

Laplace–Mellin transform of the non-holomorphic Eisenstein series

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1 Introduction and statement of the results

In this report, we give a Laplace–Mellin transform of the non-holomorphic Eisenstein series and the Fourier coefficients of the Eisenstein series respectively. The main theorem appeared in author's previous paper [5] (2007). We give a new proof by means of Mellin–Barnes type integral formulas in section 4.

Let $k \geq 0$ be an even integer, Let i be the imaginary unit, s be a complex number whose real part σ and imaginary part t . As usual, H is the upper half plane. The non-holomorphic Eisenstein series for $SL_2(\mathbb{Z})$ is defined by

$$E_k(z, s) = y^s \sum_{\{c, d\}} (cz + d)^{-k} |cz + d|^{-2s}. \quad (1)$$

Here z is a point of H , s is a complex variable and the summation is taken over $\left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right)$, a complete system of representation of $\left\{\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \in SL_2(\mathbb{Z})\right\} \setminus SL_2(\mathbb{Z})$. The right-hand side of (1) converges absolutely and locally uniformly on $\{(z, s) \mid z \in H, \operatorname{Re}(s) > 1 - \frac{k}{2}\}$, and $E_k(z, s)$ has a meromorphic continuation to the whole s -plane.

Our main result is as follows:

Theorem 1 ([5] Theorem 1) (I) Assume that $k \geq 4$. Then the Eisenstein series $E_k(z, s)$ is a C^∞ -modular form of weight k , and of bounded growth for $2 - k < \operatorname{Re}(s) < -1$ except on the poles. Further, for $1 - \frac{k}{2} < \operatorname{Re}(s) < -1$ and $n \in \mathbb{Z}_{>0}$,

$$\int_0^1 \int_0^\infty E_k(z, s) \exp(-2\pi i n \bar{z}) y^{k-2} dy dx = 0. \quad (2)$$

(II) Let $E_k(z, s) = \sum_{n=-\infty}^\infty a(n; y, s) e^{2\pi i n x}$ be the Fourier expansion of the Eisenstein series. Then Fourier coefficients of the projection of $E_k(z, s)$ to the space of holomorphic cusp forms are zero, namely,

$$(2\pi n)^{k-1} \Gamma(k-1)^{-1} \int_0^\infty a(n; y, s) e^{-2\pi n y} y^{k-2} dy = 0 \quad (3)$$

for $1 - k < \operatorname{Re}(s) < 0$ and $n \in \mathbb{Z}_{>0}$.

2 Projection to the space of cusp forms

The C^∞ -automorphic forms of bounded growth are introduced by Sturm in the study of zeta-functions of Rankin type.

The function F is called a C^∞ -modular form of weight k , if F satisfies the following conditions:

(A.1) F is a C^∞ -function from H to $\mathbb{C}$,

(A.2) $F((az+b)(cz+d)^{-1}) = (cz+d)^k F(z)$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$.

We denote by $\mathfrak{M}_k$ the set of all C^∞ -modular forms of weight k . The function $F \in \mathfrak{M}_k$ is called of bounded growth if for every $\varepsilon > 0$

$$\int_0^1 \int_0^\infty |F(z)| y^{k-2} e^{-\varepsilon y} dy dx < \infty.$$

Let k be a positive even integer and S_k be the space of cusp forms of weight k on $SL_2(\mathbb{Z})$. For $F \in \mathfrak{M}_k$ and $f \in S_k$, we define the Petersson inner product as usual

$$(f, F) = \int_{SL_2(\mathbb{Z}) \backslash H} f(z) \overline{F(z)} y^{k-2} dx dy.$$

In 1981, Sturm [7] constructed a certain kernel function by using Poincaré series, and showed the following theorem:

Theorem 2 (Sturm 1981) *Assume that $k > 2$. Let $F \in \mathfrak{M}_k$ be of bounded growth with the Fourier expansion $F(z) = \sum_{n=-\infty}^{\infty} a(n, y) e^{2\pi i n x}$. Let*

$$c(n) = (2\pi n)^{k-1} \Gamma(k-1)^{-1} \int_0^\infty a(n, y) e^{-2\pi n y} y^{k-2} dy.$$

Then $h(z) = \sum_{n=1}^{\infty} c(n) e^{2\pi i n z} \in S_k$ and

$$(g, F) = (g, h)$$

for all $g \in S_k$.

3 Fourier expansion of the Eisenstein series

Next, we recall the Fourier expansion and the growth condition of $E_k(z, s)$. Let $e(u) := \exp(2\pi i u)$ for $u \in \mathbb{C}$. For $z \in H$ and $\text{Re}(s) > 1 - \frac{k}{2}$, $E_k(z, s)$ has an expansion:

$$E_k(z, s) = y^s + a_0(s) y^{1-k-s} + \frac{y^s}{\zeta(k+2s)} \sum_{m \neq 0} \sigma_{1-k-2s}(m) a_m(y, s) e(mx), \quad (4)$$

where

$$\begin{aligned} a_0(s) &= (-1)^{\frac{k}{2}} 2\pi \cdot 2^{1-k-2s} \frac{\zeta(k+2s-1) \Gamma(k+2s-1)}{\zeta(k+2s) \Gamma(s) \Gamma(k+s)}, \\ \sigma_s(m) &= \sum_{d|m, d>0} d^s, \end{aligned}$$

and

$$a_m(y, s) = \int_{-\infty}^{\infty} e(-mu) (u+iy)^{-k} |u+iy|^{-2s} du. \quad (5)$$

We call the first two terms of (4) are the constant term of $E(z, s)$. The integral (5) is entire function in s and of exponential decay in $y|m|$. This fact gives the meromorphical continuation of $E_k(z, s)$ to the whole s -plane, and shows that the constant terms represent the y -aspect of $E(z, s)$ when y tends to ∞ . Namely, there exist positive constants A_1 and A_2 depending only on k and s such that

$$|E_k(z, s)| \leq A_1 y^{\operatorname{Re}(s)} + A_2 y^{1-\operatorname{Re}(s)-k} \quad (y \rightarrow \infty), \quad (6)$$

except on the poles. Further, the integral (5) is expressed in terms of special functions:

$$a_m(y, s) = \begin{cases} \frac{(-1)^{\frac{k}{2}} (2\pi)^{k+2s} m^{k+2s-1}}{\Gamma(k+s)} e^{-2\pi y m} \Psi(s, k+2s; 4\pi y m) & (m > 0), \\ \frac{(-1)^{\frac{k}{2}} (2\pi)^{k+2s} |m|^{k+2s-1}}{\Gamma(s)} e^{-2\pi y |m|} \Psi(k+s, k+2s; 4\pi y |m|) & (m < 0). \end{cases} \quad (7)$$

Here $\Psi(\alpha, \beta; z)$ is the confluent hypergeometric function defined for $\operatorname{Re}(z) > 0$ and $\operatorname{Re}(\alpha) > 0$ by the following

$$\Psi(\alpha, \beta; z) := \frac{1}{\Gamma(\alpha)} \int_0^{\infty} e^{-zu} u^{\alpha-1} (1+u)^{\beta-\alpha-1} du,$$

(See for example [4] §7.2.) Then we have

Proposition 1 *Assume $E_k(z, s)$ is holomorphic at $s \in \mathbb{C}$. Then, there exist positive constants A_1 and A_2 depending only on k and s such that*

$$|E_k(x+iy, s)| \leq \begin{cases} A_1 (y^{-\operatorname{Re}(s)-k} + y^{\operatorname{Re}(s)}) & (\operatorname{Re}(s) > \frac{1-k}{2}) \\ A_2 (y^{-1+\operatorname{Re}(s)} + y^{1-\operatorname{Re}(s)-k}) & (\operatorname{Re}(s) \leq \frac{1-k}{2}) \end{cases}$$

for every $y > 0$.

Remark 1. The integral (5) plays a fundamental role in the study of automorphic forms. The initial work is due to Hecke [2]. His approach is explicated in

Schoenberg's book [6] (pp. 63-68). The representation (7) was originally investigated by Maass [3] (pp. 209-211). He used the Whittaker function to express the integral (7).

Remark 2. The estimation (6) is well-known, and Proposition 1 is the consequence of (6) and modularity of $y^{\frac{k}{2}}E_k(z, s)$.

4 Proof of Theorem 1

The orthogonality of $E_k(z, s)$ and cusp forms gives the equation (2) under the convergence of the integral. Here Proposition 1 is used, however, we omit the detail of the proof of Theorem 1 (I) (see [5]). In this section, we give a new proof of Theorem 1 (II) by using the **Mellin-Barnes integral**. The following proposition is crucial to evaluate the integral in (3). (See [1], Section 6.5.)

Proposition 2 (*Mellin-Barnes integral*) $\Psi(\alpha, \beta; z)$ has an integral representation

$$\Psi(\alpha, \beta; z) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(\alpha + w)\Gamma(-w)\Gamma(1 - \beta - w)}{\Gamma(\alpha)\Gamma(\alpha - \beta + 1)} z^w dw,$$

where $|\arg(z)| \leq \frac{3}{2}\pi$, $\alpha \notin \mathbb{Z}_{\leq 0}$, $\alpha - \beta + 1 \notin \mathbb{Z}_{\leq 0}$ and the path of integration is indented so as to separate the poles of $\Gamma(\alpha + w)$ and $\Gamma(-w)\Gamma(1 - \beta - w)$.

Proof of Theorem 1 (II) By Proposition 2, we have

$$\begin{aligned} & \int_0^\infty \Psi(s, k + 2s; 4\pi ny) e^{-4\pi ny} y^{k+s-2} dy \\ &= \frac{1}{2\pi i \Gamma(s)\Gamma(1 - k - s)} \int_0^\infty \int_L \Gamma(s + w)\Gamma(-w)\Gamma(1 - k - 2s - w) \\ & \quad \times (4\pi n)^w e^{-4\pi ny} y^{w+k+s-2} dw dy \end{aligned}$$

for $s \notin \mathbb{Z}$. Here the path of integration in w is denoted by L taken from $-i\infty$ to $i\infty$ so as to separate the poles of $\Gamma(s + w)$ and $\Gamma(-w)\Gamma(1 - k - 2s - w)$. Then the interchange of the order of integration (11) is justified by Fubini's theorem in the region $1 - k < \operatorname{Re}(s) < 0$.

We also employ Barnes' lemma:

$$\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \Gamma(\alpha + w)\Gamma(\beta + w)\Gamma(\gamma - w)\Gamma(\delta - w) dw = \frac{\Gamma(\alpha + \gamma)\Gamma(\alpha + \delta)\Gamma(\beta + \gamma)\Gamma(\beta + \delta)}{\Gamma(\alpha + \beta + \gamma + \delta)},$$

where the path of integration is taken so as to separate the poles of $\Gamma(\alpha + w)\Gamma(\beta + w)$ and $\Gamma(\gamma - w)\Gamma(\delta - w)$. (See [8] Section 14.52.) By the above lemma, we have

$$\begin{aligned} & \int_0^\infty \Psi(s, k + 2s; 4\pi ny) e^{-4\pi ny} y^{k+s-2} dy \\ &= \frac{(4\pi n)^{1-k-s}}{2\pi i} \Gamma(-s) \Gamma(-1 + k + s) \cdot \Gamma(0)^{-1} \\ &= 0, \end{aligned}$$

for $1 - k < \operatorname{Re}(s) < 0$ and $s \notin \mathbb{Z}$.

If α is zero or a negative integer, $\Psi(\alpha, \beta; z)$ is a polynomial in z , which is called the (generalized) **Laguerre polynomial**. The Laguerre polynomials $L_n^a(x)$ for $n \in \mathbb{Z}_{\geq 0}$ are defined by

$$L_n^a(x) = \sum_{m=0}^n \binom{n+a}{n-m} \frac{(-x)^m}{m!} = \frac{(-1)^n}{n!} \Psi(-n, a+1; x),$$

which are known as the orthogonal polynomials associated with the scalar product

$$(\varphi_1, \varphi_2) := \int_0^\infty \varphi_1(x) \varphi_2(x) e^{-x} x^a dx$$

for $a > -1$. Especially, for positive integer n ,

$$(L_n^a, L_0^a) = \int_0^\infty L_n^a(x) e^{-x} x^a dx = 0. \quad (8)$$

By (11), we have for $s \in \mathbb{Z}_{\leq 0}$,

$$\begin{aligned} & \int_0^\infty \Psi(s, k + 2s; 4\pi ny) e^{-4\pi ny} y^{k+s-2} dy \\ &= (-s)! (-1)^s \int_0^\infty L_{-s}^{k+2s-1}(4\pi ny) e^{-4\pi ny} y^{k+s-2} dy. \end{aligned} \quad (9)$$

Applying the relation $L_n^{a-1}(x) = L_n^a(x) - L_{n-1}^a(x)$ for $(-s-1)$ -times, we resolve (10) into (9) and complete the proof of Theorem 1 (II). $\square$

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