

On a remark of strongly convex functions of order β and convex of order α

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Abstract

For analytic functions $f(z)$ in the open unit disk $\mathbb{U}$, two subclasses $\mathcal{C}(\alpha, \beta)$ and $\mathcal{S}^*(\alpha, \beta)$ are introduced. The object of the present paper is to investigate a strongly starlikeness of order δ and starlikeness of order $\beta(\alpha)$ of strongly convex functions of order γ and convex of order α .

1 Introduction

Let $\mathcal{S}$ denote the set of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic and univalent in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$.

Suppose that $f(z) \in \mathcal{S}$ satisfies the following conditions

$$\left| \arg \left(1 + \frac{zf''(z)}{f'(z)} - \alpha \right) \right| < \frac{\pi}{2}\beta \quad (z \in \mathbb{U}) \quad (1)$$

or

$$\left| \arg \left(\frac{zf'(z)}{f(z)} - \alpha \right) \right| < \frac{\pi}{2}\beta \quad (z \in \mathbb{U}), \quad (2)$$

where $0 \leq \alpha < 1$ and $0 < \beta \leq 1$. If $f(z) \in \mathcal{S}$ satisfies (1), then we say that $f(z)$ is strongly convex of order β and convex of order α in $\mathbb{U}$ and we denote by $\mathcal{C}(\alpha, \beta)$ the class of such functions $f(z)$. If $f(z) \in \mathcal{S}$ satisfies (2), then $f(z)$ is said to be strongly starlike of order β and starlike of order α in $\mathbb{U}$ and we also denote by $\mathcal{S}^*(\alpha, \beta)$ the class of such functions $f(z)$.

In view of the results by MacGregor [1] and Wilken and Feng [3], it is well known that $f(z) \in \mathcal{C}(\alpha, 1)$ implies $f(z) \in \mathcal{S}^*(\beta(\alpha), 1)$ where

$$\beta(\alpha) = \begin{cases} \frac{1-2\alpha}{2^{2-2\alpha}(1-2^{2\alpha-1})} & (0 \leq \alpha < 1; \alpha \neq \frac{1}{2}) \\ \frac{1}{2\log 2} & (\alpha = \frac{1}{2}). \end{cases}$$

In the present paper, we discuss some properties for $f(z)$ concerning with the classes $\mathcal{C}(\alpha, \beta)$ and $\mathcal{S}^*(\alpha, \beta)$.

2 Main result

To consider our problems, we have to recall here the following lemma due to Nunokawa [2].

Lemma 1. *Let $p(z)$ be analytic in $\mathbb{U}$ with $p(0) = 1$ and $p(z) \neq 0$ in $\mathbb{U}$. Suppose that there exists a point $z_0 \in \mathbb{U}$ such that*

$$|\arg p(z)| < \frac{\pi}{2}\alpha \quad (|z| < |z_0|)$$

and

$$|\arg p(z_0)| = \frac{\pi}{2}\alpha$$

where $\alpha > 0$. Then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\alpha$$

where

$$k \geq \frac{1}{2} \left(a + \frac{1}{a} \right) \quad \text{where} \quad \arg p(z_0) = \frac{\pi}{2}\alpha$$

and

$$k \leq -\frac{1}{2} \left(a + \frac{1}{a} \right) \quad \text{where} \quad \arg p(z_0) = -\frac{\pi}{2}\alpha,$$

where

$$p(z_0)^{1/\alpha} = \pm ia \quad (a > 0).$$

Now, we derive

Theorem 1. *If $f(z) \in \mathcal{C}(\gamma, \alpha)$ with $0 \leq \alpha < 1$, then $f(z) \in \mathcal{S}^*(\delta, \beta(\alpha))$, where $0 < \delta < 1$ and*

$$(i) \quad \text{if } k = \frac{1}{2} \left(a + \frac{1}{a} \right) \geq 1 \text{ and}$$

$$\tan^{-1} \frac{\delta k}{(\beta(\alpha) + 1)(\beta(\alpha) - \alpha)} \geq \frac{\pi}{2}\delta,$$

then we put

$$\begin{aligned}\gamma &= \delta + \frac{2}{\pi} \text{Tan}^{-1} \frac{R_0 \sin(\Theta - \frac{\pi}{2}\delta)}{1 + R_0 \cos(\Theta - \frac{\pi}{2}\delta)}, \\ \Theta &= \text{Tan}^{-1} \frac{\delta k}{(\beta(\alpha) + 1)(\beta(\alpha) - \alpha)}, \\ R_0 &= \frac{\delta}{2(1 - \beta(\alpha)^2)} \left\{ \left(\frac{1 + \delta}{1 - \delta} \right)^{\frac{1-\delta}{2}} + \left(\frac{1 - \delta}{1 + \delta} \right)^{\frac{1+\delta}{2}} \right\},\end{aligned}$$

(ii) if $k = \frac{1}{2} \left(a + \frac{1}{a} \right) \geq 1$ and

$$\text{Tan}^{-1} \frac{\delta k}{(\beta(\alpha) + 1)(\beta(\alpha) - \alpha)} < \frac{\pi}{2} \delta,$$

then we put

$$\gamma = \text{Tan}^{-1} \frac{\delta k}{(\beta(\alpha) + 1)(\beta(\alpha) - \alpha)},$$

(iii) if $k = \frac{1}{2} \left(a + \frac{1}{a} \right) \leq -1$ and

$$\text{Tan}^{-1} \frac{\delta k}{(\beta(\alpha) + 1)(\beta(\alpha) - \alpha)} \leq -\frac{\pi}{2} \delta,$$

then we put

$$\begin{aligned}\gamma &= -\delta - \frac{2}{\pi} \text{Tan}^{-1} \frac{R_0 \sin(\Theta + \frac{\pi}{2}\delta)}{1 + R_0 \cos(\Theta + \frac{\pi}{2}\delta)}, \\ \Theta &= \text{Tan}^{-1} \frac{\delta k}{(\beta(\alpha) + 1)(\beta(\alpha) - \alpha)}, \\ R_0 &= \frac{\delta}{2(1 - \beta(\alpha)^2)} \left\{ \left(\frac{1 + \delta}{1 - \delta} \right)^{\frac{1-\delta}{2}} + \left(\frac{1 - \delta}{1 + \delta} \right)^{\frac{1+\delta}{2}} \right\}\end{aligned}$$

(iv) if $k = \frac{1}{2} \left(a + \frac{1}{a} \right) \leq -1$ and

$$\text{Tan}^{-1} \frac{\delta k}{(\beta(\alpha) + 1)(\beta(\alpha) - \alpha)} > -\frac{\pi}{2} \delta,$$

then we put

$$\gamma = \text{Tan}^{-1} \frac{\delta k}{(\beta(\alpha) + 1)(\beta(\alpha) - \alpha)}.$$

Proof. Let us define the function $p(z)$ by

$$p(z) = \frac{zf'(z)}{f(z)}.$$

Then we have that $p(z)$ is analytic in $\mathbb{U}$ and $p(0) = 1$. It follows that

$$p(z) + \frac{zp'(z)}{p(z)} = 1 + \frac{zf''(z)}{f'(z)},$$

and

$$\begin{aligned} 1 + \frac{zf''(z)}{f'(z)} - \alpha &= p(z) - \beta(\alpha) + \frac{zp'(z)}{p(z)} + \beta(\alpha) - \alpha \\ &= (p(z) - \beta(\alpha)) \left(1 + \frac{z(p(z) - \beta(\alpha))'}{p(z) - \beta(\alpha)} \frac{1}{p(z)} + \frac{\beta(\alpha) - \alpha}{p(z) - \beta(\alpha)} \right). \end{aligned}$$

Therefore, we see that

$$\begin{aligned} &\arg \left(1 + \frac{zf''(z)}{f'(z)} - \alpha \right) \\ &= \arg(p(z) - \beta(\alpha)) + \arg \left(1 + \frac{z(p(z) - \beta(\alpha))'}{p(z) - \beta(\alpha)} \frac{1}{p(z)} + \frac{\beta(\alpha) - \alpha}{p(z) - \beta(\alpha)} \right). \end{aligned}$$

If there exists a point z_0 , $|z_0| < 1$ such that

$$|\arg(p(z) - \beta(\alpha))| < \frac{\pi}{2}\delta \quad (|z| < |z_0|)$$

and

$$|\arg(p(z_0) - \beta(\alpha))| = \frac{\pi}{2}\delta,$$

then, let us put

$$q(z) = \frac{p(z) - \beta(\alpha)}{1 - \beta(\alpha)}, \quad q(0) = 1.$$

Applying Lemma 1, we have that

$$\frac{z_0 q'(z_0)}{q(z_0)} = \frac{z_0 q'(z_0)}{p(z_0) - \beta(\alpha)} = i\delta k$$

where

$$q(z_0)^{\frac{1}{\delta}} = \left(\frac{p(z_0) - \beta(\alpha)}{1 - \beta(\alpha)} \right)^{\frac{1}{\delta}} = \pm ia \quad (a > 0)$$

and

$$k \geq \frac{1}{2} \left(a + \frac{1}{a} \right) \geq 1 \quad \text{when} \quad \arg(p(z) - \beta(\alpha)) = \frac{\pi}{2}\delta$$

and

$$k \leq -\frac{1}{2} \left(a + \frac{1}{a} \right) \leq -1 \quad \text{when} \quad \arg(p(z) - \beta(\alpha)) = -\frac{\pi}{2}\delta.$$

At first, let us consider the case $\arg(p(z_0) - \beta(\alpha)) = \frac{\pi}{2}\delta$, then it follows that

$$\begin{aligned}
& \arg\left(1 + \frac{z_0 f''(z_0)}{f'(z_0)} - \alpha\right) \\
&= \arg(p(z_0) - \beta(\alpha)) \left(1 + \frac{z_0 p'(z_0)}{p(z_0) - \beta(\alpha)} \frac{1}{p(z_0)} + \frac{\beta(\alpha) - \alpha}{p(z_0) - \beta(\alpha)}\right) \\
&= \frac{\pi}{2}\delta + \arg\left(1 + \frac{i\delta k}{(\beta(\alpha) + (ia)^\delta)(1 - \beta(\alpha))} + \frac{\beta(\alpha) - \alpha}{(1 - \beta(\alpha))(ia)^\delta}\right) \\
&\geq \frac{\pi}{2}\delta + \arg\left(1 + \frac{i\delta k}{(\beta(\alpha) + 1)(1 - \beta(\alpha))(ia)^\delta} + \frac{\beta(\alpha) - \alpha}{(1 - \beta(\alpha))(ia)^\delta}\right) \\
&= \frac{\pi}{2}\delta + \arg\left(1 + e^{-i\frac{\pi}{2}\delta} \left(\frac{i\delta k}{(\beta(\alpha) + 1)(1 - \beta(\alpha))a^\delta} + \frac{\beta(\alpha) - \alpha}{(1 - \beta(\alpha))a^\delta}\right)\right).
\end{aligned}$$

Next, let us consider

$$\begin{aligned}
\arg\left(\frac{i\delta k}{(\beta(\alpha) + 1)(1 - \beta(\alpha))a^\delta} + \frac{\beta(\alpha) - \alpha}{(1 - \beta(\alpha))a^\delta}\right) &= \text{Tan}^{-1}\left(\frac{\frac{\delta k}{(\beta(\alpha) + 1)(1 - \beta(\alpha))a^\delta}}{\frac{\beta(\alpha) - \alpha}{(1 - \beta(\alpha))a^\delta}}\right) \\
&= \text{Tan}^{-1}\left(\frac{\delta k}{(\beta(\alpha) - \alpha)(\beta(\alpha) + 1)}\right).
\end{aligned}$$

On the other hand, applying the some method in the result by Nunokawa [2], we have

$$\begin{aligned}
& \left|\frac{\beta(\alpha) - \alpha}{(1 - \beta(\alpha))a^\delta} + \frac{i\delta k}{(\beta(\alpha) + 1)(1 - \beta(\alpha))a^\delta}\right| \\
&> \left|\frac{\delta}{2(1 - \beta(\alpha)^2)a^\delta} \left(a + \frac{1}{a}\right)\right| \\
&= \frac{\delta}{2(1 - \beta(\alpha))a^\delta} \left(a^{1-\delta} + \frac{1}{a^{1+\delta}}\right) \\
&\geq \frac{\delta}{2(1 - \beta(\alpha)^2)} \left(\left(\frac{1+\delta}{1-\delta}\right)^{\frac{1-\delta}{2}} + \left(\frac{1+\delta}{1-\delta}\right)^{\frac{1+\delta}{2}}\right) \\
&= R_0
\end{aligned} \tag{3}$$

say.

Therefore, for the case

$$\arg(p(z_0) - \beta(\alpha)) = \frac{\pi}{2}\delta$$

and

$$\text{Tan}^{-1}\left(\frac{\delta k}{(\beta(\alpha) - \alpha)(\beta(\alpha) + 1)}\right) \geq \frac{\pi}{2}\delta,$$

we have

$$\begin{aligned}
& \arg \left(1 + \frac{z_0 f''(z_0)}{f'(z_0)} - \alpha \right) \\
&= \arg(p(z_0) - \beta(\alpha)) + \arg \left(1 + \frac{z_0 p'(z_0)}{p(z_0) - \beta(\alpha)} \frac{1}{p(z_0)} + \frac{\beta(\alpha) - \alpha}{p(z_0) - \beta(\alpha)} \right) \\
&\geq \frac{\pi}{2} \delta + \arg \left(1 + R_0 e^{i(\Theta - \frac{\pi}{2} \delta)} \right) \\
&= \frac{\pi}{2} \delta + \tan^{-1} \frac{R_0 \sin(\Theta - \frac{\pi}{2} \delta)}{1 + R_0 \cos(\Theta - \frac{\pi}{2} \delta)},
\end{aligned}$$

where

$$\Theta = \tan^{-1} \frac{\delta k}{(\beta(\alpha) + 1)(\beta(\alpha) - \alpha)}.$$

This is the contradiction for the condition of the theorem.

On the other hand, for the case

$$\arg(p(z_0) - \beta(\alpha)) = \frac{\pi}{2} \delta$$

and

$$\Theta = \tan^{-1} \left(\frac{\delta k}{(\beta(\alpha) - \alpha)(\beta(\alpha) + 1)} \right) < \frac{\pi}{2} \delta,$$

putting $a \rightarrow +0$ in (3), we easily have

$$\begin{aligned}
& \arg \left(1 + \frac{z_0 f''(z_0)}{f'(z_0)} - \alpha \right) \\
&= \arg(p(z_0) - \beta(\alpha)) + \arg \left(1 + \frac{z_0 p'(z_0)}{p(z_0) - \beta(\alpha)} \frac{1}{p(z_0)} + \frac{\beta(\alpha) - \alpha}{p(z_0) - \beta(\alpha)} \right) \\
&\geq \frac{\pi}{2} \delta + \arg e^{i(\Theta - \frac{\pi}{2} \delta)} \\
&= \Theta \\
&= \tan^{-1} \frac{\delta k}{(\beta(\alpha) - \alpha)(\beta(\alpha) + 1)}.
\end{aligned}$$

This is also the contradiction for the theorem.

For the case

$$\arg(p(z_0) - \beta(\alpha)) = -\frac{\pi}{2} \delta$$

and

$$\tan^{-1} \left(\frac{\delta k}{(\beta(\alpha) - \alpha)(\beta(\alpha) + 1)} \right) \leq -\frac{\pi}{2} \delta,$$

where

$$k \leq -\frac{1}{2} \left(a + \frac{1}{a} \right)$$

and

$$\left(\frac{p(z_0) - \beta(\alpha)}{1 - \beta(\alpha)} \right)^{\frac{1}{\delta}} = -ia \quad (a > 0),$$

or for the case

$$\arg(p(z_0) - \beta(\alpha)) = -\frac{\pi}{2}\delta$$

and

$$\tan^{-1} \left(\frac{\delta k}{(\beta(\alpha) - \alpha)(\beta(\alpha) + 1)} \right) < -\frac{\pi}{2}\delta,$$

where

$$k \leq -\frac{1}{2} \left(a + \frac{1}{a} \right)$$

and

$$\left(\frac{p(z_0) - \beta(\alpha)}{1 - \beta(\alpha)} \right)^{\frac{1}{\delta}} = -ia \quad (a > 0),$$

applying the same method as the above, we have the contradiction for the theorem. Therefore the proof of the theorem is completed. $\square$

Remark 1. We have to say that Theorem 1 is not sharp. To consider the sharp result, we need another method. Therefore we leave this problem for our discussion.

References

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