

Notes on Imperfect Repair by Fumio OHI Osaka University

0. Introduction

A component with failure time distribution $F(t) = 1 - e^{-\Lambda(t)}$ is repaired at failure. Each repair results in minimal repair or perfect repair. Let N be a positive integer valued random variable denoting the number of repairs performed till the perfect repair, i.e., $N = k$ means the event that $k-1$ minimal repairs were performed and the repair for the k -th failure was perfect and the component returned to the good-as-new-state. After the perfect repair the process is renewed. Time for repair is assumed to be negligible.

The dynamic process of the component is governed by a non-homogeneous Poisson process $\{N(t), t \geq 0\}$ with mean value function $\Lambda(t)$. Then T_N means the time that the component returns to the good-as-new-state, where $T_k = \inf \{t | N(t) = k\}$ $k \geq 1$, the time that the k -th failure occurs supposed that the repairs for the previous $k-1$ failures were minimal.

In this paper we study monotonic properties of T_N and other stochastic quantities, e.g., steady-state-distributions and so on. Our results may be of interest in renewal theory as well as in reliability theory.

1. Preliminaries

$\{N(t), t \geq 0\}$: a non-homogeneous Poisson Process with differentiable mean value function $\Lambda(t)$,

$$\Pr\{N(t) = k\} = e^{-\Lambda(t)} \frac{[\Lambda(t)]^k}{k!},$$

$$\lambda(t) = \frac{d}{dt} \Lambda(t),$$

$$T_k = \inf \{t \mid N(t) = k\}, \quad k = 1, 2, \dots,$$

$$T_0 \equiv 0.$$

The following theorem is easily proved.

Theorem 1. For $k \geq 0$, $\ell \geq 1$,

- (1) $\Pr[T_{k+\ell} - T_k > x \mid T_k = y] = \Pr[N(x+y) - N(y) < \ell],$
- (2) $\Pr[T_{k+\ell} - T_k > x] = \int_0^\infty \Pr[N(x+y) - N(y) < \ell] d\Pr[T_k \leq y],$
- (3) $E[T_{k+\ell} - T_k \mid T_k = y] = \int_0^\infty \Pr[N(x+y) - N(y) < \ell] dx,$
- (4) $E[T_{k+\ell} - T_k] = \int_0^\infty \int_0^\infty \Pr[N(x+y) - N(y) < \ell] dx d\Pr[T_k \leq y]. \quad \square$

Corollary 2. Letting $\ell=1$ in the previous theorem, for $k \geq 1$,

- (1) $\Pr[T_{k+1} - T_k > x \mid T_k = y] = e^{-[\Lambda(x+y) - \Lambda(y)]} = \frac{e^{-\Lambda(x+y)}}{e^{-\Lambda(y)}},$
- (2) $\Pr[T_{k+1} - T_k > x] = \int_0^\infty \frac{e^{-\Lambda(x+y)}}{e^{-\Lambda(y)}} d\Pr[T_k \leq y],$
- (3) $E[T_{k+1} - T_k \mid T_k = y] = \int_0^\infty \frac{e^{-\Lambda(x+y)}}{e^{-\Lambda(y)}} dx,$
- (4) $E[T_{k+1} - T_k] = \int_0^\infty \int_0^\infty \frac{e^{-\Lambda(x+y)}}{e^{-\Lambda(y)}} dx d\Pr[T_k \leq y]. \quad \square$

In the sequel we use the following lemmas, of which proof is easy and then omitted.

Lemma 3.

Let $g(x) \downarrow$, $g(x) \geq 0$ for $x \geq 0$, $f(x) \uparrow$, $f(x) \geq 0$ for $x \geq 0$. If two distribution functions F_1 and F_2 satisfy $F_1(0^-) = F_2(0^-) = 0$ and $F_1(x) \leq F_2(x)$ for $x \geq 0$, then

$$\int_0^\infty g(x) dF_1(x) \leq \int_0^\infty g(x) dF_2(x) \quad \text{and} \quad \int_0^\infty f(x) dF_1(x) \geq \int_0^\infty f(x) dF_2(x),$$

supposing that the integrations finitely exist. $\square$

Lemma 4. For $\lambda \geq \mu$, $\sum_{k=0}^n e^{-\lambda} \frac{\lambda^k}{k!} \leq \sum_{k=0}^n e^{-\mu} \frac{\mu^k}{k!}$ holds for $\forall n \geq 0$. $\square$

Theorem 5. For $k \geq 0$, $\ell \geq 1$,

- (1) $1 - e^{(t)} : \text{IFR} \Rightarrow \Pr[T_{k+\ell} - T_k > x] \downarrow_k \text{ for } \forall x \geq 0,$

- (2) $1-e^{-\Lambda(t)}$: DMRL $\Rightarrow E[T_{k+\ell}-T_k] \downarrow_k$,
 (3) $1-e^{-\Lambda(t)}$: NBU $\Rightarrow \Pr[T_{k+\ell}-T_k > x] \leq \Pr[T_\ell > x]$ for $\forall x \geq 0$,
 (4) $1-e^{-\Lambda(t)}$: NBUE $\Rightarrow E[T_{k+\ell}-T_k] \leq \ell E[T_1]$.

Proof. We notice that $T_k \uparrow_k$ a.s.

- (1) $1-e^{-\Lambda(t)}$: IFR $\Leftrightarrow \Lambda(x+y)-\Lambda(y) \uparrow_y \Rightarrow \Pr[N(x+y)-N(y) < \ell] \downarrow_y$ (by Lemma 4)
 $\Rightarrow \Pr[T_{k+\ell}-T_k > x] \downarrow_k$ (by Theorem 1 (2) and Lemma 3).

(2) $E[T_{k+\ell}-T_k] = \sum_{i=1}^{\ell-1} E[T_{k+i+1}-T_{k+i}]$ is decreasing in k , since each term of the right hand side is decreasing in k by Corollary 2 and Lemma 3.

(3) By Lemma 4, $\Pr[N(x+y)-N(y) < \ell] \leq \Pr[N(x) < \ell] = \Pr[T_\ell > x]$. Then (3) is obvious.

(4) $E[T_{k+\ell}-T_k] = \sum_{i=0}^{\ell-1} E[T_{k+i+1}-T_{k+i}]$ and $E[T_{k+i+1}-T_{k+i}] \leq \int_0^\infty e^{-\Lambda(x)} dx = E[T_1]$ by the assumption and Corollary 2 (4). Then we have the inequality.

□

2. Monotonic Properties of T_k

Let N be a positive integer valued r.v. independent with $\{N(t), t \geq 0\}$. In this section we study monotonic properties of T_N .

Theorem 6. (1) Suppose that Z_i ($i \geq 1$) are i.i.d. r.v.'s with common distribution function same to the one of T_1 , and are independent with N .

- (1) $1-e^{-\Lambda(t)}$: NBU $\Rightarrow \Pr[T_N > t] \leq \Pr[\sum_{i=1}^N Z_i > t]$.
 (2) $1-e^{-\Lambda(t)}$: NBUE $\Rightarrow E[T_N] \leq E[N]E[T_1]$.

Proof. (2) is obvious from Theorem 5 (4).

(1) It is sufficient to prove $\Pr[T_k > t] \leq \Pr[Z_1 + \dots + Z_k > t]$ for $k \geq 1$. We prove the inequality by the mathematical induction on k .

$$\begin{aligned} \Pr[T_{k+1} > t] &= \int \Pr[T_{k+1} > t | T_k = x] d\Pr[T_k \leq x] \\ &= \int \Pr[T_{k+1} - T_k > t - x | T_k = x] d\Pr[T_k \leq x] \\ &\leq \int \Pr[Z_{k+1} > t - x] d\Pr[T_k \leq x] \quad (\text{by Corollary 2 (1) and th} \end{aligned}$$

assumption)

$$\leq \int \Pr[Z_{k+1} > t-x] d\Pr[Z_1 + \dots + Z_k \leq x] \quad (\text{by the inductive assumption and Lemma 3})$$

$$= \Pr[Z_1 + \dots + Z_{k+1} > t]. \quad \square$$

3. Stochastic Comparisons of T_N and $T_{N'}$

$$\begin{aligned} \text{Let } \bar{F}_N(t) &= \Pr[T_N > t] = \sum_{k=0}^{\infty} \Pr[N(t)=k] \Pr[N > k], \\ f_N(t) &= \frac{d}{dt} \Pr[T_N \leq t] = \sum_{k=0}^{\infty} \Pr[N(t)=k] \lambda(t) \Pr[N=k+1], \\ \lambda_N(t) &= \frac{f_N(t)}{\bar{F}_N(t)}. \end{aligned}$$

In this section N and N' are positive integer valued r.v.'s independent with $\{N(t), t \geq 0\}$

Theorem 7. (1) $\frac{\Pr[N=\ell]}{\Pr[N=k]} \leq \frac{\Pr[N'=\ell]}{\Pr[N'=k]}$ for $\ell < k \Rightarrow \frac{f_N(x+\Delta)}{f_N(x)} \leq$

$\frac{f_{N'}(x+\Delta)}{f_{N'}(x)}$ for $x \geq 0$ and $\Delta \geq 0$.

$$\begin{aligned} (2) \frac{\Pr[N=k+1]}{\Pr[N > k]} \leq \frac{\Pr[N'=k+1]}{\Pr[N' > k]} \text{ for } k \geq 1 &\Leftrightarrow \frac{\Pr[N > \ell]}{\Pr[N > k]} \geq \frac{\Pr[N' > \ell]}{\Pr[N' > k]} \text{ for } k < \ell \\ \Rightarrow \frac{\bar{F}_N(t+\Delta)}{\bar{F}_N(t)} \geq \frac{\bar{F}_{N'}(t+\Delta)}{\bar{F}_{N'}(t)} \text{ for } t \geq 0, \Delta \geq 0 &\Leftrightarrow \lambda_N(t) \leq \lambda_{N'}(t) \text{ for } t \geq 0. \end{aligned}$$

$$(3) \Pr[N > \ell] \geq \Pr[N' > \ell] \text{ for } \ell \geq 1 \Rightarrow \bar{F}_N(t) \geq \bar{F}_{N'}(t) \text{ for } t \geq 0.$$

Proof. (1)

$$\begin{aligned} &\left| \frac{\sum_{k=0}^{\infty} \Pr[N(x+\Delta)=k] \lambda(x+\Delta) \Pr[N=k+1]}{\sum_{k=0}^{\infty} \Pr[N(x)=k] \lambda(x) \Pr[N=k+1]} - \frac{\sum_{k=0}^{\infty} \Pr[N(x+\Delta)=k] \lambda(x+\Delta) \Pr[N'=k+1]}{\sum_{k=0}^{\infty} \Pr[N(x)=k] \lambda(x) \Pr[N'=k+1]} \right| \\ &= \sum_{k < \ell} \left| \frac{\Pr[N(x+\Delta)=k] \lambda(x+\Delta)}{\Pr[N(x)=k] \lambda(x)} - \frac{\Pr[N(x+\Delta)=\ell] \lambda(x+\Delta)}{\Pr[N(x)=\ell] \lambda(x)} \right| \left| \frac{\Pr[N=k+1]}{\Pr[N=\ell+1]} - \frac{\Pr[N'=k+1]}{\Pr[N'=\ell+1]} \right| \\ &\leq 0. \end{aligned}$$

(2) The equivalent relations of (2) is obvious.

$$\left| \frac{\bar{F}_N(t+\Delta)}{\bar{F}_N(t)} - \frac{\bar{F}_{N'}(t+\Delta)}{\bar{F}_{N'}(t)} \right| = \sum_{k < \ell} \left| \frac{\Pr[N(t+\Delta)=k]}{\Pr[N(t)=k]} - \frac{\Pr[N(t+\Delta)=\ell]}{\Pr[N(t)=\ell]} \right| \left| \frac{\Pr[N > k]}{\Pr[N > \ell]} - \frac{\Pr[N' > k]}{\Pr[N' > \ell]} \right|$$

≥ 0 .

The relation (3) is easily proved by using Lemma 3. $\square$

We present simple bounds for the distribution and the expectation of T_N .

Corollary 8. Let $q_m = \inf_k \Pr[N > k+1 | N > k]$, $q_M = \sup_k \Pr[N > k+1 | N > k]$, and N_m and N_M be positive integer valued r.v.'s independent with $\{N(t), t \geq 0\}$ such that $\Pr[N_m = k] = q_m^{k-1}(1-q_m)$, $\Pr[N_M = k] = q_M^{k-1}(1-q_M)$. Since $\Pr[N_m > k] \leq \Pr[N > k] \leq \Pr[N_M > k]$ for $k \geq 1$, by Theorem 7 we have $\Pr[T_{N_m} > t] \leq \Pr[T_N > t] \leq \Pr[T_{N_M} > t]$ for $t \geq 0$ and $E[T_{N_m}] \leq E[T_N] \leq E[T_{N_M}]$. $\square$

Remark 9. Theorem 7 (1) (2) (3) show that stochastically-larger-relations between N and N' are preserved to the same stochastic relations between T_N and $T_{N'}$, without any assumption on $1-e^{-\Lambda(t)}$. $\square$

Theorem 10. $1-e^{-\Lambda(t)}$:DMRL, $\frac{\sum_{k=j}^{\infty} \Pr[N \geq k]}{E[N]} \geq \frac{\sum_{k=j}^{\infty} \Pr[N' \geq k]}{E[N']}$ for $j \geq 1$
 $\Rightarrow \frac{E[T_N]}{E[N]} \leq \frac{E[T_{N'}]}{E[N']}$.

Proof. Since $1-e^{-\Lambda(t)}$ is DMRL, $E[T_j - T_{j-1}]$ is decreasing in j by Theorem 5 (2). Then using Lemma 3,

$$\frac{E[T_N]}{E[N]} = \sum_{j=1}^{\infty} E[T_j - T_{j-1}] \frac{\Pr[N \geq j]}{E[N]} \leq \sum_{j=1}^{\infty} E[T_j - T_{j-1}] \frac{\Pr[N' \geq j]}{E[N']} = \frac{E[T_{N'}]}{E[N']} . \square$$

Remark 11. The following relation holds.

$$\frac{\Pr[N=\ell]}{\Pr[N=k]} \geq \frac{\Pr[N'=\ell]}{\Pr[N'=k]} \text{ for } k \leq \ell \Rightarrow \frac{\Pr[N \geq \ell]}{\Pr[N \geq k]} \geq \frac{\Pr[N' \geq \ell]}{\Pr[N' \geq k]} \text{ for } k \leq \ell$$

$$\Rightarrow \frac{\sum_{k=j}^{\infty} \Pr[N \geq k]}{E[N]} \geq \frac{\sum_{k=j}^{\infty} \Pr[N' \geq k]}{E[N']} \text{ for } j \geq 1 . (1)$$

$$\Rightarrow \Pr[N \geq k] \geq \Pr[N' \geq k] \text{ for } k \geq 1 . (2)$$

There is generally no relation between (1) and (2). $\square$

Remark 12. It is easily verified that if $\Pr[N \leq 2] = \Pr[N' \leq 2] = 1$, $\Pr[N=2] \geq \Pr[N'=2]$ and $1-e^{-\Lambda(t)}$ is NBU, then $\frac{E[T_N]}{E[N]} \leq \frac{E[T_{N'}]}{E[N']}$. $\square$

Lemma 13. For $a_1 \geq a_2 \geq \dots$, $\frac{a_1 + \dots + a_n}{n} \geq \frac{a_1 + \dots + a_{n+1}}{n+1}$ holds for $n \geq 1$. The proof is easy and omitted. $\square$

Theorem 14. $1 - e^{-\Lambda(t)}$:DMRL and $\Pr[N > k] \geq \Pr[N' > k]$ for $k \geq 1$

$$\Rightarrow E\left[\frac{T_N}{N}\right] \leq E\left[\frac{T_{N'}}{N'}\right].$$

Proof. Since $1 - e^{-\Lambda(t)}$ is DMRL, $E[T_j - T_{j-1}]$ is decreasing in j by Theorem 5 (2). Then by Lemma 13, $E\left[\frac{T_k}{k}\right] = \frac{\sum_{j=1}^k E[T_j - T_{j-1}]}{k}$ is decreasing in k . Then Theorem 14 is obvious by Lemma 3. $\square$

4. Stochastic Comparisons of Steady-State-Distributions

$\{N^j(t), t \geq 0\}$ ($j \geq 1$) : independent non-homogeneous Poisson

processes with common mean value function $\Lambda(t)$,

$$T_k^j = \inf \{t | N^j(t) = k\}$$

N_j ($j \geq 1$) : independent positive integer valued r.v.'s with common distribution same to the one of N ,

i.e., $\{N^j(t), t \geq 0\}$ ($j \geq 1$) are replicas of $\{N(t), t \geq 0\}$ and N_j ($j \geq 1$) are replicas of N . We assume that $\{N^j(t), t \geq 0\}$ ($j \geq 1$) are independent with N_j ($j \geq 1$).

We define a counting process $\{M(t), t \geq 0\}$ as

$$M(t) = \sum_{j=1}^{n-1} N_j + N^n(t - \sum_{j=1}^{n-1} T_{N_j}^j) \quad \text{if} \quad \sum_{j=1}^{n-1} T_{N_j}^j \leq t \leq \sum_{j=1}^n T_{N_j}^j.$$

We notice that $T_{N_j}^j$ ($j \geq 1$) are i.i.d. random variables with common distribution function $F_N(t)$, $1 - F_N(t) = \sum_{k=0}^{\infty} \Pr[N(t)=k] \Pr[N > k]$. In this section we consider stochastic quantities with respect to $\{M(t), t \geq 0\}$, which means the number of repairs performed in $[0, t]$.

Let's define

$$Z(t) = T_{N^n(t - \sum_{j=1}^{n-1} T_{N_j}^j) + 1}^n - (t - \sum_{j=1}^{n-1} T_{N_j}^j) \quad \text{if} \quad \sum_{j=1}^{n-1} T_{N_j}^j \leq t \leq \sum_{j=1}^n T_{N_j}^j,$$

which means the time to the next failure from the time epoch t .

Theorem 15.

$$\lim_{t \rightarrow \infty} \Pr[Z(t) > x] = \int_0^{\infty} \frac{\bar{F}_N(y)}{E[T_N]} \cdot \frac{e^{-\Lambda(x+y)}}{e^{-\Lambda(y)}} dy.$$

Proof. Simple calculation verifies the above equality. Since $\Pr[Z(t) > x] = \int_0^t \Pr[T_{N(x-y)+1} > t-y+x, t-y < T_N] d\{\sum_{n=1}^{\infty} (F_N)^{(n-1)}(y)\}$, where $(F_N)^{(n-1)}$ is the $(n-1)$ -fold convolution of F_N , then by the basic renewal theory we have

$$\lim_{t \rightarrow \infty} \Pr[Z(t) > x] = \frac{1}{E[T_N]} \int_0^{\infty} \Pr[T_{N(y)+1} > y+x, y < T_N] dy.$$

Noticing that $\Pr[T_{N(y)+1} > y+x, y < T_N] = \bar{F}_N(y) \Pr[N(y+x) - N(y) = 0]$, the theorem is proved. $\square$

We write the steady-state-distribution of $Z(t)$ as H , i.e.,

$$\bar{H}_N(x) = 1 - H_N(x) = \int_0^{\infty} \frac{\bar{F}_N(y)}{E[T_N]} \cdot \frac{e^{-\Lambda(x+y)}}{e^{-\Lambda(y)}} dy.$$

Theorem 16. (1) The density function $h_N(x)$ and the failure rate function $r_N(x)$ of $H_N(x)$ are

$$h_N(x) = \int_0^{\infty} \frac{\bar{F}_N(y)}{E[T_N]} \cdot \frac{e^{-\Lambda(x+y)}}{e^{-\Lambda(y)}} \cdot \lambda(x+y) dy,$$

$$r_N(x) = \frac{\int_0^{\infty} \bar{F}_N(y) \cdot \frac{e^{-\Lambda(x+y)}}{e^{-\Lambda(y)}} \cdot \lambda(x+y) dy}{\int_0^{\infty} \bar{F}_N(y) \cdot \frac{e^{-\Lambda(x+y)}}{e^{-\Lambda(y)}} dy}.$$

$$(2) \quad \frac{\Pr[N > \ell]}{\Pr[N > k]} \leq \frac{\Pr[N' > \ell]}{\Pr[N' > k]} \quad \text{for } k < \ell, \quad 1 - e^{-\Lambda(t)} \text{ has PF}_2\text{-density}$$

$$\Rightarrow \frac{h_N(t+\Delta)}{h_N(t)} \geq \frac{h_{N'}(t+\Delta)}{h_{N'}(t)} \quad \text{for } t > 0, \Delta > 0.$$

$$(3) \quad \frac{\Pr[N > \ell]}{\Pr[N > k]} \leq \frac{\Pr[N' > \ell]}{\Pr[N' > k]} \quad \text{for } k < \ell, \quad 1 - e^{-\Lambda(t)} : \text{IFR}$$

$$\Rightarrow r_N(x) \leq r_{N'}(x) \quad \text{for } \forall x > 0 \Rightarrow H_N(x) \leq H_{N'}(x) \quad \text{for } \forall x > 0.$$

Proof. Differentiating $H_N(x)$, (1) is easily obtained.

$$\begin{aligned}
 (2) & \left| \int_0^\infty \bar{F}_N(y) \cdot \frac{e^{-\Lambda(x+\Delta+y)}}{e^{-\Lambda(y)}} \cdot \lambda(x+\Delta+y) dy - \int_0^\infty \bar{F}_{N'}(y) \cdot \frac{e^{-\Lambda(x+\Delta+y)}}{e^{-\Lambda(y)}} \cdot \lambda(x+\Delta+y) dy \right| \\
 & \left| \int_0^\infty \bar{F}_N(y) \cdot \frac{e^{-\Lambda(x+y)}}{e^{-\Lambda(y)}} \cdot \lambda(x+y) dy - \int_0^\infty \bar{F}_{N'}(y) \cdot \frac{e^{-\Lambda(x+y)}}{e^{-\Lambda(y)}} \cdot \lambda(x+y) dy \right| \\
 & = \int_{y_1 < y_2} \left| \frac{\bar{F}_N(y_1) - \bar{F}_{N'}(y_1)}{\bar{F}_N(y_1) \bar{F}_{N'}(y_1)} \right| \left| \frac{e^{-\Lambda(x+\Delta+y_1)}}{e^{-\Lambda(y_1)}} \cdot \lambda(x+\Delta+y_1) - \frac{e^{-\Lambda(x+\Delta+y_2)}}{e^{-\Lambda(y_2)}} \cdot \lambda(x+\Delta+y_2) \right| \\
 & \quad \left| \frac{e^{-\Lambda(x+y_1)}}{e^{-\Lambda(y_1)}} \cdot \lambda(x+y_1) - \frac{e^{-\Lambda(x+y_2)}}{e^{-\Lambda(y_2)}} \cdot \lambda(x+y_2) \right| \\
 & \geq 0.
 \end{aligned}$$

Using Basic Composition Theorem, (3) is proved similarly to the proof of (2). $\square$

We define

$$Z^*(t) = \sum_{j=1}^n T_{N_j}^j - t \quad \text{if} \quad \sum_{j=1}^{n-1} T_{N_j}^j \leq t \leq \sum_{j=1}^n T_{N_j}^j,$$

which means the time to the next perfect repair from the time epoch t .

It is well known that

$$\lim_{t \rightarrow \infty} \Pr[Z^*(t) \leq x] = \frac{1}{E[T_N]} \int_0^x \bar{F}_N(u) du.$$

We write the right hand side of the above equality as $H_N^*(x)$.

Theorem 17. $\frac{\Pr[N > \ell]}{\Pr[N > k]} \leq \frac{\Pr[N' > \ell]}{\Pr[N' > k]}$ for $k < \ell$

$$\Rightarrow \frac{\bar{H}_N^*(t+\Delta)}{\bar{H}_N^*(t)} \leq \frac{\bar{H}_{N'}^*(t+\Delta)}{\bar{H}_{N'}^*(t)} \quad \text{for } t > 0, \Delta > 0 \Rightarrow H_N^*(t) \geq H_{N'}^*(t) \quad \text{for } t > 0.$$

Proof.

$$\left| \frac{\int_{t+\Delta}^\infty \bar{F}_N(u) du}{\int_t^\infty \bar{F}_N(u) du} - \frac{\int_{t+\Delta}^\infty \bar{F}_{N'}(u) du}{\int_t^\infty \bar{F}_{N'}(u) du} \right| = \left| \frac{\int_{t+\Delta}^\infty \bar{F}_N(u) du}{\int_t^{t+\Delta} \bar{F}_N(u) du} - \frac{\int_{t+\Delta}^\infty \bar{F}_{N'}(u) du}{\int_t^{t+\Delta} \bar{F}_{N'}(u) du} \right| \leq 0. \quad \square$$

Noticing that $\bar{H}_N(x) = \int_0^\infty \frac{e^{-\Lambda(x+y)}}{e^{-\Lambda(y)}} dH_N^*(y)$, we have by Theorem

17 and Lemma 3 :

Theorem 18. $\frac{\Pr[N > \ell]}{\Pr[N > k]} \leq \frac{\Pr[N' > \ell]}{\Pr[N' > k]}$ for $k \leq \ell$, $1 - e^{-\Lambda(t)} : \text{DMRL}$
 $\Rightarrow \int_0^\infty \bar{H}_N(x) dx \geq \int_0^\infty \bar{H}_{N'}(x) dx$. $\square$

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