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Estimates of Dirichlet Eigenvalues for Degenerate Elliptic Operators

By

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Abstract

Let Ω be a bounded open domain in $\mathbb{R}^n$ with smooth boundary and $X = (X_1, X_2, \cdots, X_m)$ be a system of real smooth vector fields defined on Ω with the boundary $\partial\Omega$ which is non-characteristic for X. If X satisfies the Hörmander's condition, then the vector fields is finite degenerate and the sum of square operator $\triangle_X = \sum_{j=1}^{m} X_j^2$ is a finitely degenerate elliptic operator, otherwise the operator $-\triangle_X$ is called infinitely degenerate. If λ_j is the j^{th} Dirichlet eigenvalue for $-\triangle_X$ on Ω, then this paper shall study the lower bound estimates for λ_j. Firstly, by using the sub-elliptic estimate directly, we shall give a simple lower bound estimates of λ_j for general finitely degenerate $\triangle_X$ which is polynomial increasing in j. Secondly, if $\triangle_X$ is so-called Grushin type degenerate elliptic operator, then we can give a precise lower bound estimates for λ_j. Finally, by using logarithmic regularity estimate, for infinitely degenerate elliptic operator $\triangle_X$ we prove that the lower bound estimates of λ_j will be logarithmic increasing in j.

§ 1. Introduction and Main Results

Let $\tilde{\Omega}$ be an open domain in $\mathbb{R}^n$ $(n \geq 2)$. For the systems of real smooth vector fields $X = (X_1, X_2, \cdots, X_m)$ defined on $\tilde{\Omega}$, we introduce following function space:

$$H_X^1(\tilde{\Omega}) = \{u \in L^2(\tilde{\Omega}) \mid X_j u \in L^2(\tilde{\Omega}), \quad j = 1, \cdots, m\},$$

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which is a Hilbert space with norm $\|u\|^2_{H^1_X} = \|u\|^2_{L^2}+\|Xu\|^2_{L^2}$, $\|Xu\|^2_{L^2} = \sum_{j=1}^n \|X_j u\|^2_{L^2}$. Let $\Omega \subset\subset \tilde{\Omega}$ be a bounded open subset with boundary $\partial\Omega$, here we assume that $\partial\Omega$ is C^∞ smooth and non characteristic for the system of vector fields X. Next, the subspace $H^1_{X,0}(\Omega)$ is defined as a closure of $C_0^\infty(\Omega)$ in $H^1_X(\tilde{\Omega})$, which is also a Hilbert space.

Let $I = (j_1, \cdots, j_k)$ with $1 \le j_i \le m$, we denote $|I| = k$. We say that the vector fields $X = \{X_1, X_2, \cdots, X_m\}$ satisfy the Hörmander's condition (cf. Hörmander, Acta Math.(1967)) if X together with their commutators

$$X_I = [X_{j_1}, [X_{j_2}, [X_{j_3}, \cdots [X_{j_{k-1}}, X_{j_k}] \cdots]]],$$

up to some fixed length $|I| \le Q$ span the tangent space at each point of Ω. Here Q is called the Hörmander index of X on Ω, which is the smallest positive integer for the Hörmander condition to be satisfied.

Let $\triangle_X = \sum_{j=1}^m X_j^2$, we consider the following Dirichlet eigenvalues problem in $H^1_{X,0}(\Omega)$,

$$\begin{cases} -\triangle_X u = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

We know that there is a sequence of discrete eigenvalues for the problem (1.1), which can be ordered, after counting (finite) multiplicity, as

$$0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots \le \lambda_k \le \cdots, \text{ and } \lambda_k \to +\infty.$$

Classical Case:

If $X = \{\partial_{x_1}, \cdots, \partial_{x_n}\}$, $\triangle_X$ is the Laplacian $\triangle$. In this case, H. Weyl's asymptotic formula [36] deduced that $\lambda_k \sim C_n(\frac{k}{|\Omega|_n})^{\frac{2}{n}}$, where $\{\lambda_k\}_{k\ge1}$ are Dirichlet eigenvalues, $|\Omega|_n$ is the Lebesgue measure of Ω and $C_n = (2\pi)^2 B_n^{-\frac{2}{n}}$ with B_n being the volume of the unit ball in $\mathbb{R}^n$.

Later, Pólya [34] proved that the above asymptotic relation is in fact a one-sided inequality if Ω is a plane domain which tiles $\mathbb{R}^2$ (and his proof also works in $\mathbb{R}^n$) and he conjectured that, for any domain in $\mathbb{R}^n$, the inequality holds,

$$\lambda_k \ge C_n \Big(\frac{k}{|\Omega|_n}\Big)^{\frac{2}{n}}, \text{ for any } k \ge 1. \tag{1.2}$$

In this direction, Lieb [27] proved an inequality like (1.2) for any domain in $\mathbb{R}^n$ but with a constant $\tilde{C_n}$ that differs from the constant C_n by a factor. Later in 1983, by using the Fourier transformation approach, Li and Yau [26] gave a simple proof for the lower bounds and obtain

$$\sum_{i=1}^k \lambda_i \ge \frac{nC_n}{n+2} k^{\frac{n+2}{n}} |\Omega|_n^{-\frac{2}{n}}, \text{ for any } k \ge 1. \tag{1.3}$$

In 2006, A.D. Melas [29] gave more detail estimate for the lower bounds:

$$\sum_{i=1}^{k}\lambda_i \geq \frac{nC_n}{n+2}k^{\frac{n+2}{n}}|\Omega|_n^{-\frac{2}{n}}+O(k),\ \text{for all } k\geq 1. \tag{1.4}$$

Upper bounds for Classical Case:

In 1994, under some geometry conditions, Kröger [25] proved the following upper bounds for Dirichelt eigenvalues of Laplacian, namely

$$\sum_{i=1}^{k}\lambda_i \leq \frac{nC_n}{n+2}k^{\frac{n+2}{n}}|\Omega|_n^{-\frac{2}{n}}+\bar{C}_n k^{\frac{n+1}{n}}. \tag{1.5}$$

That means the Li-Yau's lower bounds (1.3) would be optimal in general.

In this direction, also we have the following PPW inequality [Polya, Payne and Weinberger,1955]:

$$\lambda_{k+1}-\lambda_k \leq \frac{4}{nk}\sum_{i=1}^{k}\lambda_i. \tag{1.6}$$

HP inequality [Hile and Protter,1980]:

$$\frac{nk}{4}\leq \sum_{i=1}^{k}\frac{\lambda_i}{\lambda_{k+1}-\lambda_i}, \tag{1.7}$$

and Yang inequality [Yang, H.C.,1991]:

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \leq \frac{4}{n}\sum_{i=1}^{k}\lambda_i(\lambda_{k+1}-\lambda_i). \tag{1.8}$$

From the Yang inequality above, one can deduce the following upper bounds for Dirichlet eigenvalues of Laplacian:

$$\lambda_{k+1}\leq \left(1+\frac{4}{n}\right)\frac{1}{k}\left(\sum_{i=1}^{k}\lambda_i\right). \tag{1.9}$$

The relations of all these four inequalities above are summarized as [Ashbaugh, Mark S,2002]:

$$(1.8)\Rightarrow(1.9)\Rightarrow(1.7)\Rightarrow(1.6).$$

Lower bounds for finitely degenerate Case:

Degenerate elliptic operators have been widely studied by many authors. In the finitely degenerate case, Hörmander [18] proved the hypoellipticity of the degenerate elliptic operators $\triangle_X$; Nagel, Stein and Waingers [33] introduced the subelliptic balls and Carnot-Carathéodory metrics defined by degenerate vector fields X; Jerison and Sánchez-Calle [20, 35] studied fundamental solutions and estimates of the heat kernel for the degenerate elliptic operators $\triangle_X$; Métivier [28] gained the asymptotic results for Dirichlet eigenvalues under the Métivier's condition below; Xu [37] established the regularity theory for the degenerate elliptic operators $\triangle_X$ in quasilinear case. More results for degenerate elliptic operators can be found in [1]-[13] and [15, 16, 19, 21, 22].

By using the similar approach in [26], we can prove the following lower bounds of Dirichlet eigenvalues for general finitely degenerate elliptic operators.

Theorem 1.1. *Let X be finitely degenerate on Ω with Hörmander index $1 \le Q < +\infty$, λ_j be the j^{th} Dirichlet eigenvalue of the problem* (1.1), *then*

$$\sum_{j=1}^{k} \lambda_j \ge C_1 k^{1+\frac{2}{Qn}} - \widetilde{C}(Q)k, \ \textit{for all } k \ge 1,$$

where $C_1 = \frac{nQ(2\pi)^{\frac{2}{Q}}}{C(Q)\cdot(nQ+2)(|\Omega|_n B_n)^{\frac{2}{nQ}}}$, $C(Q)$ and $\widetilde{C}(Q)$ are the constants in Proposition 2.1 below, B_n is the volume of the unit ball in $\mathbb{R}^n$, $|\Omega|_n$ is the volume of Ω.

Remark 1.2. (1) Since $k\lambda_k \ge \sum_{j=1}^{k} \lambda_j$, then Theorem 1.1 shows that the Dirichlet eigenvalues λ_k satisfy

$$\lambda_k \ge C_1 k^{\frac{2}{Qn}} - \widetilde{C}(Q), \ \text{for all } k \ge 1. \tag{1.10}$$

(2) If $\triangle_X = \triangle$ is Laplacian, then

$$Q = 1, \ C(Q) = 1, \ \widetilde{C}(Q) = 0, \ \text{and } C_1 = \frac{n}{n+2}(2\pi)^2 B_n^{-2/n} |\Omega|_n^{-2/n}.$$

Thus the result of Theorem 1.1 is the same as the Li-Yau's estimate (1.3).

Furthermore, if X satisfies the Hörmander's condition on Ω with the Hörmander index Q, then for each $1 \le j \le Q$ and $x \in \Omega$, we denote $V_j(x)$ as the subspace of the tangent space $T_x(\Omega)$ which is spanned by the vector fields $\{X_J\}$ with $|J| \le j$. If the dimension of $V_j(x)$ is constant ν_j in a neighborhood of each $x \in \Omega$, then we say the system of the vector fields X satisfies the so called Métivier's condition on Ω and the Métivier index is defined as

$$\nu = \sum_{j=1}^{Q} j(\nu_j - \nu_{j-1}), \ \text{here } \nu_0 = 0, \tag{1.11}$$

where ν is also called the Hausdorff dimension of Ω related to the subelliptic metric induced by the vector fields X. Under the Métivier's condition above, Métivier [28] proved the following asymptotic result

$$\lambda_k \approx k^{\frac{2}{\nu}}, \ \text{as } k \to +\infty, \tag{1.12}$$

where Métivier index ν is defined by (1.11). Thus if $\triangle_X$ is finitely degenerate, then its Dirichlet eigenvalue λ_k will be polynomial increasing in k. However, when Hörmander index $Q > 1$, the increasing order of k in the lower bounds (1.10) is $\frac{2}{Qn}$, which is smaller than the order of k in the Métivier's asymptotic formula (1.12). That means the lower bounds of Dirichlet eigenvalues in (1.10) are not precise. Indeed, one example below with $Q = 2$ gives a precise lower bounds of Dirichlet eigenvalues.

Let us consider the Kohn Laplacian on Heisenberg group Ω in $\mathbb{R}^{2N+1}$, namely $X_j = \partial_{x_j} + 2y_j\partial_t$, $Y_j = \partial_{y_j} - 2x_j\partial_t$, for $j = 1, \cdots, N$, and $\triangle_X = \sum_{j=1}^{N}(X_j^2 + Y_j^2)$ is the Kohn Laplacian. Then Hörmander's condition and Métivier's condition are satisfied with $Q = 2$ and $\nu = 2N + 2$. Thus Hansson and Laptev [17] proved that

$$\lambda_k \geq \Big(\frac{(2\pi)^{N+1}(N+1)^{N+2}}{2C_N(N+2)^{N+1}|\Omega|}\Big)^{\frac{1}{N+1}} k^{\frac{1}{N+1}}, \text{ for } k \geq 1, \tag{1.13}$$

where $n = 2N + 1$, $Qn = 2(2N + 1)$, $\nu = 2(N + 1) = Q + n - 1$ and $C_N = \sum_{n_1,\cdots,n_N \geq 0} \frac{1}{(2(n_1+\cdots+n_N)+N)^{N+1}}$. The increasing order of k in (1.13) is the same to that in Métivier's result (1.12).

Métivier's condition is important on studying finitely degenerate elliptic operator. However, there are a lot of vector fields in which the Métivier's condition will be not satisfied. Thus we need to introduce the following generalized Métivier index. By using the same notation we denote here $\nu_j(x) = \dim V_j(x)$, $\nu(x) = \sum_{j=1}^{Q} j(\nu_j(x) - \nu_{j-1}(x))$, with $\nu_0(x) = 0$. Then we define

$$\tilde{\nu} = \max_{x\in\Omega} \nu(x), \tag{1.14}$$

as the generalized Métivier index. Thus a degenerate vector fields X always has the generalized Métivier index $\tilde{\nu}$ on Ω even if the Métivier's condition will be not satisfied for X. Observe $\tilde{\nu} = \nu$ if the Métivier's condition is satisfied.

Thus, for some Grushin type vector fields, which satisfy Hörmander's condition with $1 < Q < +\infty$ and the so-called generalized Métivier's condition. we can get the following precise lower bounds for the corresponding Dirichlet eigenvalues.

Theorem 1.3. *Let $X = (\partial_{x_1}, \cdots, \partial_{x_{n-1}}, x_1^l\partial_{x_n})$, $l \in \mathbb{Z}^+$, $n \geq 2$, be a Grushin type vector fields on Ω, and $\Omega \cap \{x_1 = 0\} \neq \emptyset$. Then X satisfies the Hörmander's condition with the Hörmander index $Q = l + 1$. Also the generalized Métivier index $\tilde{\nu} = Q + n - 1$. Suppose λ_j be the j^{th} Dirichlet eigenvalue of the problem* (1.1), *then*

$$\sum_{j=1}^{k} \lambda_j \geq C_2 k^{1+\frac{2}{\tilde{\nu}}} - \widetilde{C}(Q)k, \text{ for all } k \geq 1,$$

where

$$C_2 = \frac{A_Q}{\hat{C}(Q)n(n+Q+1)}\Big(\frac{(2\pi)^n}{|\Omega|_n\omega_{n-1}Q}\Big)^{\frac{2}{n+Q-1}}(n+Q-1)^{\frac{n+Q+1}{n+Q-1}},$$

and

$$\hat{C}(Q)=C(Q)+\min\{1,Q-1\}>0,\quad A_Q=\begin{cases}\min\{1,n^{\frac{3-Q}{2}}\}, & Q\geq 2,\\ n, & Q=1;\end{cases}$$

$C(Q)$ and $\widetilde{C}(Q)$ are the constants in Proposition 2.1 below, ω_{n-1} is the area of the unit sphere in $\mathbb{R}^n$, $|\Omega|_n$ is the volume of Ω.

Remark 1.4. (1) The result of Theorem 1.3 gives that

$$\lambda_k\geq C_2k^{\frac{2}{\tilde{\nu}}}-\widetilde{C}(Q),\ \text{for all } k\geq 1.$$

(2) In case of Hörmander index $Q>1$, we know that $Q+n-1\leq\tilde{\nu}<nQ$. Thus the lower bounds of Theorem 1.3 is optimal.
(3) If $l=0$ in Theorem 1.3, then we have

$$\triangle_X=\triangle,\ Q=1,\ \hat{C}(Q)=1,\ \widetilde{C}(Q)=0,\ A_Q=n,\ C_2=\frac{n}{n+2}(2\pi)^2B_n^{-2/n}|\Omega|_n^{-2/n}.$$

Thus the result of Theorem 1.3 will be the same to Li-Yau's estimate (1.3).

Next, we can prove the similar result for the Grushin type vector fields with two degenerate directions.

Theorem 1.5. *Let $X=(\partial_{x_1},\cdots,\partial_{x_{n-2}},x_i^p\partial_{x_{n-1}},x_j^q\partial_{x_n})$, $n\geq 3$, $p,\ q\in\mathbb{Z}^+$, $i,\ j\in\{1,2,\cdots,n-2\}$, be the Grushin type vector fields on Ω with two degenerate directions. If $\Omega\cap\{x_i=0\}\neq\emptyset$ and $\Omega\cap\{x_j=0\}\neq\emptyset$. Then X satisfies the Hörmander's condition on Ω with $Q=\max\{p,q\}+1$ and the generalized Métivier index $\tilde{\nu}=n+p+q$. Suppose λ_j be the j^{th} Dirichlet eigenvalue of the problem (1.1), then*

$$\sum_{j=1}^{k}\lambda_j\geq C_3k^{1+\frac{2}{\tilde{\nu}}}-\widetilde{C}(p,q)k,\ \text{for all } k\geq 1,$$

where

$$C_3=\frac{2^n}{3\bar{C}(p,q)}n^{-\frac{n+p+q+2}{2}}\Big(\frac{n+p+q}{(p+1)(q+1)\omega_{n-1}}\Big)^{\frac{n+p+q+2}{n+p+q}}\Big(\frac{(2\pi)^n}{|\Omega|_n}\Big)^{\frac{2}{n+p+q}},$$

$\bar{C}(p,q)=C(p+1)+C(q+1)+2>0$, $\widetilde{C}(p,q)=\max\{\widetilde{C}(p+1),\widetilde{C}(q+1)\}$. $C(p+1)$, $C(q+1)$, $\widetilde{C}(p+1)$ and $\widetilde{C}(q+1)$ are the constants in (2.1) below, ω_{n-1} is the area of the unit sphere in $\mathbb{R}^n$, $|\Omega|_n$ is the volume of Ω.

Remark 1.6. (1) The result of Theorem 1.5 implies that

$$\lambda_k\geq C_3k^{\frac{2}{\tilde{\nu}}}-\widetilde{C}(p,q),\ \text{for all } k\geq 1.$$

(2) The result of Theorem 1.5 can be deduced to the more general Grushin type degenerate vector fields with k degenerate directions

$$X = \{\partial_{x_1}, \cdots \partial_{x_{n-k}}, f_1(\bar{x})\partial_{x_{n-k+1}}, \cdots, f_k(\bar{x})\partial_{x_n}\},$$

where $\bar{x} = (x_1, \cdots, x_{n-k})$ for $2 \le k < n$, $f_j(\bar{x})$ $(1 \le j \le k)$ are smooth functions with finite order zero point in Ω.

Lower bounds for infinitely degenerate Case:

Let $X = \{X_1, X_2, \cdots, X_m\}$ be C^∞ smooth vector fields defined in $\tilde{\Omega}$ $(\subset \mathbb{R}^n)$. We assume that X doesn't satisfy Hörmander condition, then $\triangle_X$ is infinitely degenerate elliptic operator. For example $X = (\partial_{x_1}, \cdots, \partial_{x_{n-1}}, \varphi(x_1)\partial_{x_n})$, where $\varphi(x_1) = e^{-1/|x_1|^s}$ for $x_1 \neq 0$ $(s > 0)$, and $\varphi(0) = 0$.

For infinitely degenerate elliptic operators, Hörmander index $Q = +\infty$. However if we let $Q \to +\infty$ in the lower bound estimate (1.10) (or in Métivier's asymptotic estimate (1.12)). Then we can get nothing information for the lower bounds of the Dirichlet eigenvalues in the infinitely degenerate cases.

Motivation of infinitely degenerate elliptic operators came from complex geometry (cf. [22]). In 1981, Fefferman-Phong proved that, for degenerate elliptic operators P, the sub-elliptic estimate holds iff P is the operator with finite order degeneracy (e.g. for sum of square operator, the Hörmander condition is satisfied). That means, at points of infinite type, the sub-elliptic estimate will be not satisfied. However, there were a lot of examples in complex geometry in which the boundary of pseudo-convex domain has singular points with infinite type degeneracy (see [23, 24]).

In 1987, Y. Morimoto [30, 31] (also see M. Christ's result [14] for general case in 1997) proved that, if infinitely degenerate elliptic operator satisfies the so-called logarithmic regularity estimate, then it is hypo-elliptic. Later in 2000, by using sub-elliptic multipliers method, J. Kohn gave a purely geometrical condition for the hypo-ellipticity at points of infinite type degeneracy on the boundary of pseudo-convex domain.

Here we suppose that X satisfies the finite type of Hörmander's condition on Ω except a union of smooth surfaces Γ, the surface Γ is non characteristic for X, and the vector fields X satisfies the following logarithmic regularity estimate for all $u \in C_0^\infty(\Omega)$,

$$\|(\log \Lambda)^s u\|^2_{L^2(\Omega)} \le C_0\Big[\int_\Omega |Xu|^2 dx + \|u\|^2_{L^2(\Omega)}\Big], \tag{1.15}$$

where $s > 1$, $C_0 > 0$ and $\Lambda = (e^2 + |\nabla|^2)^{\frac{1}{2}}$. Then the result of [32] gave that there is a sequence of discrete Dirichlet eigenvalues for the problem (1.1), which can be ordered, after finite multiplicity, as $0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots \le \lambda_k \le \cdots$, and $\lambda_k \to +\infty$. However, up to now, there is no any result (even no asymptotic result) on the lower bounds of the eigenvalues. The result of Theorem 1.7 below tells us that in this case the Dirichlet eigenvalues λ_k will be at least logarithmic increasing in k.

Theorem 1.7. *Suppose that the infinitely degenerate vector fields X satisfies the logarithmic regularity estimate* (1.15) *with $s > 1$, λ_j is the j^{th} eigenvalue of the problem* (1.1)*. Then*

$$\sum_{j=1}^{k} \lambda_j \geq C_4 k(\log k)^{2s} - k, \text{ for all } k \geq k_0,$$

where $k_0 = [\frac{2^{2s} e^n B_n |\Omega|_n}{C_0 \pi^n}] + 1$, $C_4 = (2^n - 1)\Big(C_0 2^{n+4s}(|\log \frac{|\Omega|_n B_n}{(2\pi)^n}|^{2s} + n^{2s})\Big)^{-1}$, B_n is the volume of the unit ball in $\mathbb{R}^n$, $|\Omega|_n$ is the volume of Ω, s and C_0 are given in (1.15).

Remark 1.8. (1) Since $k\lambda_k \geq \sum_{j=1}^{k} \lambda_j$, then the result of Theorem 1.7 gives that

$$\lambda_k \geq C_4 (\log k)^{2s} - 1, \text{ for all } k \geq k_0.$$

(2) We know that the logarithmic regularity estimate (1.15) with $s > 1$ implies that the infinitely degenerate elliptic operator $\triangle_X$ is hypo-elliptic (cf. [14, 30]).

Our paper is organized as follows. In Section 2, we introduce some preliminaries which include subelliptic estimates and logarithmic regularity estimate. In Section 3, we prove Theorem 1.1. The proofs for Theorem 1.3 and Theorem 1.5 will be given in Section 4. Finally we shall prove Theorem 1.7 in Section 5.

§ 2. Preliminaries

§ 2.1. Subelliptic estimates

Proposition 2.1 (cf. [15]). *The system of vector fields $X = (X_1, \cdots, X_m)$ satisfies the Hörmander's condition on Ω with Hörmander index is $Q \geq 1$, if and only if the following subelliptic estimate*

$$\big\||\nabla|^{1/Q} u\big\|^2_{L^2(\Omega)} \leq C(Q)(\|Xu\|^2_{L^2(\Omega)} + \widetilde{C}(Q)\|u\|^2_{L^2(\Omega)}), \tag{2.1}$$

holds for all $u \in C_0^\infty(\Omega)$. Where $\nabla = (\partial_{x_1}, \cdots, \partial_{x_n})$, $|\nabla|^{1/Q}$ is a pesudo-differential operator with the symbol $|\xi|^{1/Q}$, the constants $C(Q) > 0$ and $\widetilde{C}(Q) \geq 0$.

Proposition 2.2. *If X belongs to the system of vector fields in Theorem 1.3, then we have the following subelliptic estimate*

$$\sum_{i=1}^{n-1} \big\|\partial_{x_i} u\big\|^2_{L^2(\Omega)} + \big\||\partial_{x_n}|^{1/Q} u\big\|^2_{L^2(\Omega)} \leq \hat{C}(Q)(\|Xu\|^2_{L^2(\Omega)} + \widetilde{C}(Q)\|u\|^2_{L^2(\Omega)}), \tag{2.2}$$

for all $u \in C_0^\infty(\Omega)$. Where $|\partial_{x_n}|^{1/Q}$ is a pesudo-differential operator with the symbol $|\xi_n|^{1/Q}$, $\hat{C}(Q) = C(Q) + \min\{1, Q-1\} > 0$, $C(Q)$ and $\widetilde{C}(Q)$ are the constants in (2.1).

Proof. First, for $Q=1$, then $\triangle_X=\triangle$ and the estimate (2.2) is obvious. If $Q>1$ and $u\in C_0^\infty(\Omega)$, from Plancherel's formula, we have

$$\begin{aligned}\big\||\partial_{x_n}|^{1/Q}u\big\|^2_{L^2(\Omega)}&=\big\||\xi_n|^{1/Q}\hat{u}\big\|^2_{L^2(\mathbb{R}^n)}\\&\le\big\||\xi|^{1/Q}\hat{u}\big\|^2_{L^2(\mathbb{R}^n)}=\big\||\nabla|^{1/Q}u\big\|^2_{L^2(\Omega)}.\end{aligned}\tag{2.3}$$

Also,

$$\sum_{i=1}^{n-1}\big\|\partial_{x_i}u\big\|^2_{L^2(\Omega)}\le\|Xu\|^2_{L^2(\Omega)}.\tag{2.4}$$

Combining (2.1), (2.3) and (2.4), we can gain the subelliptic estimate (2.2). □

Proposition 2.3. *If X belongs to the system of vector fields in Theorem 1.5, then we have the following subelliptic estimate*

$$\begin{aligned}&\sum_{i=1}^{n-2}\big\|\partial_{x_i}u\big\|^2_{L^2(\Omega)}+\big\||\partial_{x_{n-1}}|^{\frac{1}{p+1}}u\big\|^2_{L^2(\Omega)}+\big\||\partial_{x_n}|^{\frac{1}{q+1}}u\big\|^2_{L^2(\Omega)}\\&\qquad\le\bar{C}(p,q)(\|Xu\|^2_{L^2(\Omega)}+\widetilde{C}(p,q)\|u\|^2_{L^2(\Omega)}),\end{aligned}\tag{2.5}$$

for all $u\in C_0^\infty(\Omega)$. Where $\bar{C}(p,q)=C(p+1)+C(q+1)+2>0$, $\widetilde{C}(p,q)=\max\{\widetilde{C}(p+1),\widetilde{C}(q+1)\}$. $C(p+1)$, $C(q+1)$, $\widetilde{C}(p+1)$ and $\widetilde{C}(q+1)$ are the constants in (2.1).

Proof. We consider the system of vector fields $\tilde{X}=(\partial_{x_1},\cdots,\partial_{x_{n-2}},x_i^p\partial_{x_{n-1}})$ defined on $\Omega_{x'}$, where $\Omega_{x'}$ is the projection of Ω on the direction x' and $x'=(x_1,\cdots,x_{n-1})$. Then similar to Proposition 2.2, we have

$$\sum_{i=1}^{n-2}\big\|\partial_{x_i}u\big\|^2_{L^2(\Omega_{x'})}+\big\||\partial_{x_{n-1}}|^{\frac{1}{p+1}}u\big\|^2_{L^2(\Omega_{x'})}\le\big(C(p+1)+1\big)\big(\|\tilde{X}u\|^2_{L^2(\Omega_{x'})}+\widetilde{C}(p+1)\|u\|^2_{L^2(\Omega_{x'})}\big).$$

Then

$$\begin{aligned}&\sum_{i=1}^{n-2}\big\|\partial_{x_i}u\big\|^2_{L^2(\Omega)}+\big\||\partial_{x_{n-1}}|^{\frac{1}{p+1}}u\big\|^2_{L^2(\Omega)}\\&\qquad\le\big(C(p+1)+1\big)\big(\|Xu\|^2_{L^2(\Omega)}+\widetilde{C}(p+1)\|u\|^2_{L^2(\Omega)}\big).\end{aligned}\tag{2.6}$$

Similarly, we can get

$$\begin{aligned}&\sum_{i=1}^{n-2}\big\|\partial_{x_i}u\big\|^2_{L^2(\Omega)}+\big\||\partial_{x_n}|^{\frac{1}{q+1}}u\big\|^2_{L^2(\Omega)}\\&\qquad\le\big(C(q+1)+1\big)\big(\|Xu\|^2_{L^2(\Omega)}+\widetilde{C}(q+1)\|u\|^2_{L^2(\Omega)}\big).\end{aligned}\tag{2.7}$$

Thus (2.6) and (2.7) imply the subelliptic estimate (2.5). □

§ 2.2. Examples satisfying logarithmic regularity estimates

Let $X = (X_1, X_2, \cdots, X_m)$ be an infinitely degenerate system of vector fields on an open bound domain Ω in $\mathbb{R}^n$, X_J denote the repeated commutator

$$[X_{j_1}, [X_{j_2}, [X_{j_3}, \cdots [X_{j_{k-1}}, X_{j_k}] \cdots]]],$$

for $J = (j_1, \cdots, j_k)$, $j_i \in \{1, \cdots, m\}$, and $|J| = k$. For $k \geq 1$, we take

$$G(x,k) = \min_{\xi \in \mathbb{S}^{n-1}} \sum_{|J| \leq k} |X_J(x,\xi)|^2, \qquad g(t,j,k,x_0) = G((\exp tX_j)(x_0), k),$$

where $(\exp tX_j)(x_0)$ denotes the integral curve of X_j starting from $x_0 \in \Gamma$. Here $\Gamma = \{x \in \Omega; \exists\ \xi \in \mathbb{S}^{n-1}, X_J(x,\xi) = 0, \text{ for any } J\}$, and $g_I^{j,k}(x_0) = \frac{1}{|I|}\int_I g(t,j,k,x_0)dt$ is the mean value of $g(t,j,k,x_0)$ on the interval I. One has the following result:

Proposition 2.4 (cf. [32]). *If $s > 0$, and there exists $\varepsilon_1 > 0$ such that*

$$\inf_{\delta>0, k\in\mathbb{N}, \mu>0, 1\leq j\leq m} \left\{ \sup\left(|I|^{\frac{1}{s}} |\log g_I^{j,k}(x_0)|\right); I \subset (-\mu,\mu), g_I^{j,k}(x_0) < \delta \right\} < \varepsilon_1,$$

for any $x_0 \in \Gamma$, then there exist constants $C_0 > 0$ which is independent with ε_1 and C_{ε_1} such that

$$\|(\log \Lambda)^s u\|^2_{L^2(\Omega)} \leq C_0 \varepsilon_1^{2s} \int_\Omega |Xu|^2 dx + C_{\varepsilon_1} \|u\|^2_{L^2(\Omega)},$$

for any $u \in C_0^\infty(\Omega)$.

From the result of Proposition 2.4, we can give some examples as follows in which the infinitely degenerate vector fields X will satisfy the logarithmic regularity estimate (1.15) (cf. [7, 32]).

Example 2.5. Let $s > 0$, and

$$\varphi(x_1) = \begin{cases} e^{-\frac{1}{|x_1|^{1/s}}}, & x_1 \neq 0, \\ 0, & x_1 = 0. \end{cases}$$

Then $X = (\partial_{x_1}, \cdots, \partial_{x_{n-1}}, \varphi(x_1)\partial_{x_n})$ is infinitely degenerate on the surface $\Gamma = \{x_1 = 0\}$ and X satisfies the logarithmic regularity estimate (1.15).

Example 2.6. The system of vector fields $X = (\partial_{x_1}, \cdots, \partial_{x_{n-1}}, \varphi(x_1)\partial_{x_n})$, for $n \geq 2$, where $s > 0$ and

$$\varphi(x_1) = \begin{cases} e^{-\frac{1}{|x_1 \sin(\frac{\pi}{x_1})|^{1/s}}}, & x_1 \neq 0, \\ 0, & x_1 = 0. \end{cases}$$

Then X is infinitely degenerate on $\Gamma = \bigcup_{j\in\mathbb{Z}_+}\Gamma_j$, for $\Gamma_j = \{x_1 = \frac{1}{j}\}$, $j \geq 1$, and $\Gamma_0 = \{x_1 = 0\}$.

Example 2.7. The system of vector fields $X = (\partial_{x_1}, \cdots, \partial_{x_{n-1}}, \varphi(x_1, x_2)\partial_{x_n})$, for $n \geq 3$, where $k \geq 1$, $s > 0$ and

$$\varphi(x_1, x_2) = \begin{cases} e^{-\frac{1}{|x_1|^{1/s}}} {x_2}^k, & x_1 \neq 0, \\ 0, & x_1 = 0. \end{cases}$$

Then X is infinitely degenerate on the surface $\{x_1 = 0\}$.

§ 3. Proof of Theorem 1.1

Lemma 3.1. *For the system of vector fields $X = (X_1, \cdots, X_m)$, if $\{\psi_j\}_{j=1}^k$ are the set of orthonormal eigenfunctions corresponding to the Dirichlet eigenvalues $\{\lambda_j\}_{j=1}^k$. Define*

$$\Psi(x, y) = \sum_{j=1}^{k} \psi_j(x)\psi_j(y).$$

Then for the partial Fourier transformation of $\Psi(x, y)$ in the x-variable,

$$\hat{\Psi}(z, y) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \Psi(x, y) e^{-ix\cdot z} dx,$$

we have

$$\int_\Omega \int_{\mathbb{R}^n} |\hat{\Psi}(z, y)|^2 dz dy = k, \ \textit{and} \ \int_\Omega |\hat{\Psi}(z, y)|^2 dy \leq (2\pi)^{-n} |\Omega|_n.$$

Proof. Since

$$\int_{\mathbb{R}^n} \Psi^2(x, y) dx = \int_{\mathbb{R}^n} |\hat{\Psi}(z, y)|^2 dz.$$

Hence by the orthonormality of $\{\psi_j\}_{j=1}^k$, one has

$$\int_\Omega \int_{\mathbb{R}^n} |\hat{\Psi}(z, y)|^2 dz dy = \int_\Omega \int_{\mathbb{R}^n} |\Psi(x, y)|^2 dx dy = \int_\Omega \int_\Omega |\Psi(x, y)|^2 dx dy = k.$$

On the other hand,

$$\int_{\mathbb{R}^n} |\hat{\Psi}(z, y)|^2 dy = \int_\Omega (2\pi)^{-n} |\int_{\mathbb{R}^n} \Psi(x, y) e^{-ix\cdot z} dx|^2 dy = \int_\Omega (2\pi)^{-n} |\int_\Omega \Psi(x, y) e^{-ix\cdot z} dx|^2 dy.$$

Using the Fourier expansion for the function $e^{-ix\cdot z}$, i.e.

$$e^{-ix\cdot z} = \sum_{j=1}^{\infty} a_j(z)\psi_j(x), \ \text{with } a_j(z) = \int_\Omega e^{-ix\cdot z}\psi_j(x) dx.$$

Then we know that

$$\sum_{j=1}^{\infty} |a_j(z)|^2 = \int_\Omega |e^{-ix\cdot z}|^2 dx = |\Omega|_n.$$

Thus

$$|\int_\Omega \Psi(x,y)e^{-ix\cdot z}dx| \le |\int_\Omega \sum_{j=1}^{k}\sum_{l=1}^{\infty} a_l(z)\psi_l(x)\psi_j(x)\psi_j(y)dx| = |\sum_{j=1}^{k} a_j(z)\psi_j(y)|.$$

Using the estimates above, we have

$$\int_\Omega |\hat{\Psi}(z,y)|^2 dy \le (2\pi)^{-n}\int_\Omega |\sum_{j=1}^{k} a_j(z)\psi_j(y)|^2 dy = (2\pi)^{-n}\sum_{j=1}^{k}|a_j(z)|^2 \le (2\pi)^{-n}|\Omega|_n.$$

□

Lemma 3.2. *Let f be a real-valued function defined on $\mathbb{R}^n$ with $0 \le f \le M_1$. If*

$$\int_{\mathbb{R}^n} |z|^{2/Q} f(z)dz \le M_2,$$

with $Q \in \mathbb{N}^+$, then we have the following inequality,

$$\int_{\mathbb{R}^n} f(z)dz \le \Big(\frac{nQ+2}{nQ}\Big)^{\frac{nQ}{nQ+2}} (M_1B_n)^{\frac{2}{nQ+2}} M_2^{\frac{nQ}{nQ+2}},$$

where B_n is the volume of the unit ball in $\mathbb{R}^n$.

Proof. First, we choose R such that

$$\int_{\mathbb{R}^n} |z|^{2/Q} g(z)dz = M_2,$$

where

$$g(z) = \begin{cases} M_1, & |z| < R, \\ 0, & |z| \ge R. \end{cases}$$

Then $(|z|^{2/Q} - R^{2/Q})(f(z) - g(z)) \ge 0$, hence

$$R^{2/Q}\int_{\mathbb{R}^n} (f(z)-g(z))dz \le \int_{\mathbb{R}^n} |z|^{2/Q}(f(z)-g(z))dz \le 0. \tag{3.1}$$

Now we have

$$M_2 = \int_{\mathbb{R}^n} |z|^{2/Q} g(z)dz = M_1\int_0^R r^{n-1+\frac{2}{Q}}\omega_{n-1}dr = \frac{QM_1\omega_{n-1}R^{n+\frac{2}{Q}}}{nQ+2}, \tag{3.2}$$

where ω_{n-1} is the area of the unit sphere in $\mathbb{R}^n$. From the definition of $g(z)$, we know

$$\int_{\mathbb{R}^n} g(z)dz = M_1 B_n R^n, \tag{3.3}$$

where B_n is the volume of the unit ball in $\mathbb{R}^n$.

Since $nB_n = \omega_{n-1}$, then (3.1), (3.2) and (3.3) give

$$\int_{\mathbb{R}^n} f(z)dz \le \int_{\mathbb{R}^n} g(z)dz \le \Big(\frac{nQ+2}{nQ}\Big)^{\frac{nQ}{nQ+2}} (M_1 B_n)^{\frac{2}{nQ+2}} M_2^{\frac{nQ}{nQ+2}}.$$

□

Proof of Theorem 1.1: Let $\{\lambda_k\}_{k\ge1}$ be a sequence of the Dirichlet eigenvalues for the problem (1.1), $\{\psi_k(x)\}_{k\ge1}$ be the corresponding eigenfunctions, then $\{\psi_k(x)\}_{k\ge1}$ constitute an orthonormal basis of the Sobolev space $H^1_{X,0}(\Omega)$. Using Plancherel's formula and Proposition 2.1, we have for $\Psi(x,y) = \sum_{j=1}^k \psi_j(x)\psi_j(y)$,

$$\begin{aligned}\int_{\mathbb{R}^n}\int_\Omega |z|^{2/Q}|\hat{\Psi}(z,y)|^2 dydz &= \int_{\mathbb{R}^n}\int_\Omega \Big||\nabla|^{1/Q}\Psi(x,y)\Big|^2 dydx \\ &= \int_\Omega\int_\Omega \Big||\nabla|^{1/Q}\Psi(x,y)\Big|^2 dydx,\end{aligned}\tag{3.4}$$

and

$$\begin{aligned}&\int_\Omega\int_\Omega \Big||\nabla|^{1/Q}\Psi(x,y)\Big|^2 dydx \\ &\quad\le C(Q)\Big(\int_\Omega\int_\Omega |X(x)\Psi(x,y)|^2 dxdy + \widetilde{C}(Q)\int_\Omega\int_\Omega |\Psi(x,y)|^2 dxdy\Big).\end{aligned}\tag{3.5}$$

Next, we can deduce that

$$\begin{aligned}\int_\Omega\int_\Omega |X(x)\Psi(x,y)|^2 dxdy &= \int_\Omega\Big(\sum_{l=1}^m \int_\Omega |\sum_{j=1}^k (X_l(x)\psi_j(x))\psi_j(y)|^2 dx\Big)dy \\ &= \sum_{l=1}^m\Big(\int_\Omega \sum_{j=1}^k |X_l(x)\psi_j(x)|^2\Big)dx \\ &= -\int_\Omega \sum_{j=1}^k \psi_j(x)\triangle_X\psi_j(x)dx = \sum_{j=1}^k \lambda_j.\end{aligned}\tag{3.6}$$

Thus Lemma 3.1 and the estimates (3.4), (3.5) and (3.6) imply that

$$\int_{\mathbb{R}^n}\int_\Omega |z|^{2/Q}|\hat{\Psi}(z,y)|^2 dydz \le C(Q)\Big(\sum_{j=1}^k \lambda_j + \widetilde{C}(Q)k\Big).$$

Now we choose

$$f(z)=\int_{\Omega}|\hat{\Psi}(z,y)|^2dy,\ M_1=(2\pi)^{-n}|\Omega|_n,\ M_2=C(Q)\Big(\sum_{j=1}^{k}\lambda_j+\widetilde{C}(Q)k\Big).$$

Then the results of Lemma 3.1 and Lemma 3.2 give that, for any $k\geq 1$,

$$k\leq\Big(\frac{nQ+2}{nQ}\Big)^{\frac{nQ}{nQ+2}}((2\pi)^{-n}|\Omega|_nB_n)^{\frac{2}{nQ+2}}\Big(C(Q)(\sum_{j=1}^{k}\lambda_j+\widetilde{C}(Q)k)\Big)^{\frac{nQ}{nQ+2}}.$$

That means, for any $k\geq 1$,

$$\sum_{j=1}^{k}\lambda_j\geq C_1k^{1+\frac{2}{nQ}}-\widetilde{C}(Q)k,\ \text{with}\ C_1=\frac{nQ(2\pi)^{\frac{2}{Q}}}{C(Q)\cdot(nQ+2)(|\Omega|_nB_n)^{\frac{2}{nQ}}}.$$

Theorem 1.1 is proved. □

§ 4. Proof of Theorem 1.3 and Theorem 1.5

§ 4.1. Proof of Theorem 1.3

Lemma 4.1. *Let f be a real-valued function defined on $\mathbb{R}^n$ with $0\leq f\leq M_1$, and for $Q\in\mathbb{N}^+$,*

$$\int_{\mathbb{R}^n}(\sum_{i=1}^{n-1}z_i^2+|z_n|^{2/Q})f(z)dz\leq M_2.$$

Then

$$\int_{\mathbb{R}^n}f(z)dz\leq\frac{1}{n+Q-1}(M_1Q\omega_{n-1})^{\frac{2}{n+Q+1}}\Big(\frac{n(Q+n+1)}{A_Q}\Big)^{\frac{n+Q-1}{n+Q+1}}M_2^{\frac{n+Q-1}{n+Q+1}},$$

where ω_{n-1} is the area of the unit sphere in $\mathbb{R}^n$, and

$$A_Q=\begin{cases}\min\{1,n^{\frac{3-Q}{2}}\}, & Q\geq 2,\\ n, & Q=1.\end{cases}$$

Proof. First, we choose R such that

$$\int_{\mathbb{R}^n}(\sum_{i=1}^{n-1}z_i^2+|z_n|^{2/Q})g(z)dz=M_2,$$

where

$$g(z)=\begin{cases}M_1, & \sum_{i=1}^{n-1}z_i^2+|z_n|^{2/Q}<R^2,\\ 0, & \sum_{i=1}^{n-1}z_i^2+|z_n|^{2/Q}\geq R^2.\end{cases}$$

Then $(\sum_{i=1}^{n-1} z_i^2 + |z_n|^{2/Q} - R^2)(f(z) - g(z)) \geq 0$, hence

$$R^2 \int_{\mathbb{R}^n} (f(z) - g(z))dz \leq \int_{\mathbb{R}^n} (\sum_{i=1}^{n-1} z_i^2 + |z_n|^{2/Q})(f(z) - g(z))dz \leq 0. \tag{4.1}$$

Now we have

$$\begin{aligned} M_2 &= \int_{\mathbb{R}^n} (\sum_{i=1}^{n-1} z_i^2 + |z_n|^{2/Q})g(z)dz = M_1 \int_{\widetilde{B}_R} (\sum_{i=1}^{n-1} z_i^2 + |z_n|^{2/Q})dz \\ &= M_1 Q \int_{B_R} |z|^2 |z_n|^{Q-1} dz = \frac{M_1 Q}{n} \int_{B_R} |z|^2 (\sum_{i=1}^{n} |z_i|^{Q-1})dz, \end{aligned} \tag{4.2}$$

where

$$\widetilde{B}_R = \{z \in \mathbb{R}^n, \sum_{i=1}^{n-1} z_i^2 + |z_n|^{2/Q} \leq R^2\},\ B_R = \{z \in \mathbb{R}^n, |z| \leq R\}.$$

On the other hand,

$$\sum_{i=1}^{n} |z_i|^{Q-1} = |z|^{Q-1} \sum_{i=1}^{n} \Big(\frac{|z_i|}{|z|}\Big)^{Q-1} \geq A_Q |z|^{Q-1}, \tag{4.3}$$

where

$$A_Q = \begin{cases} \min\{1, n^{\frac{3-Q}{2}}\}, & Q \geq 2, \\ n, & Q = 1. \end{cases}$$

Then (4.2) and (4.3) imply

$$M_2 \geq \frac{M_1 Q A_Q}{n} \int_{B_R} |z|^{Q+1} dz = \frac{M_1 Q A_Q \omega_{n-1}}{n(n+Q+1)} R^{n+Q+1}. \tag{4.4}$$

From the definition of $g(z)$, we know

$$\begin{aligned} \int_{\mathbb{R}^n} g(z)dz &= M_1 \int_{\widetilde{B}_R} dz = M_1 Q \int_{B_R} |z_n|^{Q-1} dz \\ &\leq M_1 Q \int_{B_R} |z|^{Q-1} dz = \frac{M_1 Q \omega_{n-1}}{n+Q-1} R^{n+Q-1}. \end{aligned} \tag{4.5}$$

Combining (4.1), (4.4) and (4.5), one has

$$\int_{\mathbb{R}^n} f(z)dz \leq \int_{\mathbb{R}^n} g(z)dz \leq \frac{1}{n+Q-1} (M_1 Q \omega_{n-1})^{\frac{2}{n+Q+1}} \Big(\frac{n(Q+n+1)}{A_Q}\Big)^{\frac{n+Q-1}{n+Q+1}} M_2^{\frac{n+Q-1}{n+Q+1}}.$$

$\square$

Proof of Theorem 1.3: Let $X=(\partial_{x_1},\cdots,\partial_{x_{n-1}},x_1^l\partial_{x_n})$, $\{\lambda_k\}_{k\geq 1}$ be a sequence of the Dirichlet eigenvalues for the problem (1.1), $\{\psi_k(x)\}_{k\geq 1}$ be the corresponding eigenfunctions, then $\{\psi_k(x)\}_{k\geq 1}$ constitute an orthonormal basis of the Sobolev space $H^1_{X,0}(\Omega)$.

Let $\Psi(x,y)=\sum_{j=1}^k\psi_j(x)\psi_j(y)$, then by using Plancherel's formula and Proposition 2.2, we have for $Q=l+1$,

$$\begin{aligned}
&\int_{\mathbb{R}^n}\int_\Omega(\sum_{i=1}^{n-1}z_i^2+|z_n|^{2/Q})|\hat{\Psi}(z,y)|^2dydz\\
&=\int_{\mathbb{R}^n}\int_\Omega\Big(\sum_{i=1}^{n-1}\big|\partial_{x_i}\Psi(x,y)\big|^2+\big||\partial_{x_n}|^{1/Q}\Psi(x,y)\big|^2\Big)dydx \qquad (4.6)\\
&\leq\hat{C}(Q)\Big(\int_\Omega\int_\Omega|X(x)\Psi(x,y)|^2dxdy+\widetilde{C}(Q)\int_\Omega\int_\Omega|\Psi(x,y)|^2dxdy\Big).
\end{aligned}$$

Next, we can deduce that

$$\begin{aligned}
\int_\Omega\int_\Omega|X(x)\Psi(x,y)|^2dxdy&=\int_\Omega\Big(\sum_{l=1}^n\int_\Omega|\sum_{j=1}^k(X_l(x)\psi_j(x))\psi_j(y)|^2dx\Big)dy \qquad (4.7)\\
&=-\int_\Omega\sum_{j=1}^k\psi_j(x)\triangle_X\psi_j(x)dx=\sum_{j=1}^k\lambda_j.
\end{aligned}$$

Thus from Lemma 3.1 and the estimates (4.6) and (4.7) above, we have

$$\int_{\mathbb{R}^n}\int_\Omega(\sum_{i=1}^{n-1}z_i^2+|z_n|^{2/Q})|\hat{\Psi}(z,y)|^2dydz\leq\hat{C}(Q)(\sum_{j=1}^k\lambda_j+\widetilde{C}(Q)k).$$

Now we choose

$$f(z)=\int_\Omega|\hat{\Psi}(z,y)|^2dy,\quad M_1=(2\pi)^{-n}|\Omega|_n,\quad M_2=\hat{C}(Q)\Big(\sum_{j=1}^k\lambda_j+\widetilde{C}(Q)k\Big).$$

Then the result of Lemma 4.1 gives that, for any $k\geq 1$,

$$k\leq\frac{1}{n+Q-1}(Q(2\pi)^{-n}|\Omega|_n\omega_{n-1})^{\frac{2}{n+Q+1}}\Big(\frac{n(Q+n+1)}{A_Q}\Big)^{\frac{n+Q-1}{n+Q+1}}\cdot\Big(\hat{C}(Q)(\sum_{j=1}^k\lambda_j+\widetilde{C}(Q)k)\Big)^{\frac{n+Q-1}{n+Q+1}}.$$

This means, for any $k\geq 1$,

$$\sum_{j=1}^k\lambda_j\geq C_2k^{1+\frac{2}{n+Q-1}}-\widetilde{C}(Q)k,$$

with

$$C_2 = \frac{A_Q}{\hat{C}(Q)n(n+Q+1)}\Big(\frac{(2\pi)^n}{|\Omega|_n\omega_{n-1}Q}\Big)^{\frac{2}{n+Q-1}}(n+Q-1)^{\frac{n+Q+1}{n+Q-1}},$$

and

$$\hat{C}(Q) = C(Q) + \min\{1, Q-1\} > 0, \quad A_Q = \begin{cases} \min\{1, n^{\frac{3-Q}{2}}\}, & Q \geq 2, \\ n, & Q = 1. \end{cases}$$

The proof of Theorem 1.3 is complete. □

§ 4.2. Proof of Theorem 1.5

Lemma 4.2. *Let f be a real-valued function defined on $\mathbb{R}^n$ with $0 \leq f \leq M_1$, and for $p,\ q \in \mathbb{Z}^+$,*

$$\int_{\mathbb{R}^n} (\sum_{i=1}^{n-2} z_i^2 + |z_{n-1}|^{\frac{2}{p+1}} + |z_n|^{\frac{2}{q+1}}) f(z)dz \leq M_2.$$

Then

$$\int_{\mathbb{R}^n} f(z)dz \leq \frac{(p+1)(q+1)\omega_{n-1}}{n+p+q} M_1^{\frac{2}{n+p+q+2}} \Big(\frac{3n^{\frac{n+p+q+2}{2}}}{2^n}\Big)^{\frac{n+p+q}{n+p+q+2}} M_2^{\frac{n+p+q}{n+p+q+2}},$$

where ω_{n-1} is the area of the unit sphere in $\mathbb{R}^n$.

Proof. First, we choose R such that

$$\int_{\mathbb{R}^n} (\sum_{i=1}^{n-2} z_i^2 + |z_{n-1}|^{\frac{2}{p+1}} + |z_n|^{\frac{2}{q+1}}) g(z)dz = M_2,$$

where

$$g(z) = \begin{cases} M_1, & \sum_{i=1}^{n-2} z_i^2 + |z_{n-1}|^{\frac{2}{p+1}} + |z_n|^{\frac{2}{q+1}} < R^2, \\ 0, & \sum_{i=1}^{n-2} z_i^2 + |z_{n-1}|^{\frac{2}{p+1}} + |z_n|^{\frac{2}{q+1}} \geq R^2. \end{cases}$$

Then $(\sum_{i=1}^{n-2} z_i^2 + |z_{n-1}|^{\frac{2}{p+1}} + |z_n|^{\frac{2}{q+1}} - R^2)(f(z) - g(z)) \geq 0$, hence

$$R^2 \int_{\mathbb{R}^n} (f(z) - g(z))dz \leq \int_{\mathbb{R}^n} (\sum_{i=1}^{n-2} z_i^2 + |z_{n-1}|^{\frac{2}{p+1}} + |z_n|^{\frac{2}{q+1}})(f(z) - g(z))dz \leq 0. \tag{4.8}$$

Now we have

$$\begin{aligned} M_2 &= \int_{\mathbb{R}^n} (\sum_{i=1}^{n-2} z_i^2 + |z_{n-1}|^{\frac{2}{p+1}} + |z_n|^{\frac{2}{q+1}}) g(z)dz = M_1 \int_{\bar{B}_R} (\sum_{i=1}^{n-2} z_i^2 + |z_{n-1}|^{\frac{2}{p+1}} + |z_n|^{\frac{2}{q+1}})dz \\ &= M_1(p+1)(q+1) \int_{B_R} |z|^2 |z_{n-1}|^p |z_n|^q dz \geq M_1(p+1)(q+1) \int_{A_R} |z|^2 |z_{n-1}|^p |z_n|^q dz, \end{aligned}$$

where

$$\bar{B}_R=\{z\in\mathbb{R}^n,\sum_{i=1}^{n-2}z_i^2+|z_{n-1}|^{\frac{2}{p+1}}+|z_n|^{\frac{2}{q+1}}\leq R^2\},\ B_R=\{z\in\mathbb{R}^n,|z|\leq R\},$$

and

$$A_R=\big\{z\in\mathbb{R}^n,|z_i|\leq\frac{R}{\sqrt{n}},i=1,2\cdots,n\big\}.$$

By a direct calculation, we have

$$\int_{A_R}|z|^2|z_{n-1}|^p|z_n|^q dz\geq\int_{A_R}|z_1|^2|z_{n-1}|^p|z_n|^q dz=\frac{2^n}{3(p+1)(q+1)}n^{-\frac{n+p+q+2}{2}}R^{n+p+q+2}.$$

Then we have

$$M_2\geq\frac{2^n M_1}{3}n^{-\frac{n+p+q+2}{2}}R^{n+p+q+2}. \tag{4.9}$$

From the definition of $g(z)$, we know

$$\begin{aligned}\int_{\mathbb{R}^n}g(z)dz&=M_1\int_{\bar{B}_R}dz=M_1(p+1)(q+1)\int_{B_R}|z_{n-1}|^p|z_n|^q dz\\&\leq M_1(p+1)(q+1)\int_{B_R}|z|^{p+q}dz\\&=\frac{M_1(p+1)(q+1)\omega_{n-1}}{n+p+q}R^{n+p+q}.\end{aligned}\tag{4.10}$$

Combining (4.8), (4.9) and (4.10), we can obtain

$$\int_{\mathbb{R}^n}f(z)dz\leq\int_{\mathbb{R}^n}g(z)dz\leq\frac{(p+1)(q+1)\omega_{n-1}}{n+p+q}M_1^{\frac{2}{n+p+q+2}}\Big(\frac{3n^{\frac{n+p+q+2}{2}}}{2^n}\Big)^{\frac{n+p+q}{n+p+q+2}}M_2^{\frac{n+p+q}{n+p+q+2}}.$$

□

Proof of Theorem 1.5: Let $X=(\partial_{x_1},\cdots,\partial_{x_{n-2}},x_i^p\partial_{x_{n-1}},x_j^q\partial_{x_n})$, $\{\lambda_k\}_{k\geq1}$ be a sequence of the Dirichlet eigenvalues for the problem (1.1), $\{\psi_k(x)\}_{k\geq1}$ be the corresponding eigenfunctions. Then $\{\psi_k(x)\}_{k\geq1}$ constitute an orthonormal basis of the Sobolev space $H^1_{X,0}(\Omega)$.

Taking $\Psi(x,y)=\sum_{j=1}^k\psi_j(x)\psi_j(y)$, then from Plancherel's formula and Proposition 2.3, we have

$$\begin{aligned}&\int_{\mathbb{R}^n}\int_\Omega(\sum_{i=1}^{n-2}z_i^2+|z_{n-1}|^{\frac{2}{p+1}}+|z_n|^{\frac{2}{q+1}})|\hat{\Psi}(z,y)|^2dydz\\&=\int_{\mathbb{R}^n}\int_\Omega\big(\sum_{i=1}^{n-2}\big|\partial_{x_i}\Psi(x,y)\big|^2+\big||\partial_{x_{n-1}}|^{\frac{1}{p+1}}\Psi(x,y)\big|^2+\big||\partial_{x_n}|^{\frac{1}{q+1}}\Psi(x,y)\big|^2\big)dydx\\&=\int_\Omega\int_\Omega\big(\sum_{i=1}^{n-2}\big|\partial_{x_i}\Psi(x,y)\big|^2+\big||\partial_{x_{n-1}}|^{\frac{1}{p+1}}\Psi(x,y)\big|^2+\big||\partial_{x_n}|^{\frac{1}{q+1}}\Psi(x,y)\big|^2\big)dydx,\end{aligned}$$

and

$$\int_\Omega\int_\Omega\Big(\sum_{i=1}^{n-2}\big|\partial_{x_i}\Psi(x,y)\big|^2+\big||\partial_{x_{n-1}}|^{\frac{1}{p+1}}\Psi(x,y)\big|^2+\big||\partial_{x_n}|^{\frac{1}{q+1}}\Psi(x,y)\big|^2\Big)dydx$$
$$\leq \bar{C}(p,q)\Big(\int_\Omega\int_\Omega|X(x)\Psi(x,y)|^2dxdy+\widetilde{C}(p,q)\int_\Omega\int_\Omega|\Psi(x,y)|^2dxdy\Big).$$

Next, we can deduce that

$$\int_\Omega\int_\Omega|X(x)\Psi(x,y)|^2dxdy=\int_\Omega\Big(\sum_{l=1}^{n}\int_\Omega|\sum_{j=1}^{k}(X_l(x)\psi_j(x))\psi_j(y)|^2dx\Big)dy$$
$$=-\int_\Omega\sum_{j=1}^{k}\psi_j(x)\triangle_X\psi_j(x)dx=\sum_{j=1}^{k}\lambda_j.$$

Thus from Lemma 3.1 and the discussions above, we have

$$\int_{\mathbb{R}^n}\int_\Omega\Big(\sum_{i=1}^{n-2}z_i^2+|z_{n-1}|^{\frac{2}{p+1}}+|z_n|^{\frac{2}{q+1}}\Big)|\hat{\Psi}(z,y)|^2dydz\leq\bar{C}(p,q)\Big(\sum_{j=1}^{k}\lambda_j+\widetilde{C}(p,q)k\Big).$$

Now we choose

$$f(z)=\int_\Omega|\hat{\Psi}(z,y)|^2dy,\quad M_1=(2\pi)^{-n}|\Omega|_n,\quad M_2=\bar{C}(p,q)\Big(\sum_{j=1}^{k}\lambda_j+\widetilde{C}(p,q)k\Big).$$

Then the result of Lemma 4.2 gives that, for any $k\geq 1$,

$$k\leq\frac{(p+1)(q+1)\omega_{n-1}}{n+p+q}\Big(\frac{3n^{\frac{n+p+q+2}{2}}}{2^n}\Big)^{\frac{n+p+q}{n+p+q+2}}((2\pi)^{-n}|\Omega|_n)^{\frac{2}{n+p+q+2}}$$
$$\cdot\Big(\bar{C}(p,q)(\sum_{j=1}^{k}\lambda_j+\widetilde{C}(p,q)k)\Big)^{\frac{n+p+q}{n+p+q+2}}.$$

This means, for any $k\geq 1$,

$$\sum_{j=1}^{k}\lambda_j\geq C_3k^{1+\frac{2}{n+p+q}}-\widetilde{C}(p,q)k,$$

where

$$C_3=\frac{2^n}{3\bar{C}(p,q)}n^{-\frac{n+p+q+2}{2}}\Big(\frac{n+p+q}{(p+1)(q+1)\omega_{n-1}}\Big)^{\frac{n+p+q+2}{n+p+q}}\Big(\frac{(2\pi)^n}{|\Omega|_n}\Big)^{\frac{2}{n+p+q}},$$

and $\bar{C}(p,q)=C(p+1)+C(q+1)+2>0$, $\widetilde{C}(p,q)=\max\{\widetilde{C}(p+1),\widetilde{C}(q+1)\}$. $C(p+1)$, $C(q+1)$, $\widetilde{C}(p+1)$ and $\widetilde{C}(q+1)$ are the constants in (2.1). Theorem 1.5 is proved. $\square$

§ 5. Proof of Theorem 1.7

Lemma 5.1. *Let f be a real-valued function defined on $\mathbb{R}^n$ and $0 \le f \le M_1$. For some $s > 0$, if*

$$\int_{\mathbb{R}^n} f(z)dz \ge 1, \text{ and } \int_{\mathbb{R}^n} (\log(e^2 + |z|^2))^{2s} f(z)dz \le M_2,$$

where $M_2 \ge 2^{4s+n}e^n M_1 B_n$, B_n is the volume of the unit ball in $\mathbb{R}^n$. Then we have the following inequality,

$$\int_{\mathbb{R}^n} f(z)dz \cdot (\log(\int_{\mathbb{R}^n} f(z)dz))^{2s} \le \frac{2^{n+2s}}{2^n - 1}(|\log(M_1 B_n)|^{2s} + n^{2s})M_2.$$

Proof. We choose a constant $R > 0$ such that

$$\int_{\mathbb{R}^n} (\log(e^2 + |z|^2))^{2s} g(z)dz = M_2,$$

where

$$g(z) = \begin{cases} M_1, & |z| < R, \\ 0, & |z| \ge R. \end{cases}$$

Since $M_2 \ge 2^{4s+n}e^n M_1 B_n$, that means $R \ge 2e$. In fact, if $R < 2e$, then

$$\begin{aligned} M_2 &= \int_{\mathbb{R}^n} (\log(e^2 + |z|^2))^{2s} g(z)dz = M_1 \omega_{n-1} \int_0^R (\log(e^2 + r^2))^{2s} r^{n-1} dr \\ &\le M_1 B_n (\log(5e^2))^{2s} (2e)^n < 2^{4s+n} e^n M_1 B_n, \end{aligned}$$

where ω_{n-1} is the area of the unit sphere in $\mathbb{R}^n$, B_n is the volume of the unit ball in $\mathbb{R}^n$ and $nB_n = \omega_{n-1}$. Thus $R \ge 2e$ and $R \ge 2\sqrt{R}$, and

$$\begin{aligned} M_2 &\ge M_1 \omega_{n-1} \int_{\frac{R}{2}}^R (\log(e^2 + r^2))^{2s} r^{n-1} dr \ge M_1 \omega_{n-1} 2^{2s} \int_{\frac{R}{2}}^R (\log r)^{2s} r^{n-1} dr \\ &\ge 2^{2s} M_1 B_n (1 - 2^{-n}) R^n \Big(\log \frac{R}{2}\Big)^{2s} \ge M_1 B_n (1 - 2^{-n}) R^n (\log R)^{2s}. \end{aligned} \tag{5.1}$$

Since $\Big[(\log(e^2 + |z|^2))^{2s} - (\log(e^2 + R^2))^{2s}\Big](f(z) - g(z)) \ge 0$, we have

$$(\log(e^2 + R^2))^{2s} \int_{\mathbb{R}^n} \big(f(z) - g(z)\big) dz \le \int_{\mathbb{R}^n} (\log(e^2 + |z|^2))^{2s} \big(f(z) - g(z)\big) dz \le 0,$$

which implies

$$\int_{\mathbb{R}^n} f(z)dz \le \int_{\mathbb{R}^n} g(z)dz. \tag{5.2}$$

From the estimate (5.2) and the fact $\int_{\mathbb{R}^n} f(z)dz \geq 1$, we can obtain

$$\begin{aligned}
&\int_{\mathbb{R}^n} f(z)dz \cdot \Big(\log(\int_{\mathbb{R}^n} f(z)dz)\Big)^{2s} \\
&\leq \int_{\mathbb{R}^n} g(z)dz \cdot \Big(\log(\int_{\mathbb{R}^n} g(z)dz)\Big)^{2s} = M_1 B_n R^n \cdot \Big[\log(M_1 B_n R^n)\Big]^{2s} \qquad (5.3)\\
&\leq 2^{2s} M_1 B_n (|\log(M_1 B_n)|^{2s} + n^{2s}) R^n (\log R)^{2s}.
\end{aligned}$$

Thus the estimates (5.1) and (5.3) give that

$$\int_{\mathbb{R}^n} f(z)dz \cdot \Big(\log(\int_{\mathbb{R}^n} f(z)dz)\Big)^{2s} \leq \frac{2^{n+2s}}{2^n - 1}(|\log(M_1 B_n)|^{2s} + n^{2s}) M_2.$$

□

Proof of Theorem 1.7: First, the problem (1.1) has a sequence of discrete eigenvalues $\{\lambda_k\}_{k\geq 1}$ and the corresponding eigenfunctions $\{\psi_k(x)\}_{k\geq 1}$ constitute an orthonormal basis of the Sobolev space $H^1_{X,0}(\Omega)$.

Taking $\Psi(x,y) = \sum_{j=1}^k \psi_j(x)\psi_j(y)$, then from Lemma 3.1, we know

$$\int_\Omega \int_{\mathbb{R}^n} |\hat{\Psi}(z,y)|^2 dzdy = k, \text{ and } \int_\Omega |\hat{\Psi}(z,y)|^2 dy \leq (2\pi)^{-n}|\Omega|_n.$$

On the other hand, by using Plancherel's formula, we can obtain

$$\int_{\mathbb{R}^n} \int_\Omega |\hat{\Psi}(z,y)|^2 (\log(e^2 + |z|^2))^{2s} dydz = \int_{\mathbb{R}^n} \int_\Omega |(\log(e^2 + |\nabla_x|^2))^s \Psi(x,y)|^2 dydx.$$

where $\nabla_x = (\partial_{x_1}, \partial_{x_2}, \cdots, \partial_{x_n})$. Next, the logarithmic regularity estimate (1.15) gives

$$\begin{aligned}
&\int_{\mathbb{R}^n} \int_\Omega |(\log(e^2 + |\nabla_x|^2))^s \Psi(x,y)|^2 dydx \\
&\quad \leq 2^{2s} C_0 (\int_\Omega \int_\Omega |X(x)\Psi(x,y)|^2 dxdy + \int_\Omega \int_\Omega |\Psi(x,y)|^2 dxdy).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\int_\Omega \int_\Omega |X(x)\Psi(x,y)|^2 dxdy &= \int_\Omega \Big(\sum_{l=1}^m \int_\Omega |\sum_{j=1}^k (X_l(x)\psi_j(x))\psi_j(y)|^2 dx\Big) dy \\
&= -\int_\Omega \sum_{j=1}^k \psi_j(x) \triangle_X \psi_j(x) dx = \sum_{j=1}^k \lambda_j.
\end{aligned}$$

Therefore, from the above calculations, we obtain

$$\int_{\mathbb{R}^n} \int_\Omega |\hat{\Psi}(z,y)|^2 (\log(e^2 + |z|^2))^{2s} dydz \leq 2^{2s} C_0 (\sum_{j=1}^k \lambda_j + k).$$

Now we choose

$$f(z) = \int_{\Omega} |\hat{\Psi}(z,y)|^2 dy,\ M_1 = (2\pi)^{-n}|\Omega|_n,\ M_2 = 2^{2s}C_0(\sum_{j=1}^{k}\lambda_j + k).$$

Then we know that $0 \le f(z) \le M_1$. If we take $k_0 = [\frac{2^{2s}e^n B_n|\Omega|_n}{C_0\pi^n}] + 1$, then for any $k \ge k_0$, we can see

$$\int_{\mathbb{R}^n} f(z)dz = k \ge 1,\ \text{and}\ M_2 \ge 2^{4s}e^n|\Omega|_n B_n\pi^{-n} = 2^{4s+n}e^n M_1 B_n.$$

Thus from the result of Lemma 5.1, for any $k \ge k_0$, we have

$$k(\log k)^{2s} \le \frac{2^{n+4s}}{2^n - 1}(|\log \frac{|\Omega|_n B_n}{(2\pi)^n}|^{2s} + n^{2s})C_0 \cdot (\sum_{j=1}^{k}\lambda_j + k).$$

That means, for any $k \ge k_0$,

$$\sum_{j=1}^{k}\lambda_j \ge C_4 k(\log k)^{2s} - k,$$

where $C_4 = (2^n - 1)\Big(C_0 2^{n+4s}(|\log \frac{|\Omega|_n B_n}{(2\pi)^n}|^{2s} + n^{2s})\Big)^{-1}$. This completes the proof of Theorem 1.7. $\square$

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