [[arXiv:1408.2228](#)].
- [53] N. Miller, *From Noether’s Theorem to Bremsstrahlung: a pedagogical introduction to large gauge transformations and classical soft theorems*, [arXiv:2112.05289](#).
- [54] M. Campiglia and A. Laddha, *Subleading soft photons and large gauge transformations*, *JHEP* **11** (2016) 012, [[arXiv:1605.09677](#)].
- [55] V. Lysov, S. Pasterski, and A. Strominger, *Low’s Subleading Soft Theorem as a Symmetry of QED*, *Phys. Rev. Lett.* **113** (2014), no. 11 111601, [[arXiv:1407.3814](#)].
- [56] A. Kundu, *A Brief Review on the Asymptotic Symmetries of Gravity in Higher Dimensions*, [arXiv:2307.02143](#).

- [57] A. Ashtekar, *Geometry and physics of null infinity*, *Surveys Diff. Geom.* **20** (2015), no. 1 99–122, [[arXiv:1409.1800](#)].
- [58] T. Mädler and J. Winicour, *Bondi-Sachs Formalism*, *Scholarpedia* **11** (2016) 33528, [[arXiv:1609.01731](#)].
- [59] R. Penrose, *Conformal treatment of infinity*, .
- [60] K. Nguyen, *Schwarzian Transformations at Null Infinity or The Unobservable Sector of Celestial Holography*, [arXiv:2201.09640](#).
- [61] K. Prabhu, *Conservation of asymptotic charges from past to future null infinity: Supermomentum in general relativity*, *JHEP* **03** (2019) 148, [[arXiv:1902.08200](#)].
- [62] G. Compère, A. Fiorucci, and R. Ruzziconi, *Superboost transitions, refraction memory and super-Lorentz charge algebra*, *JHEP* **11** (2018) 200, [[arXiv:1810.00377](#)]. [Erratum: *JHEP* **04**, 172 (2020)].
- [63] H. Furugori, N. Ogawa, S. Sugishita, and T. Waki, *Celestial two-point functions and rectified dictionary*, *JHEP* **02** (2024) 063, [[arXiv:2312.07057](#)].
- [64] D. Simmons-Duffin, *Projectors, Shadows, and Conformal Blocks*, *JHEP* **04** (2014) 146, [[arXiv:1204.3894](#)].
- [65] A. Strominger, *The dS / CFT correspondence*, *JHEP* **10** (2001) 034, [[hep-th/0106113](#)].
- [66] Y. Hikida, T. Nishioka, T. Takayanagi, and Y. Taki, *CFT duals of three-dimensional de Sitter gravity*, *JHEP* **05** (2022) 129, [[arXiv:2203.02852](#)].
- [67] R. Bousso, A. Maloney, and A. Strominger, *Conformal vacua and entropy in de Sitter space*, *Phys. Rev. D* **65** (2002) 104039, [[hep-th/0112218](#)].
- [68] D. Baumann, D. Green, A. Joyce, E. Pajer, G. L. Pimentel, C. Sleight, and M. Taronna, *Snowmass White Paper: The Cosmological Bootstrap*, *SciPost Phys. Comm. Rep.* **2024** (2024) 1, [[arXiv:2203.08121](#)].
- [69] T. S. Bunch and P. C. W. Davies, *Quantum Field Theory in de Sitter Space: Renormalization by Point Splitting*, *Proc. Roy. Soc. Lond. A* **360** (1978) 117–134.
- [70] E. Mottola, *Particle Creation in de Sitter Space*, *Phys. Rev. D* **31** (1985) 754.
- [71] B. Allen, *Vacuum States in de Sitter Space*, *Phys. Rev. D* **32** (1985) 3136.

- [72] D. Marolf and I. A. Morrison, *The IR stability of de Sitter: Loop corrections to scalar propagators*, *Phys. Rev. D* **82** (2010) 105032, [[arXiv:1006.0035](#)].
- [73] D. Marolf and I. A. Morrison, *The IR stability of de Sitter QFT: results at all orders*, *Phys. Rev. D* **84** (2011) 044040, [[arXiv:1010.5327](#)].
- [74] A. Higuchi, D. Marolf, and I. A. Morrison, *On the Equivalence between Euclidean and In-In Formalisms in de Sitter QFT*, *Phys. Rev. D* **83** (2011) 084029, [[arXiv:1012.3415](#)].
- [75] A. Ball, E. Himwich, S. A. Narayanan, S. Pasterski, and A. Strominger, *Uplifting AdS_3/CFT_2 to flat space holography*, *JHEP* **08** (2019) 168, [[arXiv:1905.09809](#)].
- [76] L. Iacobacci, C. Sleight, and M. Taronna, *From celestial correlators to AdS, and back*, *JHEP* **06** (2023) 053, [[arXiv:2208.01629](#)].
- [77] C. Sleight and M. Taronna, *Celestial Holography Revisited*, *Phys. Rev. Lett.* **133** (2024), no. 24 241601, [[arXiv:2301.01810](#)].
- [78] L. Iacobacci, C. Sleight, and M. Taronna, *Celestial holography revisited. Part II. Correlators and Källén-Lehmann*, *JHEP* **08** (2024) 033, [[arXiv:2401.16591](#)].
- [79] E. Jørstad, S. Pasterski, and A. Sharma, *Equating extrapolate dictionaries for massless scattering*, *JHEP* **02** (2024) 228, [[arXiv:2310.02186](#)].
- [80] C. Sleight and M. Taronna, *From dS to AdS and back*, *JHEP* **12** (2021) 074, [[arXiv:2109.02725](#)].
- [81] J. M. Maldacena, *Non-Gaussian features of primordial fluctuations in single field inflationary models*, *JHEP* **05** (2003) 013, [[astro-ph/0210603](#)].
- [82] M. Fukuma, S. Sugishita, and Y. Sakatani, *Propagators in de Sitter space*, *Phys. Rev. D* **88** (2013), no. 2 024041, [[arXiv:1301.7352](#)].
- [83] D. Baumann, C. Duaso Pueyo, A. Joyce, H. Lee, and G. L. Pimentel, *The Cosmological Bootstrap: Spinning Correlators from Symmetries and Factorization*, *SciPost Phys.* **11** (2021) 071, [[arXiv:2005.04234](#)].
- [84] S. Kim, P. Kraus, R. Monten, and R. M. Myers, *S-matrix path integral approach to symmetries and soft theorems*, *JHEP* **10** (2023) 036, [[arXiv:2307.12368](#)].
- [85] E. Witten, *Quantum gravity in de Sitter space*, in *Strings 2001: International Conference*, 6, 2001. [hep-th/0106109](#).

- [86] M. Hogervorst, J. Penedones, and K. S. Vaziri, *Towards the non-perturbative cosmological bootstrap*, *JHEP* **02** (2023) 162, [[arXiv:2107.13871](#)].
- [87] G. Sengor and C. Skordis, *Scalar two-point functions at the late-time boundary of de Sitter*, *JHEP* **02** (2024) 076, [[arXiv:2110.01635](#)].
- [88] K. Salehi Vaziri, *A non-perturbative construction of the de Sitter late-time boundary*, [arXiv:2412.00183](#).
- [89] S. Pasterski and H. Verlinde, *Chaos in Celestial CFT*, [arXiv:2201.01630](#).
- [90] A. Fotopoulos, S. Stieberger, T. R. Taylor, and B. Zhu, *Extended BMS Algebra of Celestial CFT*, *JHEP* **03** (2020) 130, [[arXiv:1912.10973](#)].
- [91] S. Banerjee and S. Pasterski, *Revisiting the shadow stress tensor in celestial CFT*, *JHEP* **04** (2023) 118, [[arXiv:2212.00257](#)].
- [92] S. Pasterski, A. Puhm, and E. Trevisani, *Revisiting the conformally soft sector with celestial diamonds*, *JHEP* **11** (2021) 143, [[arXiv:2105.09792](#)].
- [93] D. Kapec and P. Mitra, *Shadows and soft exchange in celestial CFT*, *Phys. Rev. D* **105** (2022), no. 2 026009, [[arXiv:2109.00073](#)].
- [94] E. A. Bergshoeff, O. Hohm, and P. K. Townsend, *Massive Gravity in Three Dimensions*, *Phys. Rev. Lett.* **102** (2009) 201301, [[arXiv:0901.1766](#)].
- [95] D. Grumiller, W. Riedler, J. Rosseel, and T. Zojer, *Holographic applications of logarithmic conformal field theories*, *J. Phys. A* **46** (2013) 494002, [[arXiv:1302.0280](#)].
- [96] A. Fiorucci, D. Grumiller, and R. Ruzziconi, *Logarithmic celestial conformal field theory*, *Phys. Rev. D* **109** (2024), no. 2 L021902, [[arXiv:2305.08913](#)].
- [97] H. Elvang and Y.-t. Huang, *Scattering Amplitudes*, [arXiv:1308.1697](#).
- [98] H. Furugori, N. Ogawa, S. Sugishita, and T. Waki, *Bridging dS/CFT and Celestial Holography via Ward-Takahashi Identities*, [arXiv:2512.10367](#).
- [99] T. He, P. Mitra, and A. Strominger, *2D Kac-Moody Symmetry of 4D Yang-Mills Theory*, *JHEP* **10** (2016) 137, [[arXiv:1503.02663](#)].
- [100] X. Zhou, *Double Copy Relation in AdS Space*, *Phys. Rev. Lett.* **127** (2021), no. 14 141601, [[arXiv:2106.07651](#)].

- [101] S. Raju, *BCFW for Witten Diagrams*, *Phys. Rev. Lett.* **106** (2011) 091601, [[arXiv:1011.0780](#)].
- [102] L. Dolan, C. R. Nappi, and E. Witten, *Conformal operators for partially massless states*, *JHEP* **10** (2001) 016, [[hep-th/0109096](#)].
- [103] J. Polchinski, *S matrices from AdS space-time*, [hep-th/9901076](#).
- [104] L. Susskind, *Holography in the flat space limit*, *AIP Conf. Proc.* **493** (1999), no. 1 98–112, [[hep-th/9901079](#)].
- [105] S. B. Giddings, *Flat space scattering and bulk locality in the AdS / CFT correspondence*, *Phys. Rev. D* **61** (2000) 106008, [[hep-th/9907129](#)].
- [106] P.-X. Hao, N. Ogawa, T. Takayanagi, and T. Waki, *Flat space holography via AdS/BCFT*, *JHEP* **10** (2025) 159, [[arXiv:2509.00652](#)].
- [107] C. Sleight and M. Taronna, *From AdS to dS exchanges: Spectral representation, Mellin amplitudes, and crossing*, *Phys. Rev. D* **104** (2021), no. 8 L081902, [[arXiv:2007.09993](#)].
- [108] L. Di Pietro, V. Gorbenko, and S. Komatsu, *Analyticity and unitarity for cosmological correlators*, *JHEP* **03** (2022) 023, [[arXiv:2108.01695](#)].
- [109] M. Loparco, J. Penedones, K. Salehi Vaziri, and Z. Sun, *The Källén-Lehmann representation in de Sitter spacetime*, *JHEP* **12** (2023) 159, [[arXiv:2306.00090](#)].
- [110] S. Elitzur, *Impossibility of Spontaneously Breaking Local Symmetries*, *Phys. Rev. D* **12** (1975) 3978–3982.
- [111] A. Blommaert, T. G. Mertens, and H. Verschelde, *Edge dynamics from the path integral — Maxwell and Yang-Mills*, *JHEP* **11** (2018) 080, [[arXiv:1804.07585](#)].
- [112] S. Carrozza and P. A. Hoehn, *Edge modes as reference frames and boundary actions from post-selection*, *JHEP* **02** (2022) 172, [[arXiv:2109.06184](#)].