

LOG-PLURICANONICAL SYSTEMS OF SMOOTH PROJECTIVE SURFACES

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1 Introduction

Let X be a smooth projective variety and K_X be a canonical divisor of X . Then X is called of general type when pluricanonical system $|mK_X|$ defines a birational embedding of X for some positive integer m . The behavior of the pluricanonical systems is important to study varieties of general type. For example, there is such a problem :

Problem 1.1 *Let X be a smooth projective variety of general type. Find a positive integer m_0 such that for every $m \geq m_0$, $|mK_X|$ gives a birational map from X into a projective space. $\square$*

In the case $\dim X = 1$, it is well known that $|3K_X|$ gives a projective embedding. In the case $\dim X = 2$, E. Bombieri proved that $|5K_X|$ gives a birational embedding([1]). Recently, H. Tsuji showed that there exists an integer ν_n which depends only on $n = \dim X$ and satisfies above problem([6,7]). But when $\dim X \geq 4$, effective value of ν_n is unknown. Even if $\dim X = 3$, the value of ν_n becomes an astronomical number, and it is supposed that the value computed in [7] is not best-possible.

I am interested in this problem for open surfaces. As long as I know, such a situation is not studied yet.

Definition 1.1 *Let X be a surface and D be a divisor with normal crossings. Then the pair (X, D) is called log-surface. If the linear system $|K_X + D|$ is big, we say that (X, D) is of log-general type. $\square$*

Now we state the problem more precisely.

Problem 1.2 *Let (X, D) be a smooth projective surface of log-general type. Find a positive integer m_0 such that for every $m \geq m_0$, $|m(K_X + D)|$ gives a birational map from X into a projective space. $\square$*

The main purpose of this paper is to answer the weaker version of this problem. Our result shows the value of m_0 for a given surface. But it depends on X and divisor D .

Theorem 1.1 *Let (X, D) be a smooth projective surface of log-general type, and let $K_X + D = P + E$ be a Zariski-decomposition of $K_X + D$, where P is a nef part and E is effective part of the decomposition. Then $|m(K_X + D)|$ defines a birational map from X into projective space unless*

$$m \geq \frac{6\sqrt{2}}{\sqrt{P^2}} + 4 .$$

$\square$

Remark 1.1 *In the case (X, D) is log-general type, $P^2 > 0$ holds ([3]).*

2 Terminology

In this section we introduce a singular hermitian metric and some results we use after. See [2] for more details.

Definition 2.1 *Let L be a holomorphic line bundle on X , h_0 be a C^∞ -hermitian metric on L , and φ be a L^1_{loc} -function on X . Then we call $h = e^{-\varphi} \cdot h_0$ a singular hermitian metric with respect to φ . φ is called a weight function of h . $\square$*

Definition 2.2 *We define a curvature current $i\Theta_h$ of a singular hermitian line bundle (L, h) as follows:*

$$i\Theta_h := i\partial\bar{\partial}\varphi + i\Theta_{h_0}$$

where $\partial\bar{\partial}$ is taken as a distribution and $i\Theta_{h_0}$ is the curvature form of (L, h_0) in usual sense.

A singular hermitian line bundle is said to be positive, if the curvature current $i\Theta_h$ becomes a measure which takes values in semipositive-defined hermitian matrix. $\square$

Next we introduce a concept of multiplier ideal sheaves. Let $U \subset X$ be an open set and $\mathcal{O}(U)$ be the set of holomorphic functions on U . Then

$$I_U(h) := \left\{ f \in \mathcal{O}(U) \mid \int_U e^{-\varphi} |f|^2 dV < +\infty \right\}$$

becomes a presheaf when U runs all open subsets of X . We put $I(h)$ as the sheafification of $I_U(h)$. $I(h)$ is called the multiplier ideal sheaf with respect to h . The following theorem which is a variant of Kodaira's vanishing theorem is due to A. Nadel ([5]).

Theorem 2.1 *Let (X, ω) be a Kähler manifold and (L, h) be a singular hermitian line bundle on X . Assume that $i\Theta_h \geq \varepsilon_0 \omega$ for some $\varepsilon_0 > 0$. Then*

$$H^q(X, \mathcal{O}_X(K_X + L) \otimes I(h)) = 0 \quad (q \geq 1).$$

□

3 Proof of Theorem 1.1

In this section, we show the outline of the proof of Theorem 1.1. The proof is made along [6, section 2], so please refer to [6] for detail.

Let (X, D) be a smooth projective surface of log-general type and $x, y \in X$, $x \neq y$ be generic two points. Assume that there exists a singular hermitian metric $h_{x,y}$ on $m(K_X + D) + D$ such that:

1. $x, y \in \text{supp } \mathcal{O}_X / I(h_{x,y})$
2. One of the x or y , say x , is an isolated point of $\text{supp } \mathcal{O}_X / I(h_{x,y})$
3. $i\Theta_h \geq \varepsilon_0 \omega$ for some $\varepsilon_0 > 0$.

We consider the long exact sequence:

$$\begin{aligned} \cdots \longrightarrow H^0(X, \mathcal{O}_X((m+1)(K_X + D))) \\ \longrightarrow H^0(X, \mathcal{O}_X((m+1)(K_X + D)) \otimes \mathcal{O} / I(h_{x,y})) \\ \longrightarrow H^1(X, \mathcal{O}_X(K_X + m(K_X + D) + D) \otimes I(h_{x,y})) \longrightarrow \cdots \end{aligned}$$

Where $H^1(X, \mathcal{O}_X(K_X + m(K_X + D) + D) \otimes I(h_{x,y})) = 0$ by Nadel's vanishing theorem, hence we get surjection and we can conclude that there exists some $\sigma \in H^0(X, \mathcal{O}_X((m+1)(K_X + D)))$ such that $\sigma(y) = 0$ and $\sigma(x) \neq 0$. This shows that $\Phi_{|(m+1)(K_X + D)|}$ separates x and y . Therefore to prove theorem 1.1, we have only to compute the value m such that we can construct a singular hermitian metric $h_{x,y}$ on $(m+1)(K_X + D)$ which satisfies the condition 1, 2 and 3 above for arbitrary distinct two points $x, y \in U$, for some nonempty Zariski open subset $U \subset X$.

3.1 Construction of $h_{x,y}$

Let $K_X + D = P + E$ be a Zariski-decomposition of $K_X + D$, where P is the nef part and E is the effective part of the decomposition. We put X° as follows:

$$X^\circ := \{ p \in X \mid p \notin Bs|mP| \text{ and for some } m, |mP| \text{ gives biholomorphic near } p \}$$

Then X° is a nonempty Zariski open set of X .

We take arbitrary $x, y \in X^\circ$ and we set $\mathcal{M}_{x,y} := \mathcal{M}_x \otimes \mathcal{M}_y$, where $\mathcal{M}_x$ and $\mathcal{M}_y$ are the maximum ideal sheaf of the points x, y respectively. By considering a cohomology exact sequence and comparing the dimension of $H^0(X, \mathcal{O}_X(mP))$ and $H^0(X, \mathcal{O}_X(mP) \otimes \mathcal{O}_X/\mathcal{M}_{x,y}^{\otimes \lceil \sqrt{\frac{P^2}{2}} \cdot (1-\varepsilon)m \rceil})$, we can show the following :

Proposition 3.1 *For arbitrary small $\varepsilon > 0$,*

$$\dim H^0(X, \mathcal{O}_X(mP) \otimes \mathcal{O}_X/\mathcal{M}_{x,y}^{\otimes \lceil \sqrt{\frac{P^2}{2}} \cdot (1-\varepsilon)m \rceil}) \geq 1$$

holds if we take m sufficiently large. $\square$

We take $\sigma_0 \in H^0(X, \mathcal{O}_X(m_0P) \otimes \mathcal{O}_X/\mathcal{M}_{x,y}^{\otimes \lceil \sqrt{\frac{P^2}{2}} \cdot (1-\varepsilon_0)m_0 \rceil})$ for sufficiently small ε_0 and sufficiently large m_0 .

If we set $h_0 := \frac{1}{|\sigma_0|^{2/m_0}}$, then h_0 is a singular hermitian metric on P with positive curvature.

We set α_0 as follows :

$$\alpha_0 := \inf \{ \alpha > 0 \mid \text{supp } \mathcal{O}_X/I(h_0^\alpha) \ni x, y \} .$$

σ_0 has zeros of order at least $\lceil \sqrt{\frac{P^2}{2}} \cdot (1-\varepsilon)m \rceil$, so we get $\alpha_0 \leq \sqrt{\frac{2}{P^2}} \cdot \frac{2}{1-\varepsilon_0}$.

Next we decrease α_0 a little bit. Then one of the following two cases occurs.

Case 1. $\text{supp } \mathcal{O}_X/I(h_0^{\alpha-\delta_0})$ does not include either x nor y .

Case 2. $\text{supp } \mathcal{O}_X/I(h_0^{\alpha-\delta_0})$ includes one of x or y , say x .

In Case 1, we can consider a minimal center of log canonical singularities at x . Let X_1 be a minimal center at x . In this case one of following two cases occurs.

Case 1-1. $\text{supp } \mathcal{O}_X/I(h_0^{\alpha-\delta_0})$ does not include x nor y .

Case 1-2. Otherwise.

We shall explain Case 1-1. (Other cases are easier to prove.)

Note that $(X_1 \cdot P) > 0$ because X_1 passes through $x \in X^\circ$.

Proposition 3.2 For arbitrary small $\varepsilon > 0$,

$$\dim H^0(X_1, \mathcal{O}_{X_1}(mP) \otimes \mathcal{O}_X / \mathcal{M}_{x,y}^{\otimes \lceil \frac{(X_1 \cdot P)}{2} \cdot (1-\varepsilon)m \rceil}) \geq 1$$

holds if we take m sufficiently large. $\square$

The proof of Proposition 3.2 is the same as the proof of Proposition 3.1.

We take $\tilde{\sigma}_1 \in H^0(X, \mathcal{O}_{X_1}(m_1 P) \otimes \mathcal{O}_X / \mathcal{M}_{x,y}^{\otimes \lceil \frac{(X_1 \cdot P)}{2} \cdot (1-\varepsilon_1)m_1 \rceil})$ for sufficiently small ε_1 and sufficiently large m_1 .

Because P is nef big, P has a decomposition $P = \mathcal{A} + \mathcal{E}$ by Kodaira's lemma. Where $\mathcal{A}$ is a $\mathbb{Q}$ -ample divisor and $\mathcal{E}$ is a $\mathbb{Q}$ -effective divisor. We take integer l_1 sufficiently large so that $L_1 := l_1 \cdot \mathcal{A}$ is $\mathbb{Z}$ -very ample. Let $\tau \in H^0(X_1, \mathcal{O}_{X_1}(L_1))$ be a section which is not zero section. then

$$\tilde{\sigma}_1 \otimes \tau \in H^0(X_1, \mathcal{O}_{X_1}(mP + L_1) \otimes \mathcal{O}_X / \mathcal{M}_{x,y}^{\otimes \lceil \frac{(X_1 \cdot P)}{2} \cdot (1-\varepsilon)m \rceil})$$

holds.

Proposition 3.3 For $m \geq 0$,

$$H^0(X, \mathcal{O}_X(mP + L_1)) \longrightarrow H^0(X_1, \mathcal{O}_{X_1}(mP + L_1))$$

is surjective if we take l_1 sufficiently large. $\square$

Proof. Set $\varphi = \alpha_0 \log \frac{h_0}{h_P}$. Where h_P is arbitrary C^∞ -hermitian metric on P . We consider $\varphi \cdot h_{L_1} \cdot h_{K_X}^{-1}$. This is a singular hermitian metric on $L_1 - K_X$. If we take l_1 sufficiently large, the curvature is strictly positive and $\mathcal{O}_X / I(\varphi) = \mathcal{O}_{X_1}$. Since P is nef, we get $H^1(X, \mathcal{O}_X(mP + L_1) \otimes I(h_{mP+L_1-K_X})) = 0$. This completes the proof. $\blacksquare$

By using this proposition, we extend $\tilde{\sigma}_1 \otimes \tau$ to

$$\sigma_1 \in H^0(X, \mathcal{O}_X((m_1 + l_1)P))$$

Let $\{\rho_j\}$ be generator of $\mathcal{O}_X((m_1 + l_1) \cdot \mathcal{A}) \otimes \mathcal{I}_X$. We put

$$h_1 := \frac{1}{(|\sigma_1|^2 + \sum |\rho_j|^2)^{1/(m_1+l_1)}}.$$

We take m_1 sufficiently large so that $\frac{l_1}{m_1} \leq \delta_0 \frac{(X_1 \cdot P)}{2}$ holds.

Proposition 3.4 Let $\alpha_1 = \inf \{ \alpha > 0 \mid \text{supp } \mathcal{O}_X / I(h_0^{\alpha_0 - \delta_0} \cdot h_1^\alpha) \ni x, y \}$. Assume x and y be regular points of X_1 . Then

$$\alpha_1 \leq \frac{2}{(X_1 \cdot P)} + O(\delta_0)$$

Proof. We can choose a neighborhood U of x and a local coordinate system (z_1, z_2) on U such that

$$U \cap X_1 = \{p \in U \mid z_1(p) = 0\} = \{(0, z_2)\}$$

holds. Then we get

$$\|\sigma_1\|^2 + \sum \|\rho_j\|^2 \leq C \cdot (|z_1|^2 + |z_2|^{2 \cdot \lceil \frac{(X_1 \cdot P)}{2} \cdot (1 - \varepsilon_1) \cdot m_1 \rceil}) ,$$

here $\|\cdot\|$ is taken with respect to some C^∞ -hermitian metric on $(m_1 + l_1)P$, and C is a constant depending on the norm $\|\cdot\|$. By the construction of σ_0 ,

$$\|\sigma_0\|^{\frac{2}{m_0} \cdot (\alpha_0 - \delta_0)} \leq O(|z_1|^{2 - \delta_0})$$

also holds on some neighborhood of generic points of $U \cap X_1$. Hence we get

$$\alpha_1 \leq \frac{(m_1 + l_1)}{m_1} \cdot \frac{2}{(X_1 \cdot P)} + O(\delta_0) .$$

From the assumption $\frac{l_1}{m_1} \leq \delta_0 \frac{(X_1 \cdot P)}{2}$, we conclude the statement of the proposition. ■

Remark 3.1 *Even if x and y are not regular points of X_1 , we can show above result is true by taking $\hat{x}$ and $\hat{y}$ as regular points of X_1 and letting $\hat{x} \rightarrow x$ and $\hat{y} \rightarrow y$.*

Lemma 3.1 $|m(K_X + D)|$ separates x and y for $m \geq \lceil \alpha_0 + \alpha_1 \rceil + 1$. □

Proof. By the equation

$$m(K_X + D) = K_X + (m - 1)P + (m - 1)E + D$$

and

$$(m - 1)P = \{(\alpha_0 - \delta_0) + \alpha_1\}P + \{m - 1 - (\alpha_0 - \delta_0 + \alpha_1)\}(\mathcal{A} + \mathcal{E}) ,$$

we can equip a singular hermitian metric $h_{x,y}$ by

$$h_{x,y} := h_0^{\alpha_0 - \delta_0} \cdot h_1^{\alpha_1} \cdot h_{\mathcal{A}}^{m-1-(\alpha_0 - \delta_0 + \alpha_1)} \cdot h_{\text{eff}} ,$$

where $h_{\mathcal{A}}$ is a C^∞ -hermitian metric of $\mathbf{Q}$ -ample divisor $\mathcal{A}$ and h_{eff} is a semipositive singular hermitian metric which comes from the other components. Then by the construction of h_0 and h_1 , $h_{x,y}$ satisfies the following conditions :

1. $x, y \in \text{supp } \mathcal{O}_X / I(h_{x,y})$
2. One of the x or y , say x , is an isolated point of $\text{supp } \mathcal{O}_X / I(h_{x,y})$.
3. $i\Theta_h \geq \varepsilon_0 \omega$ for some $\varepsilon_0 > 0$.

So there exists some $\sigma \in H^0(X, m(K_X + D))$ such that $\sigma(y) = 0$ and $\sigma(x) \neq 0$, or $\sigma(x) = 0$ and $\sigma(y) \neq 0$. This completes the proof. ■

Cororally 3.1 $|m(K_X + D)|$ separates x and y for

$$m \geq \frac{2\sqrt{2}}{\sqrt{P^2}} + \frac{2}{(X_1 \cdot P)} + 1 .$$

□

3.2 Construction of X_1 as a family

Our construction of X_1 is depending on the choice of the points x and y . Therefore it seems that the value of (X_1, P) is also depending on x and y . But in fact, (X_1, P) is independent of generic choice of $x, y \in X$. We explain it in this subsection.

Let $\Delta_X \subset X \times X$ be a diagonal set. We set $B \subset X \times X$ and $Z \subset B \times X$ as follows:

$$B := X^\circ \times X^\circ - \Delta_X$$

$$Z := \{ (z_1, z_2, z_3) \mid z_3 = x_1 \text{ or } z_2 = x_1 \} .$$

Let $p : X \times B \rightarrow X$ and $q : X \times B \rightarrow B$ be the first and second projection respectively. We consider

$$q_*(\mathcal{O}_{X \times B}(m_0 p^* P) \otimes \mathcal{I}_Z^{\otimes [\sqrt{\frac{P^2}{2}} \cdot (1-\varepsilon) m]})$$

instead of

$$H^0(X, \mathcal{O}_X(mP) \otimes \mathcal{O}_X / \mathcal{M}_{x,y}^{\otimes [\sqrt{\frac{P^2}{2}} \cdot (1-\varepsilon) m]}),$$

where $\mathcal{I}_Z$ denotes the ideal sheaf of Z . For a sufficiently large integer m_0 and sufficiently small ε , we take $\tilde{\sigma}_0$ as a nonzero global meromorphic section of

$$q_*(\mathcal{O}_{X \times B}(m_0 p^* P) \otimes \mathcal{I}_Z^{\otimes [\sqrt{\frac{P^2}{2}} \cdot (1-\varepsilon) m]}) .$$

$$\tilde{h}_0 := \frac{1}{|\tilde{\sigma}_0|^{2/m_0}} ,$$

then h_0 is a singular hermitian metric on P (but curvature current of $\tilde{h}_0$ may not be positive). We shall replace α_0 by

$$\tilde{\alpha}_0 = \inf \{ \alpha > 0 \mid \text{The generic points of } Z \subset \text{Spec}(\mathcal{O}_{X \times B}/\mathcal{I}(\tilde{h}_0^\alpha)) \} .$$

Then for every small $\delta > 0$, there exists a Zariski open subset U of B such that $\tilde{h}_0|_{X \times \{b\}}$ is well-defined for every $b \in U$, and

$$b \notin \text{Spec}(\mathcal{O}_{X \times \{b\}}/\mathcal{I}(\tilde{h}_0^{\tilde{\alpha}_0 - \delta})) ,$$

where we have identified b with distinct two points in X . By the construction of α_0 , we can see

$$b \subseteq \text{Spec}(\mathcal{O}_{X \times \{b\}}/\mathcal{I}(\tilde{h}_0^{\tilde{\alpha}_0}))$$

for every $b \in B$. Let $\tilde{X}_1$ be a minimal center of logcanonical singularities of $(X \times B, \frac{\tilde{\alpha}_0}{m_0}(\tilde{\sigma}_0))$ at the generic point of Z (although $(\tilde{\sigma}_0)$ may not be effective, but this is still meaningful in this case because of our construction of $\tilde{\sigma}_0$). Then $\tilde{X}_1 \cap q^{-1}(b)$ is almost a minimal center at $b := \{ \text{Distinct two points in } X^\circ \}$ which we construct in the last subsection. Remark that $\tilde{X}_1 \cap q^{-1}(b)$ may not be irreducible even for a general $b \in B$. But if we take a suitable finite cover

$$\phi_0 : B_0 \longrightarrow B ,$$

on the base change $X \times_B B_0$, $\hat{X}_1$ defines a family of irreducible subvarieties

$$f : \hat{X}_1 \longrightarrow U_0$$

of X parametrized by a nonempty Zariski open subset U_0 of $\phi_0^{-1}(U)$.

From above arguments, we see that $\{X_1\}$'s are numerically equivalent to each other when we move $b = (x, y) \in X^\circ \times X^\circ - \Delta_X$ generically. The intersection number (X_1, P) takes value in $\mathbb{Q}$, therefore (X_1, P) is constant if we choose $b = (x, y)$ generically. Hence we get:

Proposition 3.5 $|m(K_X + D)|$ defines a birational map from X to a projective space if

$$m \geq \frac{2\sqrt{2}}{\sqrt{P^2}} + \frac{2}{(X_1 \cdot P)} + 1 .$$

3.3 An estimate of $(X_1 \cdot P)$

To complete the proof of Theorem 1.1, we have to estimate $(X_1 \cdot P)$.

We consider the self-intersection number $(X_1)^2$. Then there are three possibilities:

Case 1. $(X_1)^2 > 0$

Case 2. $(X_1)^2 = 0$

Case 3. $(X_1)^2 < 0$

Let (x, y) and $(\hat{x}, \hat{y})$ be pair of distinct two points of X° . We put X_1 and $\hat{X}_1$ as a minimal center at (x, y) and $(\hat{x}, \hat{y})$ respectively. If we take (x, y) and $(\hat{x}, \hat{y})$ general, X_1 and $\hat{X}_1$ have no common irreducible components. Since X_1 and $\hat{X}_1$ are numerically equivalent, we get

$$(X_1)^2 = (\hat{X}_1)^2 = (X_1, \hat{X}_1) \geq 0 .$$

So we have only to consider the case $(X_1)^2 \geq 0$.

i) In the case $(X_1)^2 > 0$.

By the Hodge index theorem, we get

$$(X_1, P) \geq \sqrt{(X_1)^2} \cdot \sqrt{(P)^2} .$$

Since X_1 is an integral divisor, $(X_1)^2$ takes value in $\mathbf{Z}$. As a consequence we have $(X_1)^2 \geq 1$ and

$$\frac{1}{(X_1, P)} \leq \frac{1}{\sqrt{P^2}} .$$

So in this case the proof of Theorem 1.1 is completed.

ii) In the case $(X_1)^2 = 0$.

Let $\mathcal{N}_{X_1}$ be a normal bundle of X_1 . Then we have $\mathcal{N}_{X_1} = -X_1|_{X_1}$ and $\deg_{X_1} \mathcal{N}_{X_1} = -(X_1)^2 = 0$. So we see that the normal bundle of X_1 is trivial. Furthermore, X_1 can move. As a consequence, we can conclude existence of a fibration of X :

$$\pi : X \longrightarrow S ,$$

where S denotes some algebraic curve. By the definition of α_0 , $\alpha_0 P - \pi^*(p_x) - \pi^*(p_y)$ is a pseudoeffective line bundle on X . Here p_x and p_y denote the point $\pi(x)$ and $\pi(y)$ respectively. Because $\deg_S K_S = 2g_S - 2 \geq -2$, we have

$$\alpha_0 P \geq \pi^*(2 \text{ points in } S) \geq -\pi^* K_S$$

and

$$H^0(X, \mathcal{O}_X(m(1+\alpha_0)(K_X + D))) \supset H^0(X, \mathcal{O}_X(m(K_X + D - \pi^* K_S))) .$$

Recall that we regard $H^0(X, \mathcal{O}_X(m(K_X + D - \pi^* K_S)))$ as a subset of $H^0(X, \mathcal{O}_X(m(1+\alpha_0)(K_X + D)))$ by using natural injective map derived from the sheaf exact sequence

$$0 \longrightarrow \mathcal{O}_X(m(K_X + D - \pi^* K_S)) \longrightarrow \mathcal{O}_X(m(1+\alpha_0)(K_X + D)) ,$$

and hereafter we will often use such notation. By the definition of Zariski decomposition and above inclusion, we have the natural injection

$$\phi : H^0(X, \mathcal{O}_X(m(K_X + D - \pi^* K_S))) \hookrightarrow H^0(X, \mathcal{O}_X(m(1+\alpha_0)P)) ,$$

if we let m be an integer such that $m(1+\alpha_0)P$ is a $\mathbb{Z}$ -divisor.

The divisor $\pi_*(K_X + D - \pi^* K_S)$ is semipositive by Kawamata's semipositivity theorem[4, theorem 1], hence we get

$$H^0(S, \mathcal{O}_S(m\pi_*(K_X + D - \pi^* K_S))) \longrightarrow m\pi_*(K_X + D - \pi^* K_S) \otimes \mathcal{O}_S/\mathfrak{m}_p$$

is surjective for sufficiently large m . From the above surjection, we get

$$\begin{aligned} H^0(X, \mathcal{O}_X(m(K_X + D - \pi^* K_S))) \\ \longrightarrow H^0(\pi^{-1}(p), \mathcal{O}_{\pi^{-1}(p)}(m(K_X + D - \pi^* K_S)|_{\pi^{-1}(p)})) \end{aligned}$$

is also surjective. Since $\pi^* K_S|_{\pi^{-1}(p)}$ is trivial bundle, we have a surjection :

$$\begin{aligned} H^0(X, \mathcal{O}_X(m(K_X + D - \pi^* K_S))) \\ \longrightarrow H^0(\pi^{-1}(p), \mathcal{O}_{\pi^{-1}(p)}(m(K_X + D)|_{\pi^{-1}(p)})) . \end{aligned}$$

Let us consider $H^0(X, \mathcal{O}_X(m(1+\alpha_0)P))|_{\pi^{-1}(p)}$. By the natural injective map ϕ , we can see

$$H^0(X, \mathcal{O}_X(m(1+\alpha_0)P))|_{\pi^{-1}(p)} \supset H^0(\pi^{-1}(p), \mathcal{O}_{\pi^{-1}(p)}(m(K_X + D)|_{\pi^{-1}(p)}))$$

holds.

Let σ_1 and σ_2 be a global section of $H^0(X, \mathcal{O}_X(m(K_X + D - \pi^* K_S)))$ such that $\sigma_1|_{\pi^{-1}(p)}$ and $\sigma_2|_{\pi^{-1}(p)}$ are linearly independent. Then, if we take a general fiber $\pi^{-1}(p)$, $\phi(\sigma_1)|_{\pi^{-1}(p)}$ and $\phi(\sigma_2)|_{\pi^{-1}(p)}$ are also linearly independent.

Hence we get an inequality on dimensions of holomorphic sections:

$$\dim H^0(X, \mathcal{O}_X(m(1 + \alpha_0)P)) \Big|_{\pi^{-1}(p)} \geq \dim H^0(\pi^{-1}(p), \mathcal{O}_{\pi^{-1}(p)}(m\pi_*(K_X + D)|_{\pi^{-1}(p)})) .$$

We know the asymptotic relations :

$$\dim H^0(X, \mathcal{O}_X(m(1 + \alpha_0)P)) \Big|_{\pi^{-1}(p)} \sim m(1 + \alpha_0)(P, \pi^{-1}(p))$$

and

$$\dim H^0(\pi^{-1}(p), \mathcal{O}_{\pi^{-1}(p)}(m\pi_*(K_X + D)|_{\pi^{-1}(p)})) \sim m(K_X + D, \pi^{-1}(p)) ,$$

when we keep $m(1 + \alpha_0)P$ be integral divisor and letting m to be sufficiently large. Letting $m \rightarrow \infty$, we see

$$(1 + \alpha_0)(\pi^{-1}(p), P) \geq (\pi^{-1}(p), K_X + D) .$$

By definition, $(\pi^{-1}(p), P) = (X_1, P)$ and $(\pi^{-1}(p), K_X + D) = (X_1, K_X + D)$ holds. Hence we have

$$(1 + \alpha_0)(X_1, P) \geq (X_1, K_X + D) .$$

If we take a general fiber, $(K_X + D)|_{X_1}$ becomes a big divisor and

$$\deg_{X_1}(K_X + D) = (X_1, K_X + D) \geq 1$$

holds. Then we get an estimate for (X_1, P) :

$$1 + \alpha_0 \geq \frac{1}{(X_1, P)} .$$

Since $\alpha_0 \leq \sqrt{\frac{2}{P^2}} \cdot \frac{2}{1 - \varepsilon_0}$, then we have

$$\frac{2\sqrt{2}}{\sqrt{P^2}} + \frac{2}{(X_1 \cdot P)} + 1 \leq \frac{2\sqrt{2}}{\sqrt{P^2}} + \frac{\sqrt{2}}{\sqrt{P^2}} \cdot \frac{4}{1 - \varepsilon_0} + 3 ,$$

and this completes the proof of Theorem 1.1.

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