

**"Indirect" Time Series Analysis for
One-Dimensional Chaos Based on
Perron-Frobenius Operator:
"Generalized" Ulam-Li's Approximation to
Invariant Density**

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Abstract A unified approach to time series analysis for one-dimensional discrete chaos is given which is based on the Galerkin approximation to the Perron-Frobenius operator. The proposed method gives approximations with high accuracy to statistics of various one-dimensional chaos. Numerical results for $1/f^\delta$ power spectrum of intermittent chaos also show that the observed exponent of the FFT power spectrum of long trajectories as $f \rightarrow 0$ is in good agreement not with the Procaccia-Schuster's estimate but with our estimate.

I. Introduction

There are two kinds of time series analysis for long-time chaotic trajectories $\{x_m\}_{m=0}^\infty$ generated by a recurrence formula $x_{m+1} = \tau(x_m)$ with an ergodic transformation $\tau : I = [0, 1] \rightarrow I$. One of them is

the “*time-average technique*”, in which we evaluate certain statistics of a sample long-time trajectory $\{x_m\}_{m=0}^n$ with some initial value $x = x_0$; the other one is the “*ensemble-average technique*” under the assumption that τ is mixing with respect to an absolutely continuous invariant measure, denoted by $f^*(x)dx$. We give a unified approach to time series analysis for discrete chaos by such an ensemble-average technique.

The time-average technique which is a usual method^[4] is referred to as the “*direct method*”. On the contrary, the ensemble average technique is a kind of “*indirect methods*” because there is no need to directly calculate trajectories. Hence such an indirect method is expected to play an important role in theoretically understanding chaos. In fact, the Perron-Frobenius operator whose fixed point is $f^*(x)$ permits us to theoretically calculate the ensemble average of several statistics^{[5],[6]}. This operator, denoted by P_τ , however, gives no practically calculating method because of its infinite dimensionality. Such a situation leads us to consider an efficient algorithm for systematically calculating statistics which is based on the Galerkin approximation to P_τ on a suitable function space^{[7]–[10]}. This algorithm is referred to as a “generalized” Ulam-Li’s method^[7]. We used the word “Ulam-Li’s method” because Li^[2] gave an affirmative answer to the Ulam’s conjecture^[1] concerning a piecewise-constant approximation for $f^*(x)$.

Numerical experiments demonstrate that the proposed method can give approximations with high accuracy to statistics of various

one-dimensional chaos.

II. Perron-Frobenius Operator and Statistics of Chaos

If $y = \tau(x)$ is mixing with respect to $f^*(x)dx$, then for almost initial value $x = x_0$ sequences $\{x_m\}_{m=0}^{\infty}$ can chaotically behave. From the Birchoff individual ergodic theorem, the time average of any L_1 function $F(x)$ along a trajectory $\{x_m\}_{m=0}^{\infty}$, which is defined by

$$\overline{F} = \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{n=0}^{T-1} F(\tau^n(x)), \quad (1)$$

is equal almost everywhere to the ensemble average of $F(x)$ over I , defined by

$$\langle F \rangle = \int_I F(x) f^*(x) dx. \quad (2)$$

The direct time series analysis is based on using $\overline{F}$. However, the *sensitive dependence on initial conditions*, one of chaotic properties^[3], prevents us from precisely evaluating $\overline{F}$. On the other hand, the indirect time series analysis is based on using $\langle F \rangle$. We begin with reviewing relations between typical statistics and P_τ . The operator P_τ is defined by

$$P_\tau f(x) = \int_I \delta(x - \tau(y)) h(y) dy. \quad (3)$$

For any L_1 functions of bounded variations $g(x)$ and $h(x)$, P_τ has the important property

$$(g(x), h(\tau(x))) = (P_\tau g(x), h(x)), \quad (4)$$

where

$$(g, h) = \int_I g(x)h(x)dx. \quad (5)$$

The invariant density $f^*(x)$ which plays a key role in our indirect method is the eigenfunction of P_τ belonging to the eigenvalue 1, that is,

$$P_\tau f^*(x) = f^*(x). \quad (6)$$

The autocorrelation function is defined by

$$\rho(k) = \langle x\tau^k(x) \rangle - \langle x \rangle^2. \quad (7)$$

The first term of the rhs of this equation is rewritten as

$$\langle x\tau^k(x) \rangle = (P_\tau^k(xf^*(x)), x), \quad (8)$$

where the above property of P_τ is repeatedly used. Let $h_i(x)$ be the eigenfunction of P_τ with the eigenvalue λ_i for the eigenvalue problem^[4]

$$P_\tau h_i(x) = \lambda_i h_i(x). \quad (9)$$

If we can expand $xf^*(x)$ as

$$xf^*(x) = \sum_{i=1}^{\infty} \eta_i h_i(x), \quad (10)$$

then we have

$$\rho(k) = \sum_{i=2}^{\infty} u_i \lambda_i^k, \quad (11)$$

the Fourier Transform of which gives the power spectrum $S(\nu)$

$$S(\nu) = \sum_{i=2}^{\infty} u_i \frac{1 - \lambda_i^2}{(1 - \lambda_i z)(1 - \lambda_i z^{-1})} \quad (12)$$

where

$$\lambda_1 = 1, u_i = \eta_i(x, h_i), \text{ and } z = \exp(j2\pi\nu) \quad (13)$$

with $0 < \nu < 1$. Oono and Takahashi^{[5],[6]} demonstrated that the Fredholm theory of P_τ plays an important role in discussions of the power spectrum. It is, however, difficult to find exact solutions of eigenvalues and eigenfunctions of P_τ , primarily because P_τ has the infinite dimensionality. Such a situation led us to consider an efficient algorithm of the indirect method.

III. Galerkin Approximations to Perron-Frobenius Operator

Let Δ be a function space which is spanned by a vector basis function $\vec{\ell}(x)$. The constructing method of Δ is as follows. We divide I into N subintervals $\{I_n\}$ with partition points $\{c_i\}_{i=0}^N$ satisfying $0 = c_0 < c_1 < c_2 < \cdots < c_N = 1$ such that

$$I = \bigcup_{n=1}^N I_n, \quad I_n = [c_{n-1}, c_n]. \quad (14)$$

Our Galerkin approximations depend on the appropriate selections of $\{c_i\}_{i=0}^N$ and of $\vec{\ell}(x)$ ^{[7]-[10]}. A simple but efficient procedure, however, is omitted here for selecting $\{c_i\}_{i=0}^N$. Next, we take bases $\ell_{nk}(x)$ such as^[10]

$$\ell_{nk}(x) = p_{nk}(x)s(x)\chi_n(x), \quad 0 \leq k \leq K, \quad 1 \leq n \leq N. \quad (15)$$

In the above equation, $\chi_n(x)$ is the characteristic function of I_n and $p_{nk}(x)$ is the k -th order Legendre's polynomial which is orthogonal to each other on I_n . For most of practical usages, we use $K = 2$. When τ has a bounded invariant density, the function $s(x)$, referred to as a supplementary function, is taken to be 1. On the other hand, τ has an unbounded invariant density, $s(x)$ is chosen to be a singular function which approximates to singularities of the unbounded invariant density and the inner product (g, h) must be also replaced by the weighted inner product

$$(g, h)_w = \int_I g(x)h(x)w(x)dx \quad (16)$$

with the weighting function

$$w(x) = s^{-2}(x). \quad (17)$$

Each component $\ell_{nk}(x)$ is an appropriately chosen piecewise polynomial of at most K degree whose combination approximates to $xf^*(x)$ by the Galerkin method^[7] such as

$$xf^*(x) \simeq \mathbf{f}^t \vec{\ell}(x), \quad (18)$$

where the superscript t denotes the transpose of the vector $\mathbf{f}$. Using $\vec{\ell}(x)$, we get

$$\langle x\tau^k(x) \rangle \simeq \mathbf{f}^t (P_\tau^k \vec{\ell}(x), x). \quad (19)$$

Furthermore, using the Galerkin method with $\vec{\ell}(x)$ on Δ , we approximate to $P_\tau \vec{\ell}(x)$ such as

$$P_\tau \vec{\ell}(x) \simeq \hat{P}_\tau \vec{\ell}(x) \quad (20)$$

which leads us to readily obtain

$$\langle x \tau^k(x) \rangle \simeq \mathbf{f}^t(\hat{P}_\tau^t)^k(\vec{\ell}(x), x), \quad (21)$$

where the $N(K+1) \times N(K+1)$ matrix $\hat{P}_\tau$ is referred to as the *Galerkin-approximated matrix of the Perron-Frobenius operator* where N and K are integers to be given below. The explicit form of $\hat{P}_\tau$ is given in^[7]. Let $\mathbf{h}_i$ be the i -th right eigenvector of $\hat{P}_\tau$ with the eigenvalue λ_i for the easily tractable eigenvalue problem

$$\hat{P}_\tau \mathbf{h}_i = \lambda_i \mathbf{h}_i. \quad (22)$$

Let $\hat{\lambda}_1$ be the maximum eigenvalue of $\hat{P}$. It is easily shown that $\hat{\lambda}_1$ when the supplementary function $s(x) = 1$, namely, when both the polynomial bases and the unweighted inner product are used. But $\hat{\lambda}_1 \simeq 1$ when $s(x) \neq 1$, that is, when both the singular bases and the weighted inner product are used. For the latter case, numerical experiments show $\hat{\lambda}_1$ is nearly equal to 1 with the error less than 10^{-6} for $K = 2$ and $N = 32$. An approximate solution to the invariant density given by

$$\hat{f}^*(x) = \mathbf{h}_1^t \vec{\ell}(x) \quad (23)$$

where $\mathbf{h}_1$ is normalized such that

$$\int_I \hat{f}^*(x) dx = 1 \quad (24)$$

It is easily shown that Eq.(23) is an approximate solution to Eq.(6) by the Galerkin method and that $\hat{f}^*(x)$ when $K = 0$ gives the

results by the well-known Ulam-Li's method. Figure 1 shows convergence rates of approximate solution $\hat{f}^*(x)$ by our method^[7].

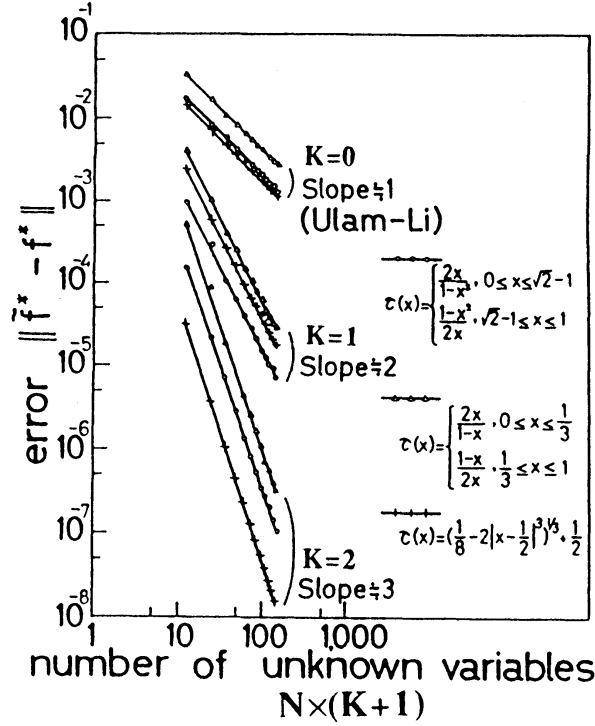


Fig. 1 Convergence rates of approximate solutions $\hat{f}^*(x)$ (the proposed method) to the bounded invariant densities $f^*(x)$ for mixing chaos in several examples.

IV. Numerical Examples

Example 1 Let

$$\tau(x) = \begin{cases} ax^z + (a+b-ab)/b & 0 \leq x \leq x_p = (1-1/b)^{1/z} \\ -b(x^z - 1) & x_p < x \leq 1 \end{cases}$$

This map can generate periodic chaos for suitable parameters. Figures 2 and 3 show $f^*(x)$ and the power spectrum $\tilde{S}_T(\nu)$ for periodic chaos of period 6 which are calculated by our method^{[8],[9]}. In this calculation, we take $\{\tau^n(0)\}_{n=1}^{30}$ as the partition points $\{c_i\}_{i=1}^{N-1}$ so

that edges of the support of $f^*(x)$ will coincide with the partition points. In the calculation of $\tilde{S}_T(\nu)$, the finite discrete Fourier transform of $\{\rho(k)\}_{k=0}^{T-1}$ ($T = 1,024 \times 6$) is used instead of using Eq.(12). On the other hand, $S_{T,m}(\nu)$ is obtained by averaging $m = 200$ discrete Fourier transforms of trajectories of length T . The spectrum $\tilde{S}_T(\nu)$ is in good agreement with $S_{T,m}(\nu)$ except for fluctuations in the latter.

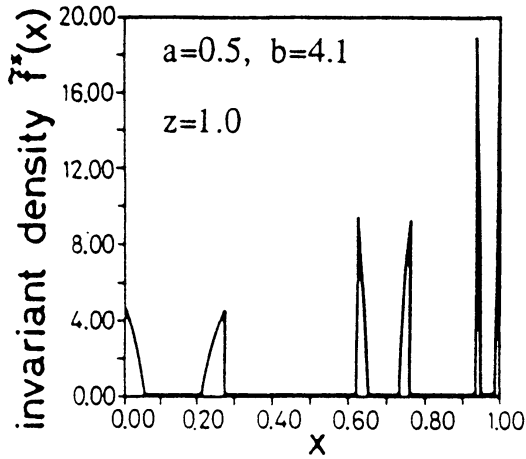


Fig. 2 Approximated invariant density $\hat{f}^*(x)$ (by our indirect method) for periodic chaos of period 6 in example 1.

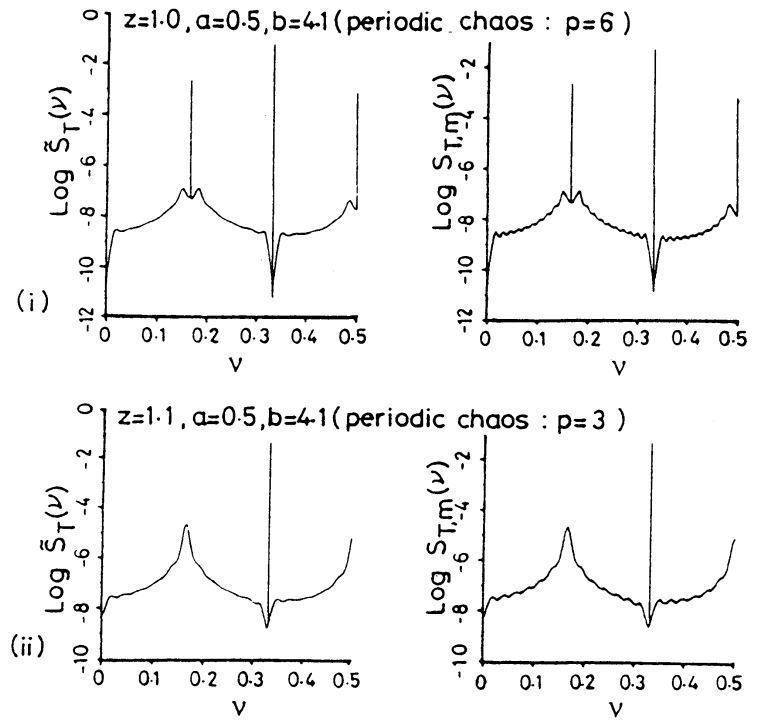


Fig. 3 Power spectra $\tilde{S}_T(\nu)$ (by our indirect method) and $\tilde{S}_{T,m}(\nu)$ (by the direct method) for periodic chaos of period 6 in example 1.

Example 2 Let

$$\tau(x) = \begin{cases} x + ux^z & 0 \leq x \leq x_p \\ (x - x_p)/(1 - x_p) & x_p < x \leq 1 \end{cases}$$

where $\tau(x_p) = 1, u > 0, 1 < z < 2$. This map generates intermittent chaos with the power spectrum $1/f^\delta$. Figure 4 shows the power spectrum $S(\nu)$ by our method^[10] (the smooth solid line) and $S_{T,m}(\nu)$ with $T = 2^{15}$ and $m = 100$ by the direct method (the fluctuated line), each of which is in good agreement each other in wide frequency range. In applying our method, we used $s(x) = x^{-(z-1)}$ because τ has the unbounded invariant density with a $(z-1)$ -th order pole at $x = 0$. In this figure, the broken line shows the Procaccia and Schuster's estimate^[11] of the spectrum when ν goes to 0 which does not coincide well with the former two.

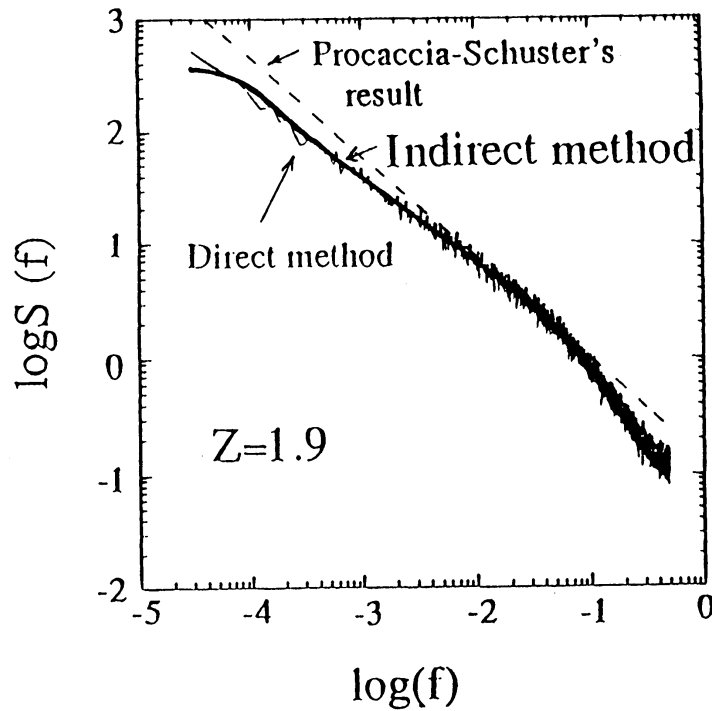


Fig. 4 Comparison of power spectra calculated by using three different methods for intermittent chaos in example 2.

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