

An error estimation of the reconstruction algorithm in computed tomography

Taizo Muroya (室屋 泰三)

Takao Hanada (花田 孝郎)

Jiro Watanabe (渡辺 二郎)

University of Electro-Communications (電気通信大学)

1 Introduction

1.1 Radon transformation

Let $f = f(x, y)$ be a piecewise continuous function on the plane with compact support, e.g., characteristic function supported on plane figures circumscribed by square, circle or asteroid. For any line $L: x \cos \theta + y \sin \theta = \xi$, let

$$\varphi(\theta, \xi) = \int_{-\infty}^{\infty} f(\xi \cos \theta + s \sin \theta, \xi \sin \theta - s \cos \theta) ds \quad (1.1)$$

where s is length measured along L . This function φ is the Radon transform of f . Let us write

$$\psi(\xi; x, y) = \frac{1}{2\pi} \int_0^{2\pi} \varphi(\theta, \xi + x \cos \theta + y \sin \theta) d\theta.$$

J. Radon [1],[2] gave the following inversion formula:

$$f(x, y) = -\frac{1}{\pi} \int_0^{\infty} \frac{\psi(\xi; x, y) - \psi(0; x, y)}{\xi^2} d\xi.$$

1.2 Approximation of the singular integral

Our problem is to make a good approximation to the singular integral

$$T(\psi) = \int_0^{\infty} \frac{\psi(\xi) - \psi(0)}{\xi^2} d\xi \quad (1.2)$$

where $\psi(\xi) = \psi(\xi; x, y)$.

Now, for step size $\delta > 0$, let us take

$$x_i = (i - 1/2)\delta, \quad I_i = [x_i, x_{i+1}), \quad i = 0, 1, 2, \dots$$

and we set

$$\psi_\delta(\xi) = \psi(i\delta), \quad \xi \in I_i, \quad i = 0, 1, 2, \dots$$

We adopt $T(\psi_\delta)$ as an approximation of the singular integral (1.2). It is easily seen that

$$T(\psi_\delta) = \frac{1}{\delta} \left\{ \sum_{i=1}^{\infty} \frac{\psi(i\delta)}{i^2 - 1/4} - 2\psi(0) \right\}$$

holds.

Our problem is to make a numerical integration formula for the singular integral $T(\psi)$. Moreover, we will investigate the order of accuracy of our integration formula when the function f is piecewise continuous.

2 Behavior of $\psi(\xi)$ near $\xi = 0$

2.1 Analytical forms of $\psi(\xi)$ near $\xi = 0$

Let us assume, for simplicity, $x = y = 0$ in (1.1).

For f , we set

$$m(r) = \frac{1}{2\pi} \int_0^{2\pi} f(r \cos \theta, r \sin \theta) d\theta, \quad r > 0.$$

Then it follows easily from (1.1) that

$$\psi(\xi) = 2 \int_\xi^\infty \frac{r \cdot m(r)}{\sqrt{r^2 - \xi^2}} dr$$

holds.

Let us treat the case where $m(r)$ is in the following functional form :

$$m(r) = \begin{cases} ar^{\alpha-1} + b, & 0 < r < R \\ 0, & r > R. \end{cases} \quad (2.1)$$

where $\alpha > 0$, $a \neq 0, b$ and $R > 0$ are constants. Let $\psi_\alpha(\xi)$ be the function $\psi(\xi)$ corresponding to the above $m(r)$ with α .

$$\begin{aligned} \psi_\alpha(\xi) &= \frac{a}{\pi} \int_\xi^R \frac{r^\alpha}{\sqrt{r^2 - \xi^2}} dr \quad (r = s\xi) \\ &= \frac{a\xi^\alpha}{\pi} \int_1^{R/\xi} \frac{s^\alpha}{\sqrt{s^2 - 1}} ds \\ &= \frac{a\xi^\alpha}{\pi} I_\alpha(\xi), \end{aligned} \quad (2.2)$$

where

$$I_\alpha = \int_1^{R/\xi} \frac{s^\alpha}{\sqrt{s^2 - 1}} ds. \quad (2.3)$$

Lemma 1 When $\alpha \neq 0$,

$$I_\alpha = \frac{1}{\alpha} \left(\frac{R}{\xi} \right)^\alpha \sqrt{1 - \left(\frac{\xi}{R} \right)^2} + \frac{\alpha - 1}{\alpha} I_{\alpha-2}(\xi).$$

Proof.

$$\frac{d}{ds} \left(\frac{1}{\alpha} \sqrt{s^{2\alpha} - s^{2\alpha-2}} \right) = \frac{s^\alpha}{\sqrt{s^2 - 1}} - \frac{\alpha - 1}{\alpha} \frac{s^{\alpha-2}}{\sqrt{s^2 - 1}}.$$

||

Lemma 2 When $\alpha < 0$,

$$I_\alpha(\xi) = \frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma(-\alpha/2)}{\Gamma((1-\alpha)/2)} + \frac{1}{\alpha} \left(\frac{\xi}{R} \right)^{-\alpha} F(1/2, -\alpha/2; 1 - \alpha/2; (\xi/R)^2). \quad (2.4)$$

Proof. By (2.3) with change of variables:

$$s = \frac{1}{\sqrt{u}}, \quad ds = -\frac{du}{2u^{3/2}}$$

we get

$$\begin{aligned} I_\alpha(\xi) &= \frac{1}{2} \int_{(\xi/R)^2}^1 u^{-1-\alpha/2} (1-u)^{-1/2} du \\ &= \frac{1}{2} \left(\int_0^1 u^{-1-\alpha/2} (1-u)^{-1/2} du - \int_0^{(\xi/R)^2} u^{-1-\alpha/2} (1-u)^{-1/2} du \right). \end{aligned} \quad (2.5)$$

For the first integral on the right-hand side of (2.5) we have

$$\int_0^1 u^{-1-\alpha/2} (1-u)^{-1/2} du = B(-\alpha/2, 1/2) = \frac{\sqrt{\pi} \Gamma(-\alpha/2)}{\Gamma(\frac{1-\alpha}{2})}. \quad (2.6)$$

For the second integral we have

$$\int_0^{(\xi/R)^2} u^{-1-\alpha/2} (1-u)^{-1/2} du = \left(\frac{\xi}{R} \right)^{-\alpha} \int_0^1 v^{-1-\alpha/2} (1 - (\xi/R)^2 v)^{-1/2} dv,$$

and using Euler's integral representation of hypergeometric function [3], when $\alpha < 0$, we obtain

$$\int_0^1 v^{-1-\alpha/2} (1 - (\xi/R)^2 v)^{-1/2} dv = -\frac{2}{\alpha} F(1/2, -\alpha/2; 1 - \alpha/2; (\xi/R)^2).$$

Consequently we get (2.4) from (2.5) – (2.6) and the above expressions.

When $\alpha = 0$

$$I_0(\xi) = \int_1^{R/\xi} \frac{ds}{\sqrt{s^2 - 1}} = \log \left[\frac{R}{\xi} \left(1 + \sqrt{1 - (\xi/R)^2} \right) \right] \quad (2.7)$$

holds.

Now we can write ψ_α in explicit forms.

When $0 < \alpha < 2$ we have by (2.2) and Lemma 1

$$\psi_\alpha(\xi) = \frac{aR^\alpha}{\pi\alpha} \sqrt{1 - (\xi/R)^2} + \frac{a(\alpha - 1)}{\pi\alpha} \xi^\alpha I_{\alpha-2}(\xi),$$

and applying Lemma 2 to $I_{\alpha-2}$,

$$\begin{aligned} \psi_\alpha(\xi) &= \frac{aR^\alpha}{\pi\alpha} \sqrt{1 - (\xi/R)^2} + \frac{a(\alpha - 1)}{2\sqrt{\pi}\alpha} \cdot \frac{\Gamma((2 - \alpha)/2)}{\Gamma((3 - \alpha)/2)} \cdot \xi^\alpha \\ &+ \frac{a(\alpha - 1)}{\pi\alpha(\alpha - 2)} \cdot R^\alpha \cdot \left(\frac{\xi}{R} \right)^2 \cdot F(1/2, 1 - \alpha/2; 2 - \alpha/2; (\xi/R)^2). \end{aligned}$$

Particularly for $\alpha = 1$ we have

$$\psi_1(\xi) = \frac{aR}{\pi} \sqrt{1 - (\xi/R)^2}.$$

For $\alpha = 2$ we have by (2.2), Lemma 1 and (2.7).

$$\begin{aligned} \psi_2(\xi) &= \frac{aR^2}{2\pi} \sqrt{1 - (\xi/R)^2} + \frac{a\xi^2}{2\pi} I_0(\xi) \\ &= \frac{aR^2}{2\pi} \sqrt{1 - (\xi/R)^2} + \frac{a\xi^2}{2\pi} \log \left[\frac{\xi}{R} \left(1 + \sqrt{1 - (\xi/R)^2} \right) \right] \end{aligned}$$

Similarly when $2 < \alpha < 4$

$$\begin{aligned} \psi_\alpha(\xi) &= \frac{aR^\alpha}{\pi\alpha} \sqrt{1 - (\xi/R)^2} \left(1 + \frac{\alpha - 1}{\alpha - 2} \left(\frac{\xi}{R} \right)^2 \right) \\ &+ \frac{a(\alpha - 1)(\alpha - 3)}{\pi\alpha(\alpha - 2)} \cdot \left[\frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma((4 - \alpha)/2)}{\Gamma((5 - \alpha)/2)} \xi^\alpha \right. \\ &\left. + \frac{R^\alpha}{\alpha - 4} \left(\frac{\xi}{R} \right)^4 \cdot F(1/2, 2 - \alpha/2; 3 - \alpha/2; (\xi/R)^2) \right] \end{aligned}$$

holds. When $\alpha = 4$

$$\psi_4(\xi) = \frac{aR^4}{4\pi} \sqrt{1 - (\xi/R)^2} \left(1 + \frac{3}{2} \left(\frac{\xi}{R} \right)^2 \right) + \frac{aR^4}{\pi} \cdot \frac{3}{4 \cdot 2} \cdot \log \left[\frac{\xi}{R} \left(1 + \sqrt{1 - (\xi/R)^2} \right) \right]$$

holds.

Thus we obtain the following functional forms of ψ_α for $\alpha > 0$:

$$\psi_\alpha(\xi) = \begin{cases} \text{const} \cdot \xi^\alpha + (\text{power series of } \xi^2) & \text{when } \alpha \neq \text{integer,} \\ (\text{power series of } \xi^2) & \text{when } \alpha \text{ is an odd integer,} \\ \text{const} \cdot \xi^\alpha \log \xi + (\text{power series of } \xi^2) & \text{when } \alpha \text{ is an even integer.} \end{cases}$$

2.2 Examples of α

Let us consider, as functions f to be reconstructed, the characteristic functions of the figures of square, disk and asteroid (see Figure 1.) When we take the dots in Figure 1 as reconstruction points, the corresponding α 's are shown in Table 1.

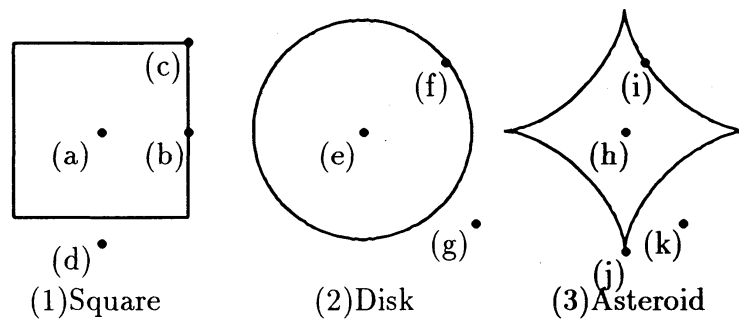


Figure 1: Examples of reconstruction points

Table 1: The values of α for reconstruction points.

figure	point	α
Square	(a) interior	1
	(b) edge	1
	(c) corner	1
	(d) exterior	1
Disk	(e) interior	1
	(f) edge	2
	(g) exterior	1
Asteroid	(h) interior	1
	(i) edge	2
	(j) cusp	3/2
	(k) exterior	1

For the above reconstruction points, we have

$$\frac{\psi(\xi) - \psi(0)}{\xi^2} = \begin{cases} (\text{power series of } \xi^2), & (\alpha = 1) \\ C \log \xi + (\text{power series of } \xi^2), & (\alpha = 2) \\ C\xi^{\alpha-2} + (\text{power series of } \xi^2), & (1 < \alpha < 2) \end{cases} \quad (2.8)$$

for $\xi > 0$ near the origin, where C is constant.

2.3 Examples of ψ

Let the reconstruction point be the origin $(0, 0)$. In case of square, $\psi(\xi)$ is given by

$$\psi(\xi) = \begin{cases} 8\left\{\frac{l}{2} \log \left| \frac{\tan(\theta_0/2 + \pi/4)}{\tan(\theta_0/2)} \right| + \xi \log |\tan \theta_0|\right\}, & |\xi| \leq l/\sqrt{2} \\ 0, & |\xi| > l/\sqrt{2} \end{cases}$$

where $\theta_0 = \cos^{-1}(\sqrt{2}\xi/l) - \pi/4$, l is length of a side of the square.

In case of disk, $\psi(\xi)$ is given by

$$\psi(\xi) = \begin{cases} 4\pi\sqrt{r^2 - \xi^2}, & |\xi| \leq r \\ 0, & |\xi| > r. \end{cases}$$

where r is the radius of the disk.

3 On the accuracy of the numerical integration of the inverse Radon transformation

Let us set

$$\begin{aligned} a(\xi) &= \frac{\psi(\xi) - \psi(0)}{\xi^2}, & \chi_\delta(\xi) &= \begin{cases} \frac{1}{x_i x_{i+1}}, & \xi \in I_i (i = 1, 2, \dots), \\ 0, & \xi \in I_0, \end{cases} \\ \bar{\psi}_\delta(\xi) &= \frac{1}{\delta} \int_{I_i} \psi(t) dt, & \xi \in I_i, & I_\delta = \int_0^1 [\chi_\delta(\xi) - \frac{1}{\xi^2}] \cdot [\psi(\xi) - \psi(0)] d\xi, \quad \text{and} \\ \bar{\psi}_\delta(\xi) &= \psi(0), & \xi \in I_0, & J_\delta = \int_1^\infty [\chi_\delta(\xi) - \frac{1}{\xi^2}] \cdot [\psi(\xi) - \psi(0)] d\xi. \\ T(\bar{\psi}_\delta) &= \int_\delta^\infty \frac{\bar{\psi}_\delta(\xi) - \psi(0)}{\xi^2} d\xi, \end{aligned}$$

Then, $T(\bar{\psi}_\delta) - T(\psi) = I_\delta + J_\delta$.

$$(A) \quad |\chi_\delta(\xi)| \leq \frac{3}{\xi^2}, \quad \xi > 0.$$

Proof. If $\xi = (i + \alpha)\delta$, $\alpha \leq \frac{1}{2}$ then

$$\delta^2 \cdot \chi_\delta(\xi) = \frac{1}{i^2 - 1/4} \leq \frac{3}{(i + \alpha)^2} = \frac{3\delta^2}{\xi^2}, \quad i \geq 1. \quad \square$$

(B) When $\xi \in I_i$ ($i \geq 1$) we have $|\xi^2 \chi_\delta(\xi) - 1| \leq \frac{1}{i - 1/2}$.

Proof. If $\xi = (i + \alpha)\delta \in I_i$ then

$$\begin{aligned} \xi^2 \chi_\delta(\xi) - 1 &= \frac{\xi^2}{x_i x_{i+1}} - 1 = \frac{(i + \alpha)^2}{i^2 - 1/4} - 1 \\ &= \frac{2\alpha i + \alpha^2 + 1/4}{i^2 - 1/4} \\ &= \frac{i + 1/2}{i^2 - 1/4} = \frac{1}{i - 1/2}. \end{aligned}$$

(C) When $x_i \leq \xi < x_{i+1}$,

$$|\chi_\delta(\xi) - \frac{1}{\xi^2}| \leq \frac{\delta}{x_i^3} \quad (i \geq 1).$$

Proof. Let $x_i \leq \xi < x_{i+1}$, in view of (B),

$$|\chi_\delta(\xi) - \frac{1}{\xi^2}| \leq \frac{1}{(i - 1/2)\xi^2} \leq \frac{1}{(i - 1/2)x_i^2} = \frac{\delta}{x_i^3}$$

(D) Let $1 \leq p' < \infty$. Then there exists a constant $C_{p'}$ independent of δ ,

$$\int_0^1 |\xi^2 \chi_\delta(\xi) - 1|^{p'} d\xi \leq \begin{cases} C_1 \cdot \delta \log \frac{1}{\delta}, & p' = 1 \\ C_{p'} \cdot \delta, & 1 < p' < \infty \end{cases}$$

Proof. Let N be the maximum of integers which are less than $\frac{1}{\delta} + \frac{1}{2}$.

$$\begin{aligned} \int_0^1 |\xi^2 \chi_\delta(\xi) - 1|^{p'} d\xi &\leq \frac{\delta}{2} + \sum_{i=1}^{1/\delta-1} \int_{x_i}^{x_{i+1}} |\xi^2 \chi_\delta(\xi) - 1|^{p'} d\xi \quad (\text{from (B)}) \\ &\leq \frac{\delta}{2} + \delta \sum_{i=1}^N \frac{1}{(i - 1/2)^{p'}}, \end{aligned}$$

from which we can easily obtain the desired result.

(E) Assume $a(\xi) \in L^p(0, 1)$, $1 < p \leq \infty$. Then

$$|I_\delta| \leq \begin{cases} C_1 \cdot \delta \log \frac{1}{\delta} \cdot \|a\|_{L^\infty(0,1)}, & p = \infty \\ C_{p'}^{1/p'} \cdot \delta^{1/p'} \cdot \|a\|_{L^p(0,1)}, & 1 < p < \infty \end{cases}$$

Proof.

$$\begin{aligned} |I_\delta| &= \left| \int_0^1 (\xi^2 \chi_\delta(\xi) - 1) a(\xi) d\xi \right| \\ &\leq \left(\int_0^1 |\xi^2 \chi_\delta(\xi) - 1|^{p'} d\xi \right)^{1/p'} \cdot \|a\|_{L^p(0,1)} \end{aligned}$$

Then (E) follows from (D) and the above inequality. $\parallel$

(F) When $\psi \in L^1(1, \infty)$ and $(i_0 - 1/2)\delta = 1$ for some integer i_0 , then we have

$$|J_\delta| \leq \delta \cdot \|\psi\|_{L^1(1, \infty)}.$$

Proof.

$$\begin{aligned} |J_\delta| &= \left| \int_1^\infty \left[\chi_\delta(\xi) - \frac{1}{\xi^2} \right] \cdot [\psi(\xi) - \psi(0)] d\xi \right| \\ &= \left| \int_1^\infty \left[\chi_\delta(\xi) - \frac{1}{\xi^2} \right] \psi(\xi) d\xi \right| \\ &\leq \delta \cdot \int_1^\infty |\psi(\xi)| d\xi \quad (\text{by (C)}). \end{aligned}$$

$\parallel$

We summarize:

Theorem 1 When $a(\xi) \in L^p(0, 1)$, $\psi(\xi) \in L^1(1, \infty)$, and $\frac{1}{p} + \frac{1}{p'} = 1$ ($1 < p \leq \infty$), we have

$$|T(\bar{\psi}_\delta) - T(\psi)| \leq \begin{cases} C\delta(\log \frac{1}{\delta} \cdot \|a\|_{L^\infty(0,1)} + \|\psi\|_{L^1(1,\infty)}), & p = \infty \\ C\delta^{1/p'}(\|a\|_{L^p(0,1)} + \delta^{1/p}\|\psi\|_{L^1(1,\infty)}), & 1 < p < \infty \end{cases} \quad (3.1)$$

where C is a constant independent of ψ and δ .

When $p = 1$ we have

Theorem 2 When $a(\xi) \in L^1(0, 1)$, $\psi(\xi) \in L^1(1, \infty)$,

$$T(\bar{\psi}_\delta) \rightarrow T(\psi) \quad (\delta \rightarrow 0).$$

Next we will estimate $T(\bar{\psi}_\delta - \psi_\delta)$.

For $\delta > 0$, we set $\xi_0 = i_0\delta$, $0 < \xi_0 < \xi_\infty$, (i_0 is integer).

We suppose that $\psi \in C^2[0, \xi_\infty)$ satisfies $\psi'(0) = 0$ and

$$\psi''(\xi_\infty - \eta) = a(\eta)\eta^\alpha, \quad 0 < \eta < \xi_\infty$$

where α is a negative constant, $a \in C^0[0, \xi_\infty]$. We set $\psi(\xi) = \psi(-\xi)$, $\xi < 0$, if necessary.

Then we have

Lemma 3.

$$T(\bar{\psi}_\delta - \psi_\delta) = O(\delta)$$

holds, when $\alpha \geq -2$.

Proof. For δ let

$$\begin{aligned} I_i &= ((i - 1/2)\delta, (i + 1/2)\delta), \\ i_\infty &= \max\{i : (i + 1/2)\delta \leq \xi_\infty\}. \end{aligned}$$

Using Taylor's theorem, we can get

$$\psi(i\delta + \eta) = \psi(i\delta) + \eta\psi'(i\delta) + \frac{\eta^2}{2}\psi''(i\delta + \theta)$$

where $0 < \theta < 1$. Thereby

$$\begin{aligned} \bar{\psi}_\delta(i\delta) &= \frac{1}{\delta} \int_{I_i} \psi(\xi) d\xi \\ &= \psi(i\delta) + \frac{\delta^2}{24}\psi''(i\delta + \theta), \quad |\theta| < \frac{1}{2}. \end{aligned}$$

Let T_0, T_1 be defined by the relations

$$\begin{aligned} T(\bar{\psi}_\delta - \psi_\delta) &= T_0 + T_1, \\ T_0 &= \frac{1}{\delta} \sum_{i=1}^{i_0} \frac{\bar{\psi}(i\delta) - \psi(i\delta)}{i^2 - 1/4} - \frac{2}{\delta} [\bar{\psi}(0) - \psi(0)], \\ T_1 &= \frac{1}{\delta} \sum_{i=i_0+1}^{i_\infty-1} \frac{\bar{\psi}(i\delta) - \psi(i\delta)}{i^2 - 1/4}. \end{aligned}$$

(i) Estimate of T_1

First we have

$$T_1 = \frac{\delta}{24} \sum_{i=i_0+1}^{i_\infty-1} \frac{\psi''(i\delta + \theta\delta)}{i^2 - 1/4}$$

and, when $i \leq i_\infty - 1$,

$$\begin{aligned} |\psi''(i\delta + \theta\delta)| &\leq \|a\|_\infty |\xi_\infty - i\delta - \theta\delta|^\alpha \\ &\leq \|a\|_\infty |\xi_\infty - \frac{\delta}{2} - i\delta|^\alpha \end{aligned}$$

therefore

$$\begin{aligned} |T_1| &\leq \frac{\delta}{24} \|a\|_\infty \sum_{i=i_0+1}^{i_\infty-1} \frac{|\xi_\infty - \delta/2 - i\delta|^\alpha}{i^2 - 1/4} \\ &\leq \frac{\delta}{12} \|a\|_\infty \sum_{i=i_0+1}^{i_\infty-1} \frac{|\xi_\infty - \delta/2 - i\delta|^\alpha}{i^2} \\ &= \frac{\delta^{3+\alpha}}{12\xi_0^2} \|a\|_\infty \sum_{i=i_0+1}^{i_\infty-1} |\xi_\infty - \delta/2 - i\delta|^\alpha. \end{aligned}$$

If we set

$$A = \frac{\xi_\infty}{\delta} + \frac{1}{2} - i_\infty, B = \frac{\xi_\infty}{\delta} - \frac{3}{2} - i_0,$$

then we have

$$\sum_{i=i_0+1}^{i_\infty-1} \left| \frac{\xi_\infty}{\delta} - \frac{1}{2} - i \right|^\alpha \leq \int_A^B x^\alpha dx + A^\alpha.$$

Hence we obtain

$$|T_1| \leq \frac{\delta^{3+\alpha}}{12\xi_0} \|a\|_\infty \left[\frac{1}{1+\alpha} (B^{1+\alpha} - A^{1+\alpha}) + A^\alpha \right]$$

holds when $\alpha \neq -1$. Since

$$\delta A = \xi_\infty + \frac{\delta}{2} - i_\infty \delta \sim \text{const} \cdot \delta, \quad (3.2)$$

$$\delta B = \xi_\infty - \frac{3}{2}\delta - \xi_0 \sim \xi_\infty - \xi_0, \quad (3.3)$$

we have

$$\delta^{1+\alpha} \left[\frac{1}{1+\alpha} (B^{1+\alpha} - A^{1+\alpha}) + A^\alpha \right] \sim \begin{cases} \text{const} & (1+\alpha > 0) \\ \text{const} \cdot \delta^{1+\alpha} & (1+\alpha < 0) \end{cases}$$

Accordingly

$$T_1 = \begin{cases} O(\delta^2) & (1+\alpha > 0) \\ O(\delta^{3+\alpha}) & (1+\alpha < 0) \end{cases}.$$

When $\alpha = 1$ we have

$$|T_1| \leq \frac{\delta^2}{12\xi_0^2} \|a\|_\infty \left[\log \frac{B}{A} + \frac{1}{A} \right].$$

Also using (3.2) and (3.3) we can show that

$$T_1 = O(\delta^2 \log \frac{1}{\delta}).$$

(ii) Estimate of T_0

$$\begin{aligned} \frac{1}{\delta} \sum_{i=1}^{i_0} \frac{|\bar{\psi}(i\delta) - \psi(i\delta)|}{i^2 - 1/4} &\leq \frac{1}{12\delta} \sum_{i=1}^{i_0} \frac{|\bar{\psi}(i\delta) - \psi(i\delta)|}{i^2} \\ &\leq \frac{\delta}{12} \cdot \|\psi''\|_{L^\infty} \frac{\pi^2}{6} \end{aligned}$$

and

$$\frac{2}{\delta} [\bar{\psi}(0) - \psi(0)] = \frac{\delta}{12} \psi''(\theta\delta) = O(\delta)$$

therefore

$$T_0 = O(\delta).$$

Hence we have

$$T(\bar{\psi}_\delta - \psi_\delta) = O(\delta).$$

We note that the statements of Theorems 1 and 2 are valid if $\bar{\psi}_\delta$ is replaced by ψ_δ , when ψ satisfies conditions of Lemma 3 and when $\alpha \geq -2$.

4 Results of numerical experiments

On our algorithm, the reconstruction was performed for characteristic functions f of figures of square, disk or asteroid.

For each reconstruction point, the absolute error of reconstruction is proportional to δ^k when δ is in a certain interval. We show graphs of absolute error in our numerical experiments. Here the abscissa is the logarithm of δ and the ordinate is the logarithm of absolute error. (See Figure 2 – Figure 4.)

In the following table (Table 2) k and k' denote the orders of the accuracy of reconstruction obtained in the numerical experiments and theoretically in the right-hand side of the inequality (3.1), respectively. To get k' we have calculated the number α in (2.1) and used (2.8) in each case. We note that the order k' is obtained without regard to the differentiability of ψ . Therefore we can only expect that $k \geq k'$ holds.

Table 2: Results of numerical experiments.

f	reconstruction point		k	k'	α
square	center	(a)	3	1	1
	edge point	(b)	3	1	1
	corner	(c)	3	1	1
	exterior point	(d)	3	1	1
circle	center	(e)	1.5	1	1
	edge point	(f)	1	$k' < 1$	2
	exterior point	(g)	2	1	1
asteroid	center	(h)	2	1	1
	edge point	(i)	1	$k' < 1$	2
	cusp	(j)	0.5	$k' < 0.5$	3/2
	exterior point	(k)	2	1	1

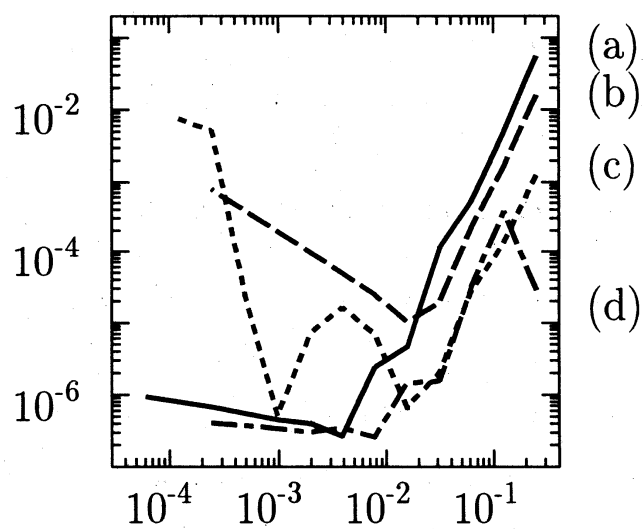


Figure 2:Square

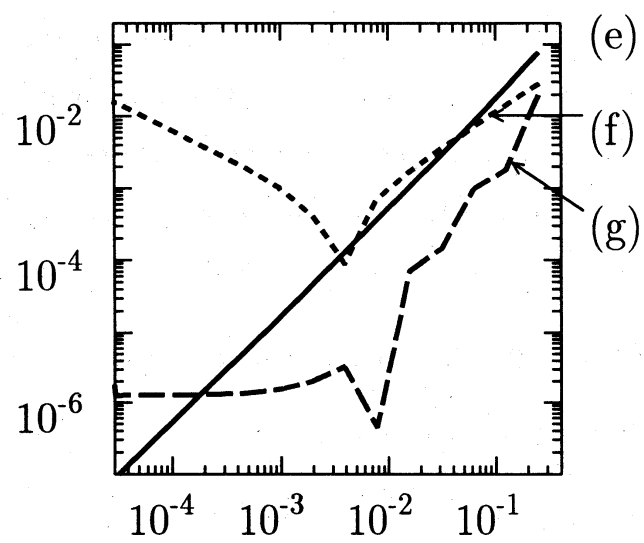


Figure 3:Disk

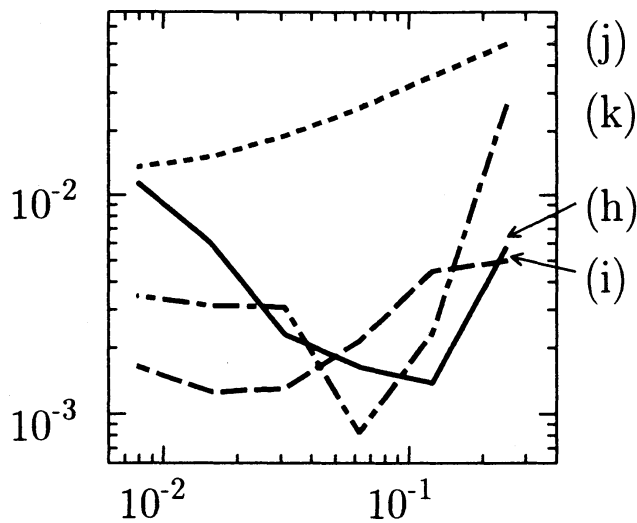


Figure 4:Asteroid

Finally let us write a summary of our results.

- (i) We have tried to make a numerical reconstruction formula for piecewise continuous functions. We have adopted a kind of rectangular rule for the inverse Radon transformation.
- (ii) In numerical experiments we have obtained orders of the accuracy of reconstruction which are greater than or equal to orders assured by the L^p error estimates.
- (iii) We have order $k = 0.5$ for the cusp of the asteroid which is the most difficult to reconstruct in our experiments, and we have order $2 \sim 3$ for ordinary continuous points.

References

- [1] J. Radon, *Über die Bestimmung von Funktionen durch ihre Integralwerte längs gewissen Mannigfaltigkeiten*, Berichte Sächsische Akademie der Wissenschaften 69, 262–277, 1917.
- [2] F. Natterer, *The Mathematics of Computerized Tomography*, John Wiley & Sons 1986.
- [3] E. T. Copson, *An Introduction to the Theory of Functions of a Complex Variable*, Oxford Univ. Press, 1935.