

Generalized Lerch formulas

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1 Generalized Lerch's formulas

The zeta-regularized product of a countable sequence $\{\lambda_k\} \subset \mathbb{C} \setminus \{0\}$ is defined by

$$\widehat{\prod}_k \lambda_k = \exp \left(-\frac{\partial}{\partial s} \sum_k \lambda_k^{-s} \Big|_{s=0} \right),$$

provided that $\Lambda(s) = \sum_k \lambda_k^{-s}$ is continued holomorphically at $s = 0$. Here the branch is chosen so that $-\pi < \arg(\lambda_k) \leq \pi$.

There are several interesting formulas which can be formulated in terms of zeta-regularized products. Typical examples are Lerch's formula

$$\widehat{\prod}_{n=0}^{\infty} (n+x) = \frac{\sqrt{2\pi}}{\Gamma(x)} \quad (1)$$

and Kronecker's limit formula

$$\widehat{\prod}_{(c,d)=1} \frac{|cz+d|}{\sqrt{y}} = (y^6 |\Delta(z)|)^{-\frac{1}{6}}. \quad (2)$$

Here $\Gamma(x)$ is Euler's gamma function and $\Delta(z) = e^{2\pi iz} \prod_{n=1}^{\infty} (1 - e^{2\pi inz})^{24}$ is Ramanujan's delta function.

In this paper, we generalize Lerch's formula.

Theorem 1 For $z_j \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$, we have

$$\widehat{\prod}_{m=0}^{\infty} \left(\prod_{j=1}^n (m + z_j) \right) = \frac{(\sqrt{2\pi})^n}{\prod_{j=1}^n \Gamma(z_j)} = \prod_{j=1}^n \left(\widehat{\prod}_{m=0}^{\infty} (m + z_j) \right).$$

As a part of Theorem 1, we can obtain the formula of Lerch, Kurokawa and Wakayama.

Corollary 1 (Lerch)

$$\widehat{\prod}_{n=0}^{\infty} ((n+x)^2 + y^2) = \frac{2\pi}{\Gamma(x+iy)\Gamma(x-iy)}.$$

Corollary 2 (Kurokawa and Wakayama [5])

$$\widehat{\prod}_{n=0}^{\infty} ((n+x)^m - y^m) = \frac{(\sqrt{2\pi})^m}{\prod_{\zeta^m=1} \Gamma(x - \zeta y)}.$$

We would like to mention that our motivation of generalizing Lerch's formula is how $\widehat{\prod}_n (a_n \cdot b_n)$ is connected with $\widehat{\prod}_n a_n \cdot \widehat{\prod}_n b_n$.

Suppose that a_n and b_n depend on some parameters X . In many examples, we know

$$\widehat{\prod}_n (a_n \cdot b_n) = e^{F(X)} \widehat{\prod}_n a_n \cdot \widehat{\prod}_n b_n \quad (3)$$

with some $F(X)$. An interesting question is to understand $F(X)$.

Theorem 1 is an example of the case where $F(X)$ vanishes in (3). In fact we have

Corollary 3 For monic polynomials $P_j(x)$ such that $P_j(m) \neq 0$ for any $m \in \{0\} \cup \mathbb{N}$, one has

$$\widehat{\prod}_{m=0}^{\infty} \left(\prod_{j=1}^n P_j(m) \right) = \prod_{j=1}^n \left(\widehat{\prod}_{m=0}^{\infty} P_j(m) \right).$$

Corollary 3 is remarkable because it is saying that $F(X) = 0$ in (3), which does not hold in general at all. We can see examples for $F(X) \neq 0$ in Corollary 4 which will be given in Section 2 and Lemma 1 of [8].

2 Two dimensional analogue and q -analogue

There are two dimensional analogue and q -analogue of Euler's gamma function, so called Barnes' double gamma functions and Jackson's q -gamma functions (see [1], [7]). Hence it is natural to seek two dimensional analogue and q -analogue of Theorem 1.

Barnes' double gamma function $\Gamma_2^*(z, (\omega_1, \omega_2))$ is defined by

$$\log \Gamma_2^*(z, (\omega_1, \omega_2)) = \frac{\partial}{\partial s} \sum_{m,l=0}^{\infty} (m\omega_1 + l\omega_2 + z)^{-s} \Big|_{s=0},$$

$$\Gamma_2^*(z, (\omega_1, \omega_2))^{-1} = \widehat{\prod}_{m,l=0}^{\infty} (m\omega_1 + l\omega_2 + z).$$

We get a two dimensional analogue of Theorem 1 by using the following result.

Theorem 2 Assume that $q_j, \tau_j, z_j \in \mathbf{C}$ satisfy that $\Re(q_j) > 0, \Re(\tau_j) > 0, \Re(z_j) > 0$, and $q_j \neq q_k, \tau_j \neq \tau_k, q_j \tau_k \neq q_k \tau_j$ for $j \neq k$. The function of s defined by

$$H_2(s) = \sum_{m,l=0}^{\infty} \prod_{j=1}^n (mq_j + l\tau_j + z_j)^{-s}$$

is continued meromorphically to all s -plane. $H_2(s)$ is holomorphic at $s = 0$ and we have the following formula for $\frac{\partial}{\partial s} H_2(s) \Big|_{s=0}$,

$$\begin{aligned} \frac{\partial}{\partial s} H_2(s) \Big|_{s=0} &= \sum_{j=1}^n \log \Gamma_2^*(z_j, (q_j, \tau_j)) \\ &+ \frac{1}{2n} \sum_{1 \leq j < k \leq n} \left\{ \frac{q_j \tau_k - \tau_j q_k}{q_j q_k} (\log q_k - \log q_j) B_2 \left(\frac{q_j z_k - q_k z_j}{q_j \tau_k - \tau_j q_k} \right) \right. \\ &\left. + \frac{q_k \tau_j - \tau_k q_j}{\tau_j \tau_k} (\log \tau_k - \log \tau_j) B_2 \left(\frac{\tau_j z_k - \tau_k z_j}{\tau_j q_k - q_j \tau_k} \right) \right\}. \end{aligned}$$

Here $B_2(x) = x^2 - x + 1/6$ is the second Bernoulli polynomial. We choose the principal branch for $\log q_i, \log \tau_i$.

This is a generalization of Shintani's result (see [12]). He treated the case $n = 2$ to give a new proof of Kronecker's limit formula (2).

We remark that in order to conclude

$$\exp \left(- \frac{\partial}{\partial s} H_2(s) \Big|_{s=0} \right) = \widehat{\prod}_{m,l=0}^{\infty} \left(\prod_{j=1}^n (mq_j + l\tau_j + z_j) \right),$$

the equation

$$\left\{ \prod_{j=1}^n (mq_j + l\tau_j + z_j) \right\}^s = \prod_{j=1}^n (mq_j + l\tau_j + z_j)^s \quad (4)$$

must hold for any $m, l \in \mathbf{N} \cup \{0\}$. We take this remark into account to give a two dimensional analogue of Theorem 1. As an example of q_j, τ_j, z_j which satisfy the equation (4) for any $m, l \in \mathbf{N} \cup \{0\}$, we can take $n = 2h, q_j, \tau_j, z_j \in \mathbf{C}, q_{h+j} = \overline{q_j}, \tau_{h+j} = \overline{\tau_j}, z_{h+j} = \overline{z_j}, j = 1, \dots, h$.

Corollary 4 Fix $q_j, \tau_j, z_j \in \mathbf{C}$ such that $\Re(q_j) > 0, \Re(\tau_j) > 0, \Re(z_j) > 0$, and $q_j \neq q_k, \tau_j \neq \tau_k, q_j \tau_k \neq q_k \tau_j$ for $j \neq k$. Suppose that (4) is satisfied for any $m, l \in \mathbf{N} \cup \{0\}$. Then we have

$$\begin{aligned} \widehat{\prod}_{m,l=0}^{\infty} \left(\prod_{j=1}^n (mq_j + l\tau_j + z_j) \right) &= e^F \prod_{j=1}^n \Gamma_2^*(z_j, (q_j, \tau_j))^{-1} \\ &= e^F \prod_{j=1}^n \left(\widehat{\prod}_{m,l=0}^{\infty} (mq_j + l\tau_j + z_j) \right), \end{aligned}$$

where

$$\begin{aligned} F &= -\frac{1}{2n} \sum_{1 \leq j < k \leq n} \left\{ \frac{q_j \tau_k - \tau_j q_k}{q_j q_k} (\log q_k - \log q_j) B_2 \left(\frac{q_j z_k - q_k z_j}{q_j \tau_k - \tau_j q_k} \right) \right. \\ &\quad \left. + \frac{q_k \tau_j - \tau_k q_j}{\tau_j \tau_k} (\log \tau_k - \log \tau_j) B_2 \left(\frac{\tau_j z_k - \tau_k z_j}{\tau_j q_k - q_j \tau_k} \right) \right\}. \end{aligned}$$

Next we present q -analogue of Theorem 1. Usually the zeta-regularized product is defined for a sequence $\{\lambda_k\} \subset \mathbf{C} \setminus \{0\}$ such that $\Lambda(s) = \sum_k \lambda_k^{-s}$ can be continued holomorphically at $s = 0$. In case $\Lambda(s)$ is meromorphic at

$s = 0$, Kurokawa and Wakayama [6] define the generalized zeta regularization by

$$\widehat{\prod}_k \lambda_k = \exp \left(- \operatorname{Res}_{s=0} \frac{\Lambda(s)}{s^2} \right).$$

They obtained several examples of such product, one of which is the following q -analogue of Lerch's formula.

Theorem 3 (Kurokawa and Wakayama [6]) *For $q > 1, x > 0$,*

$$\widehat{\prod}_{n=0}^{\infty} [n+x]_q = \frac{C_q}{\Gamma_q(x)}.$$

Here $[x]_q = \frac{q^x - 1}{q - 1}$ is the q -analogue of number x ,

$$\Gamma_q(x) = \frac{\prod_{n=1}^{\infty} (1 - q^{-n})}{\prod_{n=0}^{\infty} (1 - q^{-(x+n)})} (q-1)^{1-x} q^{\frac{x(x-1)}{2}}$$

is Jackson's q -gamma function,

$$C_q = \widehat{\prod}_{n=1}^{\infty} [n]_q = q^{-\frac{1}{12}} (q-1)^{\frac{1}{2} - \frac{\log(q-1)}{2 \log q}} \prod_{n=1}^{\infty} (1 - q^{-n}).$$

We obtain the next result which is the q -analogue of Theorem 1 including the above Theorem 3.

Theorem 4 *For $q > 1, z_j > 1$, we have*

$$\begin{aligned} \widehat{\prod}_{m=0}^{\infty} \left(\prod_{j=1}^n [m+z_j]_q \right) &= \frac{C_q^n}{\prod_{j=1}^n \Gamma_q(z_j)} q^{-\frac{1}{2n} (\sum_{j=1}^n z_j)^2 + \frac{1}{2} \sum_{j=1}^n z_j^2} \\ &= q^{-\frac{1}{2n} (\sum_{j=1}^n z_j)^2 + \frac{1}{2} \sum_{j=1}^n z_j^2} \prod_{j=1}^n \left(\widehat{\prod}_{m=0}^{\infty} [n+z_j]_q \right). \end{aligned}$$

3 Double Hurwitz zeta

For $\beta > \alpha > 0$, let $H_{\alpha,\beta}(s_1, s_2)$ be Dirichlet series defined by

$$H_{\alpha,\beta}(s_1, s_2) = \sum_{n=0}^{\infty} (n + \alpha)^{-s_1} (n + \beta)^{-s_2}.$$

This series converges absolutely for $\Re(s_1 + s_2) > 1$.

$H_{\alpha,\beta}(s_1, s_2)$ is an important object in the theory of the zeta-regularized product. For example, as we presented in Section 1, we know generalized Lerch's formula

$$\exp \left(- \frac{\partial}{\partial s} H_{\alpha,\beta}(s, s) \Big|_{s=0} \right) = \frac{2\pi}{\Gamma(\alpha)\Gamma(\beta)}.$$

We know also that the spectral zeta function $Z_n(s)$ of the unit n -sphere S^{n-1} can be written in terms of $H_{\alpha,\beta}(s_1, s_2)$ as

$$Z_n(s) = \sum_{d=0}^{n-1} T_{n,d} H_{1,n}(s-d, s), \quad (5)$$

where

$$T_{n,d} = \frac{1}{n!} \sum_{r=d+1}^n s(n, r) \binom{r}{d} (n^{r-d} - (n-2)^{r-d}),$$

$s(r, d)$ denoting the Stirling numbers of the first kind. See Lemma 2 of [4] p.202. We get the formula for the functional determinant of the Laplacian by evaluating $\frac{\partial}{\partial s} Z_n(s) \Big|_{s=0}$. See Theorem 1 of [4] p. 200.

In the results mentioned above, the main target is not $H_{\alpha,\beta}(s_1, s_2)$ itself but evaluating derivative of $H_{\alpha,\beta}(s_1, s_2)$. In this section, we analyze $H_{\alpha,\beta}(s_1, s_2)$ itself. First by applying the method described in [2], we can get the following expression for $H_{\alpha,\beta}(s_1, s_2)$.

$$H_{\alpha,\beta}(s_1, s_2) = \frac{\Gamma(s_1 + s_2)}{\Gamma(s_1)\Gamma(s_2)} \int_0^1 u^{s_2-1} (1-u)^{s_1-1} \zeta(s_1 + s_2, \alpha - (\alpha - \beta)u) du, \quad (6)$$

where $\zeta(s, x) = \sum_{n=0}^{\infty} (n+x)^{-s}$ is Hurwitz zeta function. It is very interesting to note that S. Ramanujan already treated the integral of the right hand side on (6) apart from Dirichlet series $H_{\alpha,\beta}(s_1, s_2)$. See (14) of [9] p.166.

Starting from the integral expression (6), we show the following results.

Theorem 5 $H_{\alpha,\beta}(s_1, s_2)$ can be continued meromorphically to all $s_1, s_2 \in \mathbb{C}$.

Theorem 6 For $\Re(s_1) < 0, \Re(s_2) < 0, 0 < \alpha < \beta < 1$, we have

$$\begin{aligned} H_{\alpha,\beta}(s_1, s_2) &= \frac{\Gamma(1-s_1-s_2)}{(2\pi)^{1-s_1-s_2}} \\ &\times \left\{ e^{\frac{\pi i}{2}(1-s_1-s_2)} \sum_{n=1}^{\infty} n^{s_1+s_2-1} e^{-2\pi i n \beta} {}_1F_1(s_1, s_1+s_2, 2\pi i n(\beta-\alpha)) \right. \\ &+ \left. e^{-\frac{\pi i}{2}(1-s_1-s_2)} \sum_{n=1}^{\infty} n^{s_1+s_2-1} e^{2\pi i n \alpha} {}_1F_1(s_2, s_1+s_2, 2\pi i n(\beta-\alpha)) \right\}. \quad (7) \end{aligned}$$

Here ${}_1F_1(a, b, z)$ is the confluent hypergeometric series defined by

$${}_1F_1(a, b, z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{z^n}{n!} \quad (8)$$

with $(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}$.

This is a generalization of well known Hurwitz relation for $\zeta(s, x)$.

Theorem 7 We have

$$H_{\alpha,\beta}(s_1, s_2) = \sum_{n=0}^{\infty} \frac{(s_2)_n}{n!} \zeta(s_1 + s_2 + n, \alpha) (\alpha - \beta)^n.$$

This is a special case of Main Theorem of [3]. However we can prove Main Theorem of [3] by quite different manner using the confluent hypergeometric series ${}_1F_1(a, b, z)$.

Next we give the evaluation formula of $H_{\alpha,\beta}(s_1, s_2)$. We can evaluate the values of $H_{\alpha,\beta}(s_1, s_2)$ at any integers s_1, s_2 in terms of the values of Hurwitz zeta function.

Theorem 8 For $p, q \in \mathbb{N}$, we have

$$\begin{aligned} H_{\alpha,\beta}(q, p) &= \frac{\Gamma(p+q)}{(p+q-1)\Gamma(p)\Gamma(q)} \\ &\times \left\{ \sum_{n=0}^{p+q-3} \left\{ \sum_{m=\max\{n-p+1, 0\}}^{q-1} (-1)^m \binom{q-1}{m} \binom{p+m-1}{n} \right\} \right\} \end{aligned}$$

$$\begin{aligned}
& \times \frac{n!}{(2-p-q)_n} \zeta(p+q-n-1, \beta)(\alpha-\beta)^{-n-1} \\
& - \sum_{m=0}^{q-2} (-1)^m \binom{q-1}{m} \frac{(p+m-1)!}{(2-p-q)_{p+m-1}} \zeta(q-m, \alpha)(\alpha-\beta)^{-p-m} \\
& + (-1)^{q-1} \frac{(p+q-2)!}{(2-p-q)_{p+q-2}} (\alpha-\beta)^{-p-q+1} \left(\frac{\Gamma'}{\Gamma}(\beta) - \frac{\Gamma'}{\Gamma}(\alpha) \right) \Bigg\}.
\end{aligned}$$

Here empty sum is considered as zero.

Theorem 9 For $p, q \in \mathbb{Z}$ which are not both negative, we have

$$\begin{aligned}
H_{\alpha, \beta}(-p, -q) &= \sum_{k=0}^q \binom{q}{k} (\beta - \alpha)^k \zeta(-p - q + k, \alpha) \\
&+ \sum_{k=0}^p \binom{p}{k} (\alpha - \beta)^k \zeta(-p - q + k, \beta).
\end{aligned}$$

Here empty sum is considered as zero.

Finally we mention that we can provide another approach to evaluate the determinant $\det \Delta_n$ of the Laplacian on the n -sphere S^{n-1} starting from the integral expression (6). Here $\det \Delta_n$ is defined by

$$\det \Delta_n = \exp \left(- \sum_{d=0}^{n-1} T_{n,d} \frac{\partial}{\partial s} H_{1,n}(s-d, s) \Big|_{s=0} \right).$$

See (5) for the definition of $T_{n,d}$.

Theorem 10

$$\begin{aligned}
\frac{\partial}{\partial s} H_{1,n+1}(s-d, s) \Big|_{s=0} &= \zeta'(-d) + \sum_{l=0}^d (-n)^{d-l} \binom{d}{l} \zeta'(-l, n+1) \\
&- \frac{(-n)^{d+1}}{2(d+1)} \left(\sum_{j=1}^d \frac{1}{j} \right).
\end{aligned}$$

This is simpler than Kumagai's formula given in Lemma 3 of [4] p.202. Comparing Theorem 10 and Kumagai's result, we get the following identity for harmonic numbers.

Corollary 5 *The following identity holds:*

$$2^{1-d} \sum_{l=1, \text{odd}}^d \binom{d+1}{l+1} \sum_{j=1, \text{odd}}^l \frac{1}{j} = \sum_{j=1}^d \frac{1}{j}.$$

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