

A Non-Zero-Sum Repeated Game — Criminal vs Police

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Abstract In this paper a conflict between a potential criminal offender and a law-enforcement authorities is studied. The model is a non-zero-sum n -period game with perfect information, where each player has to "act" at most a permitted times during the n periods. We formulate the game by dynamic programming and derive equilibria of the four two-period games each in an explicit form depending on the parameter values of the game. It is shown that in equilibrium the offender is more pushed to making a crime, and the defender invests more effort in law-enforcement, both in the second period than in the first. The fact that the expected payoff to the offender is non-decreasing as his illegal income coming from an unpunished crime increases, but the expected payoff to the defender is not necessarily non-increasing, is also established.

1. Formulation of the Multistage Game — Criminal vs Police.

The game is played as a repeated game over n periods between a potential criminal offender (hereafter called a criminal, or player I) and a law-enforcement authorities (hereafter called police, or player II). Being a repeated game implies that the fundamentals of the game are the same in each period. There are two pure strategies available in each period to player I: to commit a crime (C) and to act honestly (H). Similarly, player II has two pure strategies: to enforce the law (E) or to do nothing (N). If player I chooses H he earns his legal income $r > 0$ (dollars). If he chooses C, illegal income in amount of $\pi > 0$, in addition to his legal income r , may be earned. However if I's crime is detected and arrested by II, I is punished by having to pay a fine in amount of $f > 0$, and imprisoned until the end of the game. When caught in prison, I earns no income at all, of course.

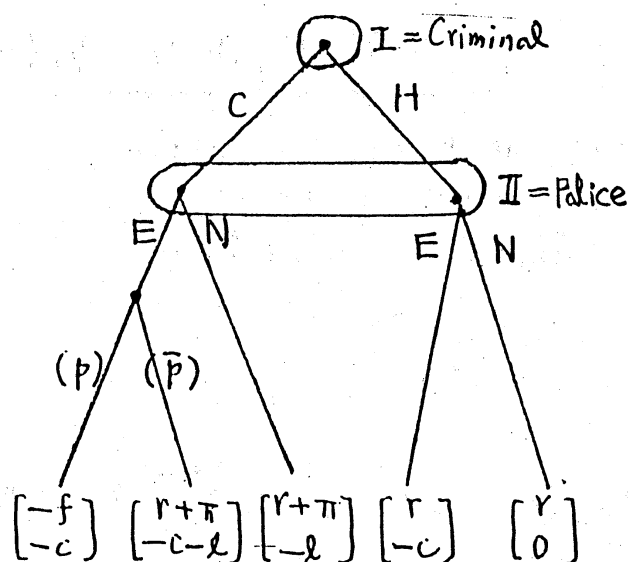
If player II chooses E, with a cost of $c > 0$ (dollars), he can (cannot) catch I's crime with probability p ($\bar{p} = 1 - p$). In case that I commits crime that goes unpunished, a loss of $l > 0$ is inflicted upon society.

So a single stage of this game has the game tree as shown by Figure 1, and is represented by a bimatrix game with payoff bimatrix (1).

We assume that $c < p\pi$ i.e. the strategy E for player II has a positive merit of choosing. This condition is very important as is seen in the proofs of the subsequent theorems.

We shall discuss the n -stage game, where player I wants to commit crime at most k of n periods, and player II attempts to prevent I's illegal act by taking enforcement action at most m times during n periods. After each period is over, the outcome in that period becomes known to both players. The total payoff during n periods is the

sum of the payoffs on each period. We assume that all of the above information is known to both players.



C: commit a crime
 H: be honest
 E: law-enforcement
 N: do nothing
 p: prob. of being punished
 f: amount of fine
 $r(\pi)$: legal (illegal) income to I (if unpunished)
 c: II's cost of law-enforcement
 l: social loss to II for an unpunished crime

		(II)	
		E	N
(I) (I)	C	$-pf + \bar{p}(r+\pi), -c + \bar{p}l$	$r+\pi, -l$
	H	$r, -c$	$r, 0$

Let $\Gamma_{k,m}(n)$ denote the game described above. (n, k, m) denotes the state of the system in which players I and II possess k and m times to take actions, respectively, and they have n periods to go as their "mission time." Let $(u_{k,m}(n), v_{k,m}(n))$ represent the equilibrium values of this non-zero-sum n -stage game $\Gamma_{k,m}(n)$. Then the Optimality Equation of dynamic programming gives a system of equations

$$(u_{k,m}(n), v_{k,m}(n)) = \text{Eq.Val.}$$

		E	N
(2)	C	$-pf + \bar{p}\{r + \pi + u_{k-1, m-1}(n-1)\},$ $-(c + \bar{p}l) + \bar{p}v_{k-1, m-1}(n-1)$	$r + \pi + u_{k-1, m}(n-1),$ $-l + v_{k-1, m}(n-1)$
	H	$r + u_{k, m-1}(n-1), \quad -c + v_{k, m-1}(n-1)$	$r + u_{k, m}(n-1), \quad v_{k, m}(n-1)$

(if the equilibrium values exist uniquely), with the boundary conditions :

$$(2a) \quad (u_{0, m}(n), v_{0, m}(n)) = (nr, 0), \quad \text{for } 1 \leq m \leq n,$$

$$(2b) \quad (u_{k, 0}(n), v_{k, 0}(n)) = (nr + k\pi, -kl), \quad \text{for } 1 \leq k \leq n,$$

$$(2c) \quad (u_{0, 0}(n), v_{0, 0}(n)) = (nr, 0), \quad \text{for } n \geq 1,$$

$$(2d) \quad u_{k, m}(0) = v_{k, m}(0) = 0, \quad \forall k, m \geq 0,$$

and

$$(2e) \quad (u_{k, m}(n), v_{k, m}(n)) = (u_{k', m'}(n), v_{k', m'}(n)), \quad \text{with } k' = k \wedge n, m' = m \wedge n.$$

The four conditions (2a) ~ (2d) imply that: (a) If II has m times of law-enforcement and his opponent has none of the opportunity of violation, then the decision-pair H-N is repeated throughout the whole period, (b) If I has k times of violating law and his opponent cannot do anything because of lack of budget, then I chooses C and H k and $n-k$ times, respectively, during the n periods, (c) If both ^{do not} players have any law-violation and law-enforcement intentions, the decision-pair H-N is repeated throughout the whole period, and (d) The problem with $n=1$ reduces to the bimatrix game with payoff matrix (1).

If release from prison and a second offense are not taken into account, we need not consider large n , and the optimality equation (2), with (2a) ~ (2e), can be, in principle solved by backward induction. The two-period games $\Gamma_{n, n}(n)$, $\Gamma_{1, 1}(n)$, $\Gamma_{1, n}(n)$ and $\Gamma_{n, 1}(n)$, all for $n=2$, are explicitly solved in the subsequent sections 2, 3, 4 and 5, respectively.

Concerning the n -period games of the kind discussed in this paper, Sakaguchi [3] studies a zero-sum game of smuggler vs customs, and Dawid, Feichtinger and Jorgensen [1] and Sakaguchi [4] investigate non-zero-sum games, with $k=m=n$, the latter being related to full-information optimal stopping games. Also Kilgour [2] studies a zero-sum game with $k=m=n$, where the offender is not restricted to the two pure strategies and is asked to choose the "level of violation" $q \in [0, 1]$ in each stage.

2. The Game $\Gamma_{n, n}(n)$.

First we consider the case $k=m=n$, which is discussed by Dawid, Feichtinger and Jorgensen [1]. Let us simply write $\Pi_n = u_{n, n}(n)$ and $\nabla_n = v_{n, n}(n)$. Then (2) becomes

$$(\bar{U}_n, \bar{V}_n) = \text{Eq. Val.} \begin{array}{|c|c|} \hline -pf + \bar{P}(r + \pi + \bar{U}_{n-1}), -(c + \bar{P}l) + \bar{P}\bar{V}_{n-1} & r + \pi + \bar{U}_{n-1}, -l + \bar{V}_{n-1} \\ \hline r + \bar{U}_{n-1}, & -c + \bar{V}_{n-1} \\ \hline \end{array} \begin{array}{|c|c|} \hline r + \pi + \bar{U}_{n-1}, -l + \bar{V}_{n-1} & r + \bar{U}_{n-1}, \bar{V}_{n-1} \\ \hline \end{array}$$

$$(3) \quad = (\bar{U}_{n-1}, \bar{V}_{n-1}) + \text{Eq. Val. } M_n$$

where M_n is a bimatrix

$$(4) \quad M_n = \begin{array}{|c|c|} \hline \bar{P}(r + \pi) - p(f + \bar{U}_{n-1}), -(c + \bar{P}l + p\bar{V}_{n-1}) & r + \pi, -l \\ \hline r, & -c \\ \hline \end{array} \begin{array}{|c|c|} \hline r + \pi, -l & r, 0 \\ \hline \end{array}$$

($n \geq 1$; $\bar{U}_0 = \bar{V}_0 = 0$)

which is identical to (1) for $n=1$.

The two theorems that follow are not new, and essentially reproduction of the main results in [1] but with more simpler description of the proofs.

Theorem 1. Let $\pi_1 = \left(\frac{p}{\bar{P}}\right)(f+r)$. Then the solution to the game $\Gamma_{1,1}(1)$ is:

Case	$0 < \pi < \pi_1$	$\pi = \pi_1$	$\pi > \pi_1$
Eq. play	$x_i^* - y_i^*$, with $x = \frac{c}{p\bar{L}}, y = \frac{\pi}{p(f+r+\pi)}$	$(zC + \bar{z}H) - E$ $\forall z \in [0,1]$	$C - E$
Eq. values	$\bar{U}_1 = r$ $\bar{V}_1 = -c/p$	$\bar{U}_1 = r$ $\bar{V}_1 = -(c + \bar{P}l\pi)$	$\bar{U}_1 = -pf + \bar{P}(r + \pi)$ $\bar{V}_1 = -(c + \bar{P}l)$

(The mixed-strategy equilibrium is denoted by $x_i^* - y_i^*$ with $x_i^* = \langle x, \pi \rangle$ and $y_i^* = \langle y, \bar{y} \rangle$. The strategy $zC + \bar{z}H$ means the mixture of the pure strategies C and H, with probabilities z and $\bar{z}$, respectively.)

Proof We have $-(c + \bar{P}l) > -l$, since we assumed $c < p\bar{L}$. So by the "circular rule" of finding the eq. of 2×2 bimatrix game, which of $-pf + \bar{P}(r + \pi)$ and r is larger becomes important. Clearly $-pf + \bar{P}(r + \pi) \leq r$, if $\pi \leq \pi_1$. If $0 < \pi < \pi_1$ there is a mixed-strategy eq. $x_i^* - y_i^*$ which together with the eq. values is found by solving

$$\begin{cases} -(c + \bar{P}l) - c\bar{z} = -l x = \bar{V}_1, \\ \{-pf + \bar{P}(r + \pi)\} y + (r + \pi)\bar{y} = r = \bar{U}_1, \end{cases}$$

If $\pi > \pi_1$, there is a pure-strategy eq. C-E. $\square$

Theorem 2 Let $\pi_2 = \left(\frac{p}{\bar{P}}\right)(f+r) + \left(\frac{p}{\bar{P}^2}\right)r$. Then the solution to the game $\Gamma_{2,2}(2)$ is:

Case	$0 < \pi < \pi_1$	$\pi_1 < \pi < \pi_2$	$\pi > \pi_2$
Eq play	$x_2^* - y_2^*$ with $x = c/(pl+c)$ $y = \pi/p(f+2r+\pi)$	$x_2^* - y_2^*$ with $x = c/p\{c+l+\bar{p}\}l$ $y = (\pi/p)\{\bar{p}f+(1+\bar{p})(r+\pi)\}$	C-E
Eq values	$U_2 = 2r$ $V_2 = -\frac{c(2pl+c)}{p(pl+c)}$	$U_2 = -\bar{p}f + \bar{p}(r+\pi) + r$ $V_2 = -\left[\frac{cl}{p\{c+(1+\bar{p})l\}} + c + \bar{p}l\right]$	$U_2 = (1+\bar{p})\{\bar{p}f + \bar{p}(r+\pi)\}$ $V_2 = -(1+\bar{p})(c + \bar{p}l)$

(The solutions for the bordering cases $\pi = \pi_1$ and $\pi = \pi_2$ are omitted. The mixed-strategy equilibrium is denoted by $x_2^* - y_2^*$ with $x_2^* = \langle x, \bar{x} \rangle$ and $y_2^* = \langle y, \bar{y} \rangle$.)

Proof Substituting the values of U_1 and V_1 , which was found in Theorem 1, into M_2 in (4) we obtain the bimatrices

$$(5_1) \quad \begin{array}{cc|cc} \bar{p}(r+\pi) - \bar{p}(f+r), & -\bar{p}l & r+\pi, & -l \\ r, & -c & r, & 0 \end{array}, \quad \text{if } 0 < \pi < \pi_1;$$

$$(5_2) \quad \begin{array}{cc|cc} \bar{p}\{-\bar{p}f + \bar{p}(r+\pi)\}, & -\bar{p}(c + \bar{p}l) & r+\pi, & -l \\ r, & -c & r, & 0 \end{array}, \quad \text{if } \pi > \pi_1.$$

Since $0 < \pi < \pi_1 \equiv (p/\bar{p})(f+r) \Rightarrow \pi < (p/\bar{p})(f+2r) \Leftrightarrow \bar{p}(r+\pi) - \bar{p}(f+r) < r$, the bimatrix (5₁) has the mixed-strategy solution and it is found, together with eq. values by solving

$$\begin{cases} -\bar{p}lx - c\bar{x} = -lx = V_2 - V_1 \\ \{\bar{p}(r+\pi) - \bar{p}(f+r)\}y + (r+\pi)\bar{y} = r = U_2 - U_1 \end{cases}$$

and giving the result mentioned in the theorem (1st column of the table)

Since $c < pl < pl(1 + 1/\bar{p}) \Leftrightarrow \bar{p}(c + \bar{p}l) < l$ and

$$\bar{p}\{-\bar{p}f + \bar{p}(r+\pi)\} \begin{cases} \leq \\ > \end{cases} r, \quad \text{if } \pi \begin{cases} \leq \\ > \end{cases} \pi_2 \equiv (p/\bar{p})(f+r) + (p/\bar{p}^2)r,$$

the bimatrix (5₂) has the mixed-strategy eq. $x_2^* - y_2^*$ if $\pi_1 < \pi < \pi_2$, and pure-strategy eq. C-E, if $\pi > \pi_2$. This mixed-strategy eq. together with eq. values is found by solving

$$\begin{cases} -\bar{p}(c + \bar{p}l)x - c\bar{x} = -lx = V_2 - V_1 \\ \bar{p}\{-\bar{p}f + \bar{p}(r+\pi)\}y + (r+\pi)\bar{y} = r = U_2 - U_1 \end{cases}$$

and giving the result mentioned in the theorem (2nd column of the table). The pure-strategy eq. C-E for $\pi > \pi_2$ gives the eq. values

$$(U_2, V_2) = (U_1 + \bar{p}\{-\bar{p}f + \bar{p}(r+\pi)\}, V_1 - \bar{p}(c + \bar{p}l)),$$

which are mentioned in the 3rd column of the table in the theorem. $\square$

Hereafter we shall omit considerations about the bordering cases, where a continuum of equilibria in the first period and correspondingly in the whole game exist. Direct calculations show that the game value for the offender is continuous, non-decreasing in π , and the game value for the defender involves the mixing parameter $z \in [0, 1]$ chosen arbitrarily by the offender.

In the game $\Gamma_{1,1}(1)$, $\pi > \pi_1 - (p/f)(f+r)$ means that committing crime for I has a positive expected payoff to I. Theorem 1 implies that I is not motivated to commit crime as long as $\pi < \pi_1$ and C-E is in equilibrium if $\pi > \pi_1$. II should take p and f larger, in order to make law-violator get low profit.

In the game $\Gamma_{2,2}(2)$, $\pi > \pi_2$ means that committing a crime for I in the first stage has a positive profit in that stage since $\bar{p}\pi > p\{f+2r+\bar{p}\pi - p(f+r)\}$. Theorem 2 implies that I is not motivated to commit crime in both stages as long as $\pi < \pi_1$, the choice-pair C-E is in eq. in the first stage if $\pi > \pi_2$, and the intermediate situation arises if $\pi_1 < \pi < \pi_2$.

3. The Game $\Gamma_{1,1}(n)$.

4 The Game $\Gamma_{1,n}(n)$.

5. The Game $\Gamma_{n,1}(n)$.

6. Remarks and a Numerical Example

1°) From Theorems 1 and 2, the two-period equilibrium play in the game $\Gamma_{2,2}(2)$ is as follows :

Case	Two-period eq play in $\Gamma_{2,2}(2)$
$0 < \pi < \pi_1$	$\{x_2^* - y_2^* ; x_1^* - y_1^*\}$ with $(x_2^*)_1 = c/(pl+c)$, $(y_2^*)_1 = \pi/[p(f+2r+\pi)]$ and $(x_1^*)_1 = c/(pl)$, $(y_1^*)_1 = \pi/[p(f+r+\pi)]$.
$\pi_1 < \pi < \pi_2$	$\{x_2^* - y_2^* ; C-E\}$, with $(x_2^*)_1 = c/[p\{c+(1+\bar{p})l\}]$, and $(y_2^*)_1 = (\pi/p)/\{\bar{p}f+(1+\bar{p})(r+\pi)\}$.
$\pi > \pi_2$	$\{C-E ; C-E\}$

(For example $\{x_2^* - y_2^* ; x_1^* - y_1^*\}$ means that players employ mixed-strategy pair $x_2^* - y_2^*$ in the 1st period, and if the game is not over (cf. the game is over if and only if it results in C-E and I is punished by II) $x_1^* - y_1^*$ is employed in the 2nd period.)

2°) For the game $\Gamma_{2,2}(2)$ we easily find, from Theorems 1 and 2, that

$$(x_2^*)_1 \leq (x_1^*)_1 \text{ and } (y_2^*)_1 \leq (y_1^*)_1 \text{ for all } \pi > 0.$$

This means that player I is more pushed to making a crime, and II invests more effort in law-enforcement, both in the 2nd period than in the 1st. From Theorems 4 and 5, the above is true in the games $\Gamma_{1,2}(2)$ and $\Gamma_{2,1}(2)$ also.

3) Throughout Theorems 2~5, the following fact is observed. As functions of $\pi > 0$ the two-period eq. payoff U_2 for I is continuous and non-decreasing, but V_2 for II is piece-wise constant and not necessarily decreasing as π increases.

4) We give, by Table 2, a numerical example of the solutions to the four two-period games for the parameters $r = \frac{1}{2}, p = \frac{2}{3}, c = 1, f = l = 2$ and therefore $c < pl$ being satisfied and $\pi_1 = 5 < \pi_2 = 8$.

Table 2. Example of Solutions to the Five Games

Game	Case	Eq. play $x^* - y^*$ in the 1st period	Eq. payoffs	Based on
$\Gamma_{1,1}(1)$	1	$\langle \frac{3}{4}, \frac{1}{4} \rangle - \langle \frac{3\pi}{2\pi+5}, \frac{5-\pi}{2\pi+5} \rangle$	$\frac{1}{2} - (-\frac{3}{2})$	Th. 1
	2	C-E	$(\frac{1}{6})(2\pi-7) - (-\frac{5}{3})$	
$\Gamma_{2,2}(2)$	1	$\langle \frac{3}{4}, \frac{4}{4} \rangle - \langle \frac{3\pi}{2(\pi+3)}, \frac{6-\pi}{2(\pi+3)} \rangle$	$1 - (-\frac{3}{4})$	Th. 2
	2'	$\langle \frac{9}{22}, \frac{13}{22} \rangle - \langle \frac{9\pi}{8(\pi+1)}, \frac{8-\pi}{8(\pi+1)} \rangle$	$(\frac{1}{3})(\pi-2) - (-\frac{82}{33})$	
	2''	C-E	$(\frac{2}{9})(2\pi-1) - (-\frac{20}{9})$	
$\Gamma_{1,1}(2)$	1	$\langle \frac{9}{11}, \frac{2}{11} \rangle - \langle \frac{3\pi}{5\pi+6}, \frac{2\pi+4}{5\pi+6} \rangle$	$\frac{3\pi^2+5\pi+6}{5\pi+6} - (-\frac{2}{11})$	Th. 3
	2	$\langle \frac{4}{5}, \frac{1}{5} \rangle - \langle \frac{2\pi+5}{4\pi+11}, \frac{2\pi+6}{4\pi+11} \rangle$	$\frac{2\pi^2+9\pi+11}{4\pi+11} - (-\frac{29}{15})$	
$\Gamma_{1,2}(2)$	1	$\langle \frac{3}{4}, \frac{1}{4} \rangle - \langle \frac{3\pi}{2(\pi+3)}, \frac{6-\pi}{2(\pi+3)} \rangle$	$1 - (-\frac{15}{8})$	Th. 4
	2	$\langle \frac{3}{4}, \frac{1}{4} \rangle - \langle \frac{2\pi+5}{2(\pi+3)}, \frac{1}{2(\pi+3)} \rangle$	$(\frac{1}{3})(\pi-2) - (-\frac{23}{12})$	
$\Gamma_{2,1}(2)$	1	$\langle \frac{9}{16}, \frac{7}{16} \rangle - \langle \frac{3\pi}{2(2\pi+3)}, \frac{\pi+6}{2(2\pi+3)} \rangle$	$\frac{3\pi^2+4\pi+6}{2(2\pi+3)} - (-\frac{21}{8})$	Th. 5
	2	$\langle \frac{1}{2}, \frac{1}{2} \rangle - \langle \frac{3\pi}{2(2\pi+3)}, \frac{\pi+6}{2(2\pi+3)} \rangle$	$\frac{10\pi^2+13\pi-12}{6(2\pi+3)} - (-\frac{8}{3})$	

(Case 1,2,2',2'' means $0 < \pi < 5, \pi > 5, 5 < \pi < 8, \pi > 8$, respectively.)

We can make sure that, for all $\pi > 0$,

$$u_{1,2}(2) \leq \left\{ \begin{array}{l} u_{1,1}(2) \\ u_{2,2}(2) \end{array} \right\} \leq u_{2,1}(2), \quad \text{for I,}$$

and

$$v_{2,1}(2) \leq \left\{ \begin{array}{l} v_{1,1}(2) \\ v_{2,2}(2) \end{array} \right\} \leq v_{1,2}(2), \quad \text{for II.}$$

Furthermore, we can check that the facts mentioned in 3) are true.

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