

RESEARCH ARTICLE

An analytical model for ground support by bolting

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Summary

Bolts have been used under various geological conditions as reinforcement to support underground openings. However, quantitative evaluations of the impacts of reinforcement bolts remain controversial. When a bolt is modeled explicitly, one challenge is determining the parameter that represents the interaction between the bolt and surrounding medium. This study aims to derive a simplified Winkler model for the ground support provided by a bolt, with the mutual interaction between the bolt and the surrounding ground analytically evaluated. Theoretical solutions are derived that rigorously consider the mutual interaction between a bolt and the surrounding ground based on two-dimensional axisymmetric elasticity. The obtained solutions are compared with those from the simplified model, in which the bolt is assumed as a rod and the surrounding ground is assumed to have a constant stiffness. The numerical results show that the simplified model can evaluate the ground-bolt interaction when the ground deformation follows the bolt. However, the model cannot accurately model pullout tests as the free surface strongly influences the result, and it is difficult to define a constant representing the surrounding ground.

KEYWORDS:

reinforcement, bolt, axisymmetric elasticity, simplified model, underground opening

1 | INTRODUCTION

Bolts have been used under various geological conditions as reinforcement to support underground openings such as tunnels, including in soft ground that can be treated as a continuum and hard rock that should be treated as a discontinuum. Previous experiences have verified the effectiveness of this reinforcement method. However, quantitative evaluations of the impacts of reinforcement bolts remain controversial. Due to the small bolt diameter and many inclusions, modeling the bolting without simplification is challenging with numerical analysis, such as the finite element method. Therefore, the numerical implementation requires some ingenuity.

There are several theoretical models for evaluating the ground support provided by bolts. Some previous studies^{1,2,3,4,5,6,7,8,9} model the bolt explicitly, such as modeling the bolt as a cylindrical rod, with the displacement of the bolt and the surrounding medium defined separately and the mutual interaction associated with the relative displacement. These models can define the boundary between the bolt and the surrounding medium. Thus, they can consider imperfect connections caused by sliding or joint openings. Alternatively, the bolt can be modeled implicitly^{10,11,12,13,14,15,16,17}. These models assume axial symmetry (*i.e.*, a circular tunnel under hydrostatic pressure) and close spacing of regularly installed bolts. The reinforcement is represented as a radial load smeared within the zone of influence of the bolts. These models can evaluate macroscopically the ground strength improvement from the bolts and incorporate the support provided into the ground reaction curve. Bobet¹⁸ approximated the

effect of bolts by treating an elastic bolt-rock composite as a material with cylindrical anisotropy and stiffnesses related to the properties of the rock and bolts.

When evaluating the provided support through numerical analysis, a model that explicitly represents bolts is more compatible. One of the challenges when using the explicit model is determining the parameter that represents the interaction between the bolt and the surrounding medium. This parameter is typically estimated from pullout tests or measurements at the construction site. However, there has been little theoretical consideration.

This study aims to derive a simplified Winkler model for the ground support provided by a bolt, where the mutual interaction between the bolt and the surrounding ground is analytically evaluated. Theoretical solutions are derived that rigorously consider the mutual interaction between a bolt and the surrounding ground based on two-dimensional axisymmetric elasticity^{19,20,21}. The obtained solutions are compared with those from the simplified model. The model validity and the parameter estimation method are presented. Furthermore, the range of influence of the bolt support is investigated.

2 | THEORY

2.1 | Problem definition

Consider a cylindrical bolt embedded in the ground, as shown in Figure 1. The ground and bolt are modeled as an elastic finite thick-walled cylinder with an infinite outer radius and an elastic finite cylinder, respectively. They are considered to be homogeneous and isotropic mediums. Tension is taken to be positive, and the ground surface ($z = \pm h$) is a free surface. The bolt radius is a and its length is $2h$. The length h is assumed to be sufficiently longer than the actual bolt length, and ground support provided by a long bolt is considered. Solutions are symmetric with respect to $z = 0$, and the domain $z \geq 0$ is used for consideration. The bolt faceplate and grout between the bolt and surrounding ground are not considered.

2.2 | Solution scheme for the detailed model

Two substructures are considered to evaluate solutions under mutual interaction: the ground system and the bolt system, which are denoted by subscripts (1) and (2), respectively. No subscript is used for the general expression.

When an arbitrarily determined initial ground displacement in the absence of the bolt $\mathbf{u}_{(1)}^i$ is given, the resultant ground displacement considering ground-bolt interaction $\mathbf{u}_{(1)}$ is

$$\mathbf{u}_{(1)} = \mathbf{u}_{(1)}^i + \mathbf{u}_{(1)}^r, \quad (1)$$

where $\mathbf{u}_{(1)}^r$ is the secondary displacement caused by surface loading $\mathbf{f}_{(1)}$ from the bolt. Superscripts i and r are used to denote 'initial' and 'reaction', respectively. The bolt displacement $\mathbf{u}_{(2)}$ is caused by surface loading from the ground $\mathbf{f}_{(2)}$.

When a no-slip condition is imposed at the ground-bolt interface, the solutions must satisfy $\mathbf{f}_{(1)} + \mathbf{f}_{(2)} = 0$ and $\mathbf{u}_{(1)} = \mathbf{u}_{(2)}$ at $r = a$. However, it is impossible to strictly or analytically satisfy such boundary conditions. Therefore, the point-matching method, where boundary conditions are only satisfied at a finite set of points at the interface, is adopted to obtain approximate solutions^{22,23,24}.

2.3 | Basic equations for axisymmetric elasticity

The displacement and stress components for any axisymmetric stress field in an elastic body of revolution can be expressed by Love's stress function $\chi(r, z)$ ¹⁹, given as

$$\left. \begin{aligned} u_r &= -\frac{1+\nu}{E} \frac{\partial^2 \chi}{\partial r \partial z} \\ u_z &= \frac{1+\nu}{E} \left\{ (1-2\nu) \nabla^2 \chi + \frac{\partial^2 \chi}{\partial r^2} + \frac{1}{r} \frac{\partial \chi}{\partial r} \right\} \end{aligned} \right\}, \quad (2)$$

$$\left. \begin{aligned} \sigma_{rr} &= \frac{\partial}{\partial z} \left(\nu \nabla^2 \chi - \frac{\partial^2 \chi}{\partial r^2} \right) \\ \sigma_{rz} &= \frac{\partial}{\partial r} \left\{ (1 - \nu) \nabla^2 \chi - \frac{\partial^2 \chi}{\partial z^2} \right\} \\ \sigma_{zz} &= \frac{\partial}{\partial z} \left\{ (2 - \nu) \nabla^2 \chi - \frac{\partial^2 \chi}{\partial z^2} \right\} \\ \sigma_{\theta\theta} &= \frac{\partial}{\partial z} \left(\nu \nabla^2 \chi - \frac{1}{r} \frac{\partial \chi}{\partial r} \right) \end{aligned} \right\}, \quad (3)$$

where $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$ denotes the Laplace operator and E and ν are the Young's modulus and Poisson's ratio, respectively. The function χ must satisfy the differential equation

$$\nabla^2 \nabla^2 \chi = 0. \quad (4)$$

2.4 | Solutions for the secondary ground field

Only the ground system is considered here. The boundary conditions for the secondary ground field are given by

$$\left. \begin{aligned} f_{r(1)}|_{z=\pm h} &= \sigma_{rz(1)}|_{z=\pm h} = 0 \\ f_{z(1)}|_{z=\pm h} &= \sigma_{zz(1)}|_{z=\pm h} = 0 \\ f_{r(1)}|_{r=a} &= -\sigma_{rr(1)}|_{r=a} = \sum_{n=0}^{\infty} F_{r,n(1)} \cos k_n z \\ f_{z(1)}|_{r=a} &= -\sigma_{rz(1)}|_{r=a} = \sum_{n=0}^{\infty} F_{z,n(1)} \sin k_n z \end{aligned} \right\}, \quad (5)$$

with

$$k_n = \frac{n\pi}{h}, \quad (6)$$

where $F_{r,n(1)}$ and $F_{z,n(1)}$ are the coefficients of a Fourier series expansion with $F_{z,0(1)}$ assumed to be zero. It should be noted that the sign of the relationship between surface loading and stress depends on the definition of positive direction.

The stress function $\chi_{(1)}$ is expressed as¹⁹

$$\begin{aligned} \chi_{(1)} &= B_{0(1)} a^2 z \log r + \sum_{n=1}^{\infty} \sin k_n z \{ A_{n(1)} K_0(k_n r) + B_{n(1)} k_n r K_1(k_n r) \} \\ &+ \int_0^{\infty} \frac{C_{(1)}(\beta)}{\beta^3 \cosh \beta h} \{ \beta z \cosh \beta z - (\beta h \coth \beta h + 2\nu_{(1)}) \sinh \beta z \} V_0(\beta r) d\beta, \end{aligned} \quad (7)$$

with

$$V_0(\beta r) = J_0(\beta r) Y_1(\beta a) - Y_0(\beta r) J_1(\beta a), \quad (8)$$

where J_0 , Y_0 , J_1 , and Y_1 are zeroth and first order Bessel functions, respectively, and K_0 and K_1 are zeroth and first order modified Bessel functions of the second kind, respectively. Coefficients $A_{n(1)}$, $B_{n(1)}$, and $C_{(1)}(\beta)$ are determined from the boundary condition in Eq. (5). Once coefficients $F_{r,n(1)}$ and $F_{z,n(1)}$ are determined, the displacement and stress at any arbitrary point in the ground can be obtained.

Displacement at a point p on the ground-bolt interface is given by

$$\left. \begin{aligned} u_{r(1)}^{r,p}|_{r=a} &= \sum_{n=0}^{\infty} \left(U_{r,rn(1)}^p F_{r,n(1)} + U_{r,zn(1)}^p F_{z,n(1)} \right) \\ u_{z(1)}^{r,p}|_{r=a} &= \sum_{n=0}^{\infty} \left(U_{z,rn(1)}^p F_{r,n(1)} + U_{z,zn(1)}^p F_{z,n(1)} \right) \end{aligned} \right\}, \quad (9)$$

where $U_{i,jn(1)}^p$ is the displacement in the i -direction at point p when the n th-order unit periodic surface loading in the j -direction ($F_{j,n(1)} = 1$) is applied on the boundary.

2.5 | Solutions for the bolt

The bolt system is only considered here. The boundary conditions for the bolt modeled as an elastic finite cylinder are given as

$$\left. \begin{aligned} f_{r(2)}|_{z=\pm h} &= \sigma_{rz(2)}|_{z=\pm h} = 0 \\ f_{z(2)}|_{z=\pm h} &= \sigma_{zz(2)}|_{z=\pm h} = q_0 \\ f_{r(2)}|_{r=a} &= \sigma_{rr(2)}|_{r=a} = \sum_{n=0}^{\infty} F_{r,n(2)} \cos k_n z \\ f_{z(2)}|_{r=a} &= \sigma_{rz(2)}|_{r=a} = \sum_{n=0}^{\infty} F_{z,n(2)} \sin k_n z \end{aligned} \right\}, \quad (10)$$

where $F_{r,n(2)}$ and $F_{z,n(2)}$ are coefficients of a Fourier series expansion with $F_{z,0(2)}$ assumed to be zero. Since it is assumed to be symmetric about the horizontal plane $z = 0$, the axial stress at the bolt ends is the same. For simplification, the stress is assumed to be constant and is given as q_0 .

The stress function $\chi_{(2)}$ is given by^{20,21}

$$\begin{aligned} \chi_{(2)} &= \frac{1}{6} B_{0(2)} z^3 + \frac{1}{2} D_{0(2)} r^2 z \\ &+ \sum_{n=1}^{\infty} \left\{ A_{n(2)} \frac{I_0(k_n r)}{I_1(k_n a)} + B_{n(2)} r \frac{I_1(k_n r)}{I_1(k_n a)} \right\} \frac{\sin(k_n z)}{k_n^2} + \sum_{j=1}^{\infty} \left\{ C_{j(2)} \frac{\sinh(\lambda_j z)}{\sinh(\lambda_j h)} + D_{j(2)} z \frac{\cosh(\lambda_j z)}{\sinh(\lambda_j h)} \right\} \frac{J_0(\lambda_j r)}{\lambda_j^2 J_0(\lambda_j a)}, \end{aligned} \quad (11)$$

where I_0 and I_1 are zeroth and first order modified Bessel functions of the first kind, respectively. Coefficients $B_{0(2)}$, $D_{0(2)}$, $A_{n(2)}$, $B_{n(2)}$, $C_{j(2)}$, and $D_{j(2)}$ are determined from the boundary condition in Eq. (10). Once coefficients $F_{r,n(2)}$, $F_{z,n(2)}$ and q_0 are determined, the displacement and stress at any arbitrary point can be calculated.

Displacement at a point p on the ground-bolt interface can then be expressed as

$$\left. \begin{aligned} u_{r(2)}^p|_{r=a} &= \sum_{n=0}^{\infty} \left(U_{r,rn(2)}^p F_{r,n(2)} + U_{r,zn(2)}^p F_{z,n(2)} \right) + U_{r,q_0(2)}^p \\ u_{z(2)}^p|_{r=a} &= \sum_{n=0}^{\infty} \left(U_{z,rn(2)}^p F_{r,n(2)} + U_{z,zn(2)}^p F_{z,n(2)} \right) + U_{z,q_0(2)}^p \end{aligned} \right\}, \quad (12)$$

where $U_{i,jn(2)}^p$ is the displacement in the i -direction at point p when the n th-order unit periodic surface loading in the j -direction ($F_{j,n(2)} = 1$) is only applied on the boundary. $U_{r,q_0(2)}^p$ and $U_{z,q_0(2)}^p$ are the displacement when the axial stress q_0 is applied at only the bolt ends.

2.6 | Solutions under ground-bolt interaction

When a no-slip condition, which results in the maximum bolt effectiveness, is imposed at the ground-bolt interface, the boundary conditions are

$$\left. \begin{aligned} f_{r(1)}|_{z=\pm h} &= f_{r(2)}|_{z=\pm h} = 0 \\ f_{z(1)}|_{z=\pm h} &= 0 \\ f_{z(2)}|_{z=\pm h} &= q_0 \\ f_{r(1)}|_{r=a} + f_{r(2)}|_{r=a} &= 0 \\ f_{z(1)}|_{r=a} + f_{z(2)}|_{r=a} &= 0 \\ u_{r(1)}^i|_{r=a} + u_{r(1)}^r|_{r=a} &= u_{r(2)}|_{r=a} \\ u_{z(1)}^i|_{r=a} + u_{z(1)}^r|_{r=a} &= u_{z(2)}|_{r=a} \end{aligned} \right\}. \quad (13)$$

Loading boundary conditions at $z = \pm h$ in Eq. (13) are always satisfied when solutions for the secondary ground field satisfy Eq. (5) and solutions for the bolt satisfy Eq. (10). Loading boundary conditions at $r = a$ in Eq. (13) are satisfied when

$$F_{r(1)} = -F_{r(2)}, \quad F_{z(1)} = -F_{z(2)}. \quad (14)$$

For displacement boundary conditions, it is impossible to strictly or analytically satisfy Eq. (13). Therefore, the point-matching method was adopted. Considering Eqs. (9), (12), and (14), the displacement boundary conditions in Eq. (13) are modified to

$$\left. \begin{aligned} u_{r(1)}^{i,p}|_{r=a} + \sum_{n=0}^{N-1} \left(U_{r,rn(1)}^p F_{r,n(1)} + U_{r,zn(1)}^p F_{z,n(1)} \right) &= - \sum_{n=0}^{N-1} \left(U_{r,rn(2)}^p F_{r,n(1)} + U_{r,zn(2)}^p F_{z,n(1)} \right) + U_{r,q_0(2)}^p \\ u_{z(1)}^{i,p}|_{r=a} + \sum_{n=0}^{N-1} \left(U_{z,rn(1)}^p F_{r,n(1)} + U_{z,zn(1)}^p F_{z,n(1)} \right) &= - \sum_{n=0}^{N-1} \left(U_{z,rn(2)}^p F_{r,n(1)} + U_{z,zn(2)}^p F_{z,n(1)} \right) + U_{z,q_0(2)}^p \end{aligned} \right\}, \quad (15)$$

where $u_{r(1)}^{i,p}|_{r=a}$ and $u_{z(1)}^{i,p}|_{r=a}$ are the initial ground displacements in the absence of a bolt for point p at $r = a$. The first N terms of the infinite series (from $n = 0$ to $N - 1$) are used to express the approximate solutions. N points in the half-space ($0 \leq z \leq \alpha h$) are selected, which are equidistantly spaced for convenience ($z^p = \alpha p h / (N - 1)$; $p = 0, 1, 2 \dots N - 1$). In the following, α is assumed to be 0.998 because the results around $z = h$ have some numerical errors caused by weak convergence of the numerical integration in Eq. (7). The resulting system of linear equations for $F_{r,n(1)}$ and $F_{z,n(1)}$ is solved by matrix inversion. Both the total number of linear equations in Eq. (15) and that of the undetermined coefficients $F_{r,n(1)}$ and $F_{z,n(1)}$ are $2N - 1$. This is because the displacement boundary condition in the z direction at z^0 is always satisfied (i.e., $u_{z(1)}^{i,p}$, $U_{z,rn}^p$, $U_{z,zn}^p$, and $U_{z,q_0(2)}^p$ are zero at $z = 0$) and $F_{z,0}$ is assumed to be zero.

2.7 | Solutions for the simplified Winkler model

The governing equation of the simplified Winkler model, in which a bolt is assumed as a rod and the surrounding ground is assumed as a constant stiffness, is given by Hyett et al.² as

$$A \frac{d\sigma_{zz(2')}}{dz} = A E_{(2)} \frac{d^2 u_{z(2')}}{dz^2} = -2\pi a f_{z(2')}, \quad (16)$$

with

$$\left. \begin{aligned} A &= \pi a^2 \\ 2\pi a f_{z(2')} &= K \mu_{(1)} \left(u_{z(1)}^i - u_{z(2')} \right) \end{aligned} \right\}, \quad (17)$$

where A is the cross-section of the cylindrical bolt, K is a constant related to the shear stiffness of the ground, and $\mu_{(1)} = 0.5E_{(1)}/(1 + \nu_{(1)})$ is the shear modulus of the ground. Solutions derived by the simplified model are denoted by the subscript (2'). This governing equation can be solved analytically or numerically when the initial axial ground displacement $u_{z(1)}^i$, constant K , and boundary conditions at the bolt ends are given. The method for obtaining an appropriate value of K is discussed later.

3 | NUMERICAL RESULTS

Material properties for the numerical calculations are given in Table 1. Two types of ground are considered: soil and rock (soft rock). A relatively large bolt radius of $a = 15.0$ mm is assumed. The first 401 terms of the infinite series (from $n = 0$ to 400) in Eq. (15) are used to express the following approximate solutions. The value of j in Eq. (11) is assumed to reach a maximum of 400.

3.1 | Solutions at the ground-bolt interface

Solutions for a no bolt installation were obtained to understand the basic response of the ground system. When only the ground system is considered, the stress state is independent of the ground's Young's modulus, and the displacement is directly proportional to the Young's modulus. Figure 2 shows the applied shear loading and the resulting axial displacement at the cavity surface $r = a$ when the ground is soil. Two cases are shown: where only $F_{z,3(1)}$ is applied and where only $F_{z,9(1)}$ is applied at

$r = a$. The displacement amplitude for $F_{z,9(1)}$ is smaller than that for $F_{z,3(1)}$, even under the same shear loading (1.00 MPa). The distribution of the applied shear loading and the resultant displacement almost match for both cases, although the displacement shows some influence from the ground surface (free surface) at $z = 15.0$ m. In the vicinity of the surface, the displacement tends to be larger than when there is no surface. The reach of the ground surface influence is approximately one-quarter of the wavelength for the applied loading.

Solutions for the case with no surrounding ground were obtained to understand the basic response of the bolt system. Figure 3 shows the axial bolt displacement at $r = a$ when the shear loading is applied at $r = a$. Two cases are shown: where only $F_{z,3(2)}$ is given and where only $F_{z,9(2)}$ is given. The solutions obtained by the simplified model are also plotted. The loading conditions at the bolt ends are assumed to be zero. The solutions of the two models are in good agreement. The displacement increases in proportion to z and is substantially larger than $u_{z(1)}$, as shown in Figure 2. Comparing the fluctuating component of $u_{z(1)}$ and $u_{z(2)}$, $u_{z(2)}$ is larger for $F_{z,3(1)} = 1.00$ MPa and smaller for $F_{z,9(1)} = 1.00$ MPa.

In the following, solutions considering the ground-bolt interaction are obtained. Two separate initial ground displacement in the absence of the bolt $u_{(1)}^i$ are assumed

$$\left. \begin{aligned} u_{r(1)}^i|_{r=a} &= 0 \\ u_{z(1)}^i|_{r=a} &= \frac{U_0 z}{(r_0 + h - z)h} \end{aligned} \right\}, \quad (18)$$

and

$$\left. \begin{aligned} u_{r(1)}^i|_{r=a} &= 0 \\ u_{z(1)}^i|_{r=a} &= U_0 H(z - 0.5h) \end{aligned} \right\}, \quad (19)$$

where U_0 and r_0 are the displacement amplitude and radius of the underground opening, respectively, and are assumed to be 1.00 mm and 1.00 m. $H()$ is the Heaviside step function. The former represents a monotonically increasing displacement and the latter represents an abruptly changing displacement. These initial displacements were determined with reference to a study by Hyett et al.².

Figure 4 shows the solutions at the interface $r = a$ when the ground is soil and the initial displacement increases monotonically. The solutions obtained by the simplified model assuming three distinct values of K are also plotted. The rate of reducing displacement from the bolt installation at the ground surface is approximately 50 %, and the maximum axial stress of the bolt is approximately 25 MPa. Solutions from the detailed model and those from the simplified model assuming $K = 1.50$ match well except for shear loading near the free surface. The shear loading $f_{z(2)}$ is a maximum near the free surface even though $f_{z(2)}|_{z=\pm h} = 0$ is strictly imposed. It should be noted that the plotted maximum value on the x -axis in the figures is not 15.00 m but 14.97 m since the range of the obtained solutions is from 0 m to $0.998h$ m.

Figure 5 shows the solutions when the ground is rock. The rate of reducing displacement from the bolt installation at the ground surface is approximately 20 %, and the maximum axial stress of the bolt is approximately 70 MPa. Compared to the soil, the rate of reducing displacement decreases and the maximum axial stress increases. Solutions from the detailed model and those from the simplified model assuming $K = 2.00$ match well except for shear loading near the free surface. The appropriate value of K for rock is larger than for soil.

Figure 6 shows the solutions when the ground is soil and the initial displacement changes abruptly. Compared with Figure 4, the maximum axial stress and shear loading are approximately four times larger even under an identical initial displacement of 1 mm. The abrupt change introduces a large concentration of stress to the bolt. There are some differences between the solutions from the detailed model and those from the simplified model. The appropriate value of K to evaluate the axial displacement is 1.50, and that for axial stress is around 1.50 and 2.00. In contrast, the appropriate value of K to evaluate the shear loading is difficult to determine. The maximum value of the shear loading for the simplified model is less than a quarter of that for the detailed model, even if K is assumed to be 2.00.

Figure 7 shows the solutions at the ground-bolt interface for a 100 MPa pullout axial stress with no initial ground displacement. This is a simulation of a pullout test. There are differences between the solutions from the detailed model and those from the simplified model. The appropriate value of K to evaluate the axial displacement, axial stress, and shear loading appears to be 1.00, around 1.00 and 1.50, and 1.50, respectively.

3.2 | Influence range of a bolt support

The solutions obtained from the detailed model can be used to evaluate the range of influence of the bolt support. Figure 8 shows the radial distributions of axial shear stress $\sigma_{rz(1)}$ and axial displacement $u_{z(1)}$ in soil when there is no bolt and shear loading is applied at $r = a$. The difference in the stress distribution between the two applied loading cases is not very significant. The radial stress distribution can be approximated to be inversely proportional to r . The displacement attenuates slower than the stress. The displacement for $F_{z,9(1)}$ attenuates faster than that for $F_{z,3(1)}$.

Figure 9 shows solutions for the secondary ground field when the initial displacement increases monotonically. The reinforcement from the bolt is concentrated around the bolt itself. The axial shear stress $\sigma_{rz(1)}$ becomes nearly zero at 0.15 m, which is ten times the bolt radius. The range of influence of displacement is larger than that of the stress. The range of influence of the bolt support is larger when the ground is soil than when it is rock.

4 | DISCUSSION

The amplitude of the ground and bolt displacements under unit periodic loading depends on the spatial frequency k_n of the loading, as shown in Figures 2 and 3. As expected from Eq. (16), the shear stiffness of the bolt is proportional to k_n for the periodic loading. The shear stiffness of the ground also increases with increasing k_n although the formulation is more complicated. Assuming that the ground is infinite with a cylindrical cavity, the shear stiffness of the ground is given by a closed-form expression (see Appendix A). The solutions from the simplified model can evaluate the ground-bolt interaction when a constant K is appropriately selected, as shown in Figures 4 and 5. The problem is how to estimate the constant K . It can be estimated from the relationship between the assumed spatial frequency using trial calculation and the dominant spatial frequency of the resulting shear loading.

Figure 10 shows the flowchart for obtaining an appropriate value of K . K'_n is the ground stiffness for periodic shear loading with a spatial frequency k'_n , given in Appendix A. Once a value of $K = K'_n$ is assumed, $u_{z(2')}$ and $f_{z(1')}|_{r=a} (= -f_{z(2')})$ can be obtained by solving Eq. (16). The dominant spatial frequency k''_n can be calculated from the discrete Fourier transform of $f_{z(1')}|_{r=a}$. When the values of k'_n and k''_n are similar, the results are consistent, and K'_n can be interpreted as the appropriate value of K .

As an example, Figure 11 (a) shows the Fourier amplitude spectra of $f_{z(1')}|_{r=a}$ for several values of K . In this case, the ground is soil and the initial displacement increases monotonically. When n is 1, the assumed spatial frequency k'_1 is 0.417, the constant K (equal to K'_1) is 1.32, and k''_1 , which is the dominant spatial frequency of $f_{z(1')}$ for $K = 1.32$, is 0.834. In this case, k'_1 and k''_1 are not matched. Conversely, when n is 2, k'_2 is 0.834, K is 1.54, and k''_2 is 0.834. The assumed spatial frequency k'_2 and the dominant spatial frequency k''_2 are matched. Therefore, 1.54 is estimated as the appropriate value of K . This estimation is consistent with Figure 4. When the ground is rock, the appropriate value of K becomes 1.98, as shown in Figure 11 (b). This result is also consistent with Figure 5. When the initial displacement increases monotonically, an appropriate value of K can be estimated. The appropriate value of K for rock is higher than that for soil. This is because the distribution of the solutions for rock is localized as compared with those for soil, and the dominant spatial frequency for rock is higher.

When the ground is soil and initial displacement changes abruptly, the appropriate value of K becomes 1.54, as shown in Figure 12 (a). Reasonable solutions can be obtained although the maximum axial stress and shear loading are underestimated, as shown in Figure 6. K is assumed constant regardless of the location, and the underestimation is inevitable when sudden changes occur. Such an abrupt change is unlikely to occur in practice. Therefore, the simplified model can evaluate the ground-bolt interaction when the ground deformation follows the bolt. For a 100 MPa pullout axial stress, the appropriate value of K is 1.32, as shown in Figure 12 (b). The simplified model cannot accurately estimate the result of the pullout tests since the free surface strongly influences the result. The constant K used for the simplified model is derived under the assumption of infinite ground with a cylindrical cavity, and it is difficult to define the constant.

The appropriate value of K when the pullout axial stress is given is smaller than when the initial ground displacement is given. This is because the change in shear loading along the bolt as obtained by the pullout test is more gradual and the dominant spatial frequency for the pullout test is lowered. The parameter that represents the interaction between the bolt and the surrounding medium is often estimated from pullout tests. It is suggested that the estimated value should be slightly increased when the ground support from bolts is evaluated by numerical analysis.

In this study, grout between the bolt and surrounding ground is not considered. The reinforcement provided by a bolt strongly depends on the cross-sectional rigidity $AE_{(2)}$, as predicted in Eq. (16). Based on this result, it is presumed that the effect of the

grout, which is present between the ground and bolt, can be included by increasing the cross-sectional rigidity of the bolt, as treated by Zuo et al.²⁵.

As shown in Figure 9, the range of influence of the displacement is considerably larger than the bolt radius. However, the shear stress becomes nearly zero at a distance of ten times the bolt radius. These results overestimate the range of influence as compared with the actual results since the bolt is longer than in actuality and slippage at the interface is ignored. The reinforcement provided by a bolt concentrates around the bolt itself. Therefore, attention should be given when using a homogenized model to represent bolt installation at regular intervals.

5 | CONCLUSIONS AND REMARKS

In this study, theoretical solutions were derived that consider in detail the mutual interaction between a bolt and the surrounding ground. The obtained solutions were compared with those from the simplified model. The validity of the model and the parameter estimation method was presented. In addition, the range of influence of a bolt support was also investigated. The following conclusions are drawn:

- (1) The simplified Winkler model, in which a bolt is assumed as a rod and the surrounding ground is assumed to have a constant stiffness, can evaluate the ground-bolt interaction when the constant is appropriately selected.
- (2) An appropriate value of the constant can be estimated from the relationship between the assumed spatial frequency using trial calculation and the dominant spatial frequency of the resulting shear loading.
- (3) When the initial ground displacement changes abruptly, reasonable solutions can be obtained although the simplified model underestimates the maximum bolt axial stress and shear loading.
- (4) The simplified model cannot accurately model pullout tests as the free surface strongly influences the result, and it is difficult to define a constant representing the surrounding ground.
- (5) The range of influence of the displacement is considerably larger than the bolt radius. However, the shear stress becomes nearly zero at a distance of ten times the bolt radius. Attention should be given when using the homogenized model to represent a bolt installation at regular intervals.

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APPENDIX

A STIFFNESS MATRIX OF INFINITE GROUND WITH A CYLINDRICAL CAVITY

The stress function $\chi_{(1)}$ for infinite ground with a cylindrical cavity is expressed as

$$\chi_{(1)} = \sum_{n=1}^{\infty} \{A_{n(1)}K_0(k'_n r) + B_{n(1)}k'_n r K_1(k'_n r)\} \sin k'_n z, \quad (A1)$$

with

$$k'_n = \frac{2\pi n}{h}. \quad (A2)$$

From the stress function $\chi_{(1)}$ and Eqs. (2) and (3), the general solution for the displacement and stress can be derived as

$$\left. \begin{aligned} u_{r(1)} &= \sum_{n=1}^{\infty} \frac{k_n'^2}{2\mu_{(1)}} \{A_{n(1)}K_1(k_n'r) + B_{n(1)}k_n'rK_0(k_n'r)\} \cos k_n'z \\ u_{z(1)} &= \sum_{n=1}^{\infty} \frac{k_n'^2}{2\mu_{(1)}} [A_{n(1)}K_0(k_n'r) + B_{n(1)}\{-4(1-\nu_{(1)})K_0(k_n'r) + k_n'rK_1(k_n'r)\}] \sin k_n'z \\ \sigma_{rr(1)} &= -\sum_{n=1}^{\infty} k_n'^3 \left[A_{n(1)} \left\{ K_0(k_n'r) + \frac{K_1(k_n'r)}{k_n'r} \right\} + B_{n(1)} \{(-1+2\nu_{(1)})K_0(k_n'r) + k_n'rK_1(k_n'r)\} \right] \cos k_n'z \\ \sigma_{rz(1)} &= -\sum_{n=1}^{\infty} k_n'^3 [A_{n(1)}K_1(k_n'r) + B_{n(1)}\{(k_n'rK_0(k_n'r) - (2-2\nu_{(1)})K_1(k_n'r)\}] \sin k_n'z \end{aligned} \right\}, \quad (A3)$$

where $\mu_{(1)}$ is the shear modulus of the ground.

At the inner wall of a cylindrical cavity, the displacement $u_{(1)}$ and surface loading $f_{(1)}$ can be expressed in the form of a Fourier series as

$$\left. \begin{aligned} u_{r(1)}|_{r=a} &= \sum_{n=1}^{\infty} U_{r,n(1)} \cos k_n'z \\ u_{z(1)}|_{r=a} &= \sum_{n=1}^{\infty} U_{z,n(1)} \sin k_n'z \\ f_{r(1)}|_{r=a} &= -\sigma_{rr(1)}|_{r=a} = \sum_{n=1}^{\infty} F_{r,n(1)} \cos k_n'z \\ f_{z(1)}|_{r=a} &= -\sigma_{rz(1)}|_{r=a} = \sum_{n=1}^{\infty} F_{z,n(1)} \sin k_n'z \end{aligned} \right\}, \quad (A4)$$

where $U_{r,n(1)}$, $U_{z,n(1)}$, $F_{r,n(1)}$, and $F_{z,n(1)}$ are constants. Relationships for these constants can be derived from Eqs. (A3) and (A4) as

$$\begin{pmatrix} F_{r,n(1)} \\ F_{z,n(1)} \end{pmatrix} = \begin{pmatrix} K_{rr,n(1)} & K_{rz,n(1)} \\ K_{rz,n(1)} & K_{zz,n(1)} \end{pmatrix} \begin{pmatrix} U_{r,n(1)} \\ U_{z,n(1)} \end{pmatrix}, \quad (A5)$$

and

$$\left. \begin{aligned} K_{rr,n(1)} &= \Delta \left\{ (3-2\nu_{(1)})K_0^2(k_n'a) + \frac{4(1-\nu_{(1)})}{k_n'a} K_0(k_n'a)K_1(k_n'a) - K_1^2(k_n'a) \right\} \\ K_{rz,n(1)} &= \Delta \left\{ k_n'aK_0^2(k_n'a) + (2-2\nu_{(1)})K_0(k_n'a)K_1(k_n'a) - k_n'aK_1^2(k_n'a) \right\} \\ K_{zz,n(1)} &= \Delta(2-2\nu_{(1)})K_1^2(k_n'a) \\ \Delta &= \frac{2\mu_{(1)}k_n'}{k_n'aK_0^2(k_n'a) + 4(1-\nu_{(1)})K_0(k_n'a)K_1(k_n'a) - k_n'aK_1^2(k_n'a)} \end{aligned} \right\}. \quad (A6)$$

The coefficient matrix of the right-hand side in Eq. (A5) is the stiffness matrix of the ground.

Figure 13 shows a comparison of the stiffness components of the ground. $K_{rz,n(1)}$ and $K_{zz,n(1)}$ increase with increasing $k_n'a$, whereas $K_{rr,n(1)}$ is constant. $K_{zz,n(1)}$ is larger than $K_{rz,n(1)}$, and the axial ground displacement is generally considerably larger than the radial ground displacement when considering the bolt support. Therefore, the effect of $K_{rz,n(1)}$ can be neglected and the ground response can be simplified using $K_{zz,n(1)}$. When $k_n'a$ is not zero, the value of $K_{zz,n(1)}$ can be defined. Therefore, it is not necessary to define the radius of influence of a single bolt R , as introduced by Cai et al.⁵.

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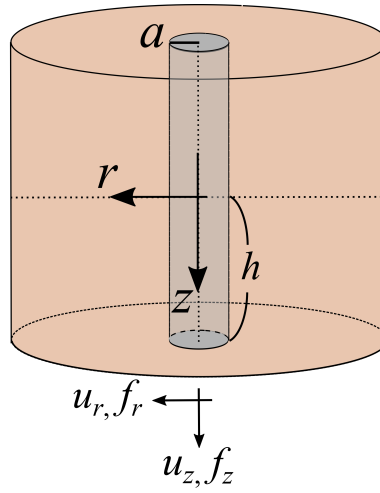
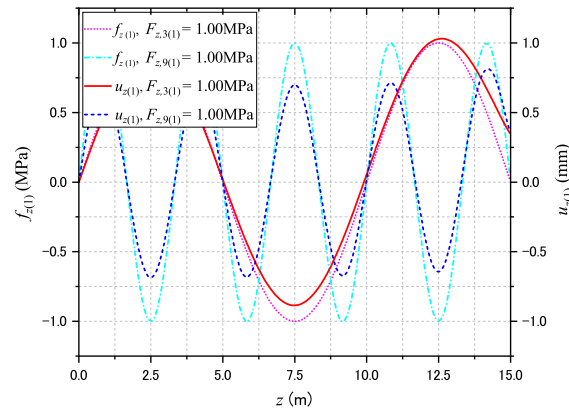
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TABLE 1 Material properties.

Parameters	Soil	Rock	Bolt
Young's modulus (GPa)	0.200	2.00	200
Poisson's ratio	0.400	0.400	0.300
Bolt length h (m)			15.0
Bolt radius a (m)			0.015

**FIGURE 1** Axisymmetric model of a bolt and the surrounding ground with an infinite outer radius.**FIGURE 2** Applied shear loading and the resulting axial displacement at $r = a$ when the ground is soil and there is no bolt.

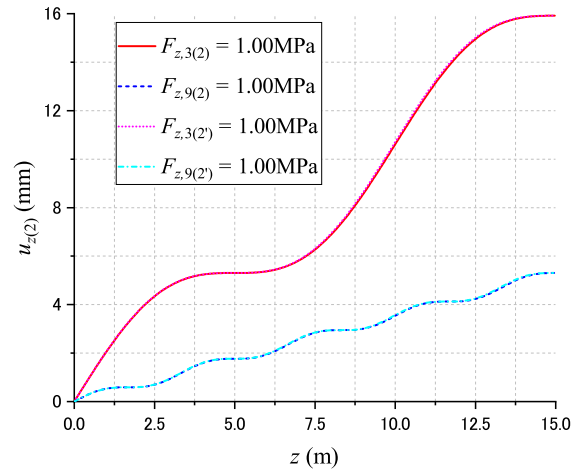


FIGURE 3 Axial bolt displacement at $r = a$ when there is no surrounding ground and shear loading is applied at $r = a$.

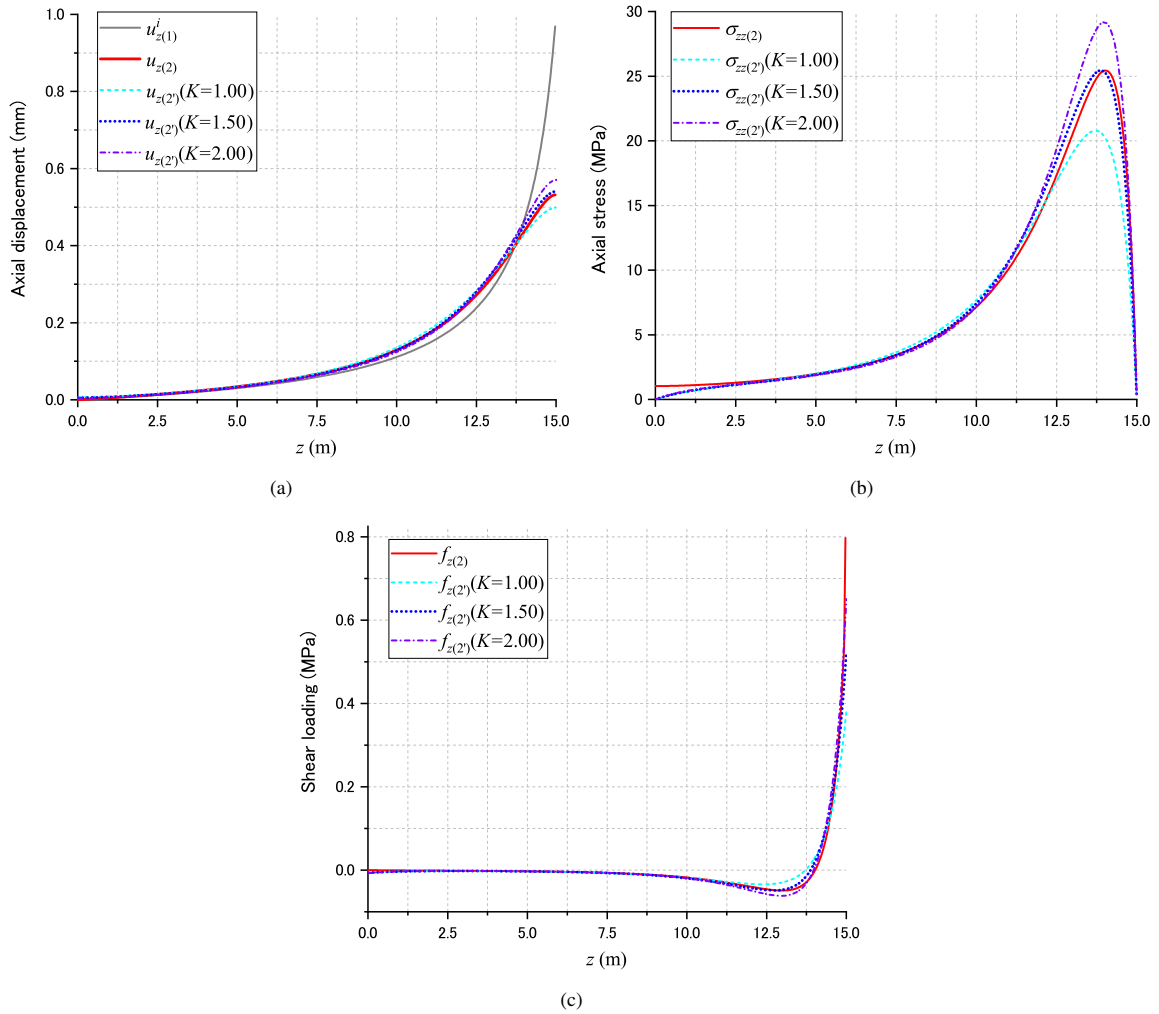


FIGURE 4 Solutions at $r = a$ when the ground is soil and the initial displacement increases monotonically: (a) axial displacement, (b) axial stress, and (c) shear loading.

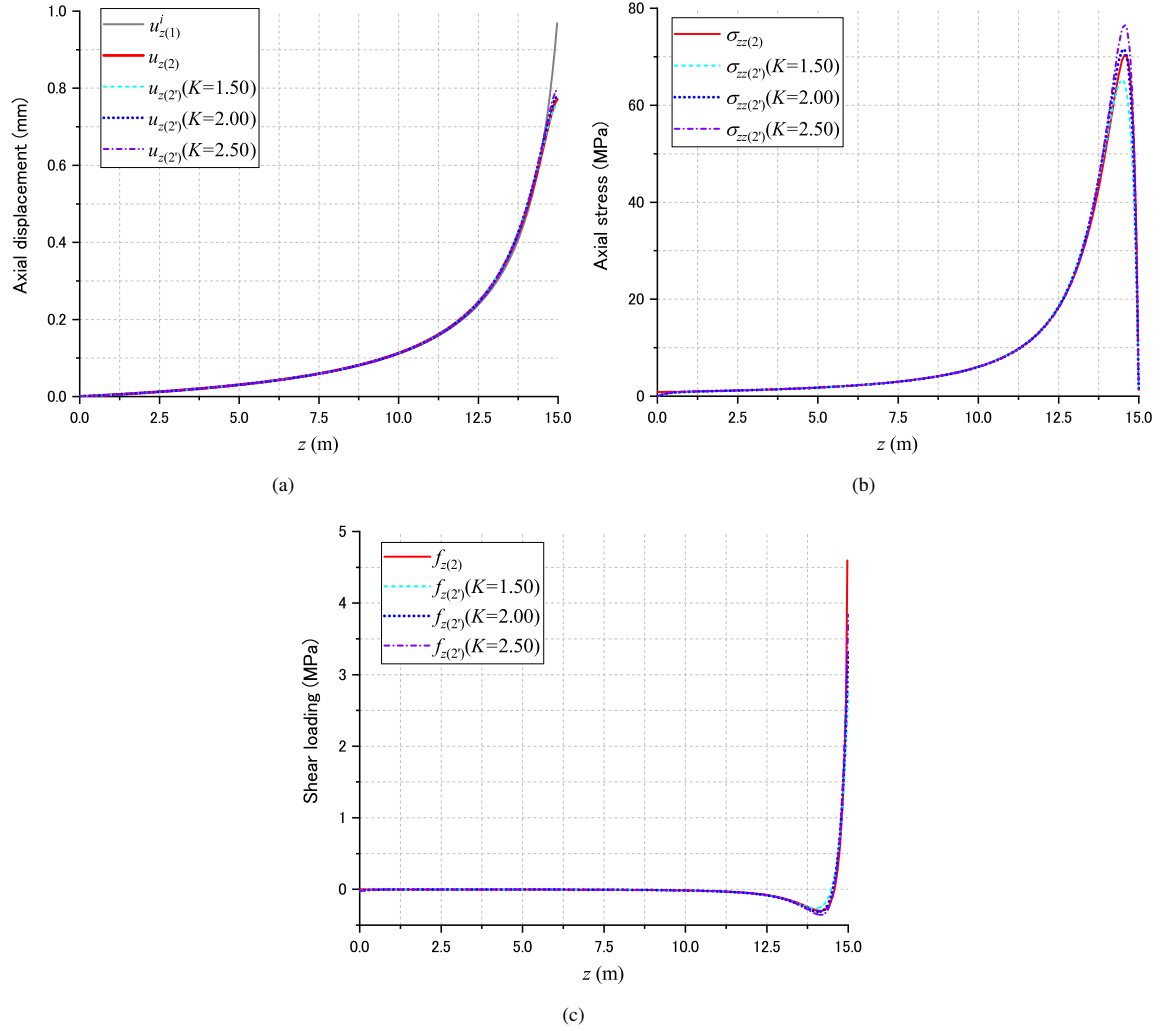


FIGURE 5 Solutions at $r = a$ when the ground is rock and the initial displacement increases monotonically: (a) axial displacement, (b) axial stress, and (c) shear loading.

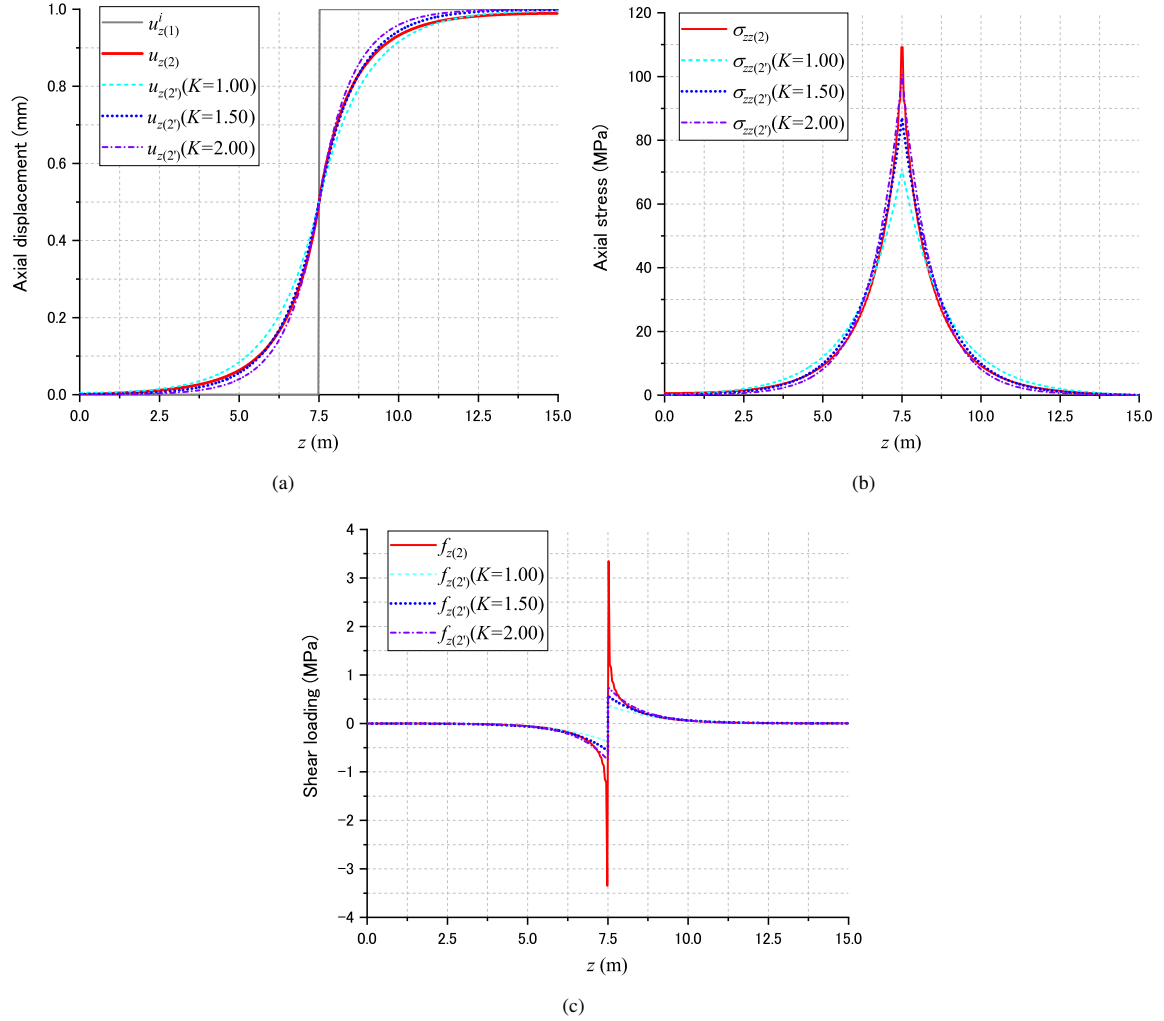


FIGURE 6 Solutions at the ground-bolt interface when the ground is soil and the initial displacement changes abruptly: (a) axial displacement, (b) axial stress, and (c) shear loading.

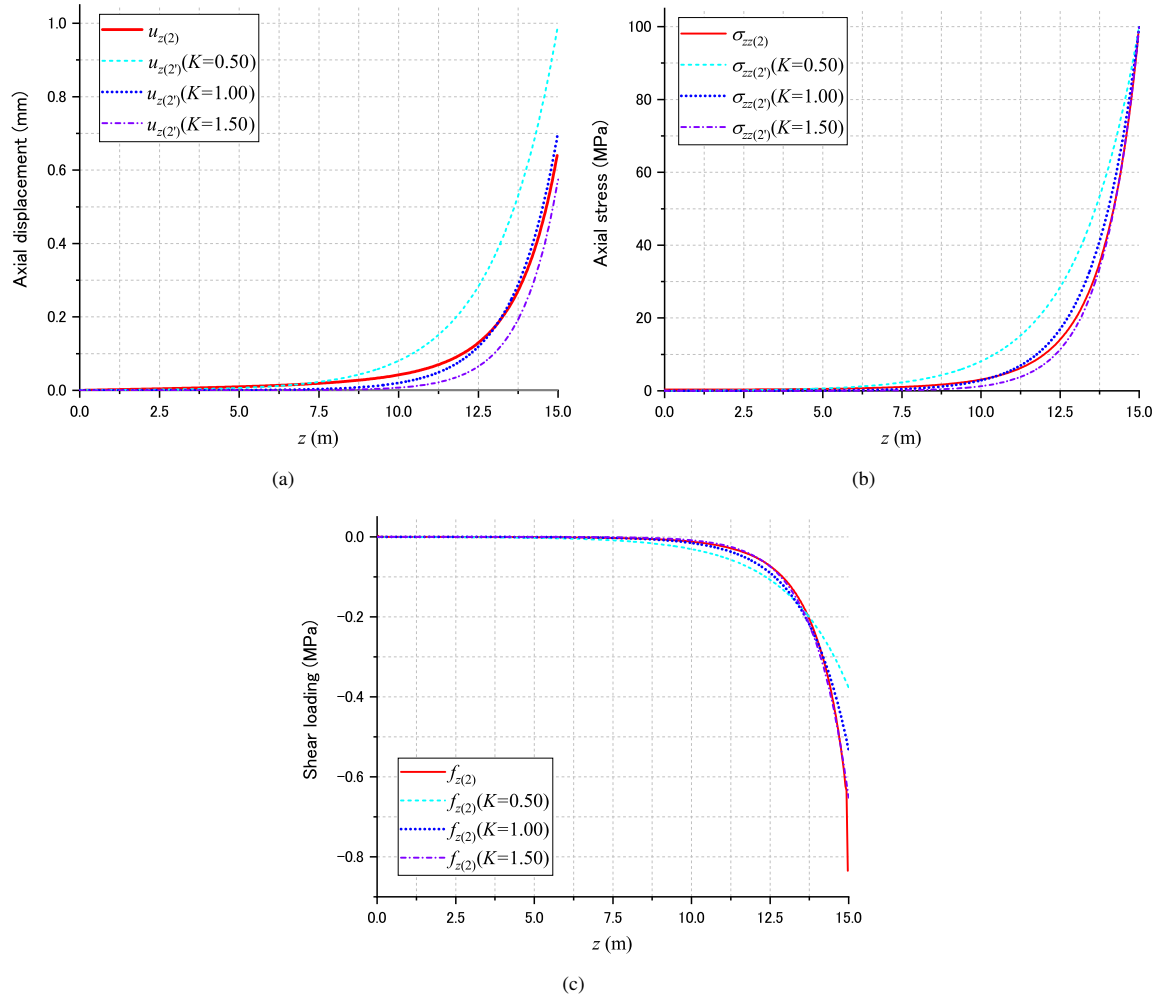


FIGURE 7 Solutions at the ground-bolt interface for a 100 MPa pullout axial stress with no initial ground displacement when the ground is soil: (a) axial displacement, (b) axial stress, and (c) shear loading.

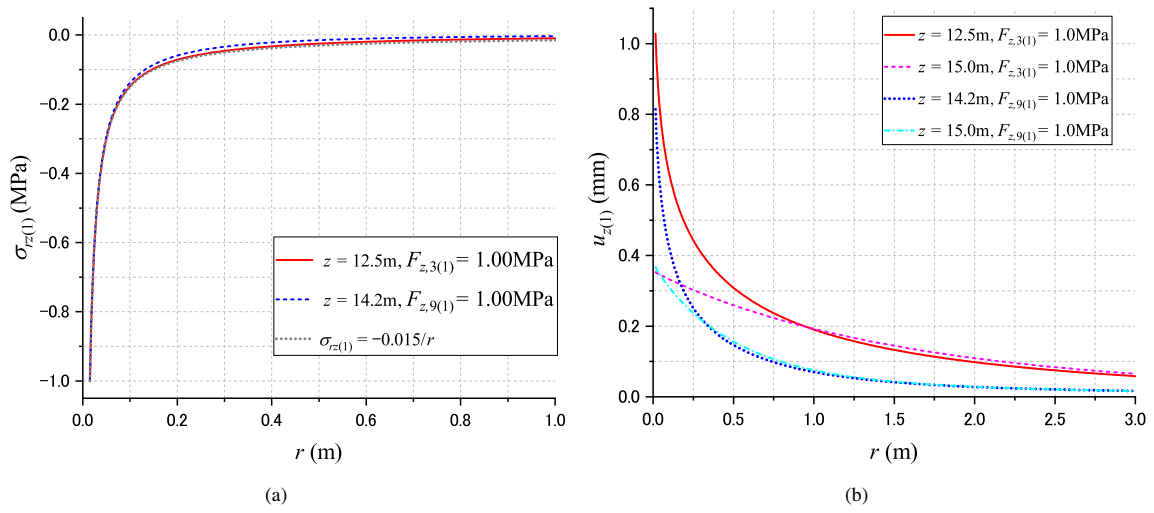


FIGURE 8 Radial distributions of the (a) axial shear stress and (b) axial displacement in the soil when there is no bolt and shear loading is applied at $r = a$.

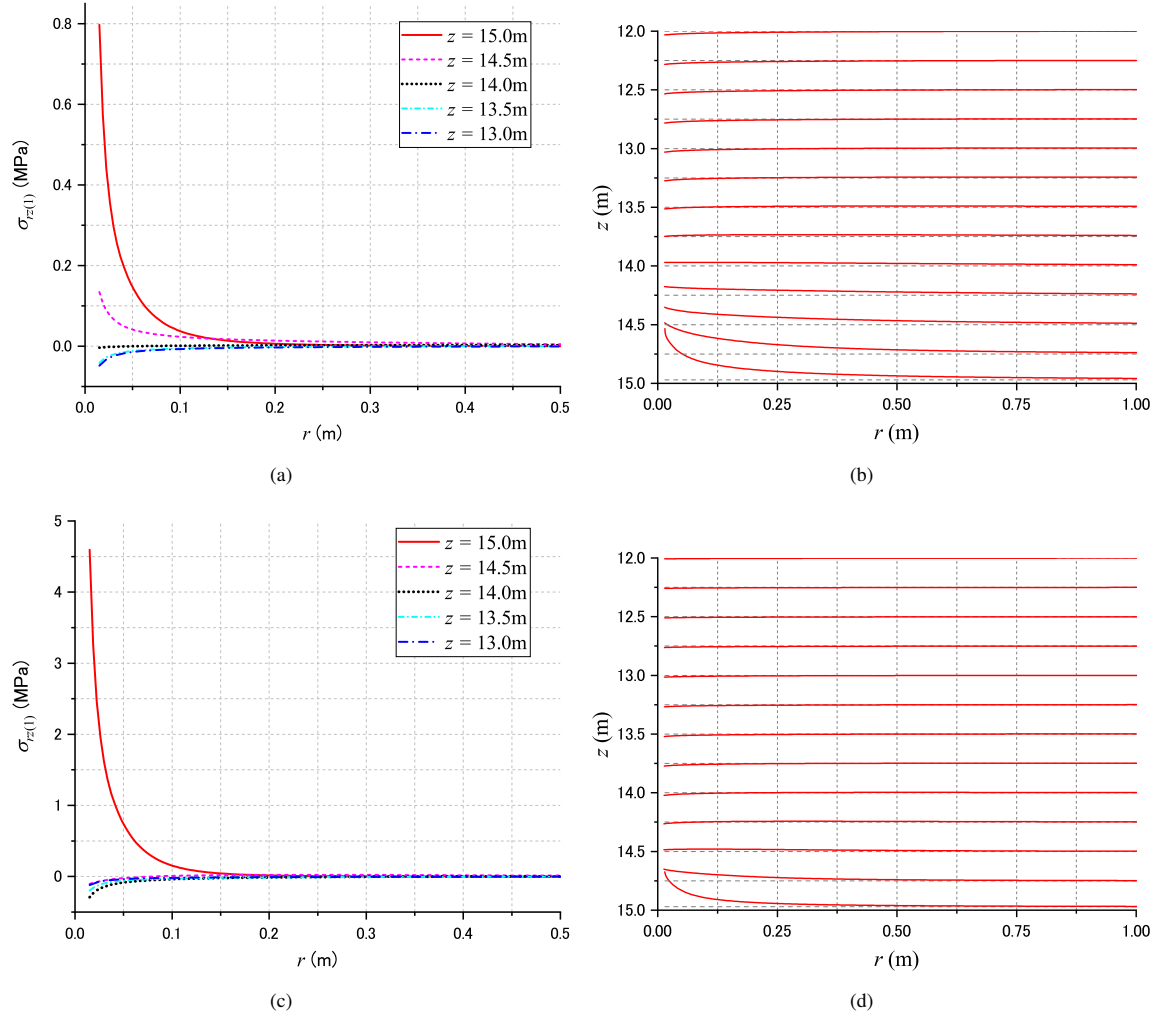


FIGURE 9 Solutions of the secondary ground field when the initial displacement increases monotonically: (a) axial shear stress and (b) axial displacement magnified 1000 times in the soil, and (c) shear stress and (d) axial displacement magnified 1500 times in the rock.

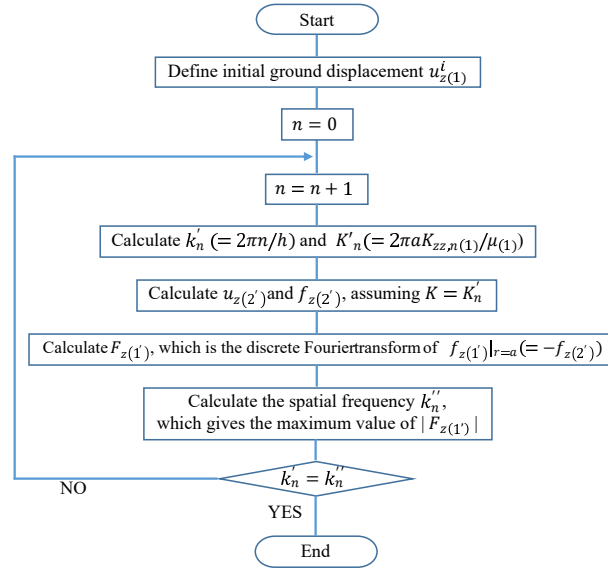


FIGURE 10 Flowchart to obtain solutions an appropriate value of K for the simplified model.

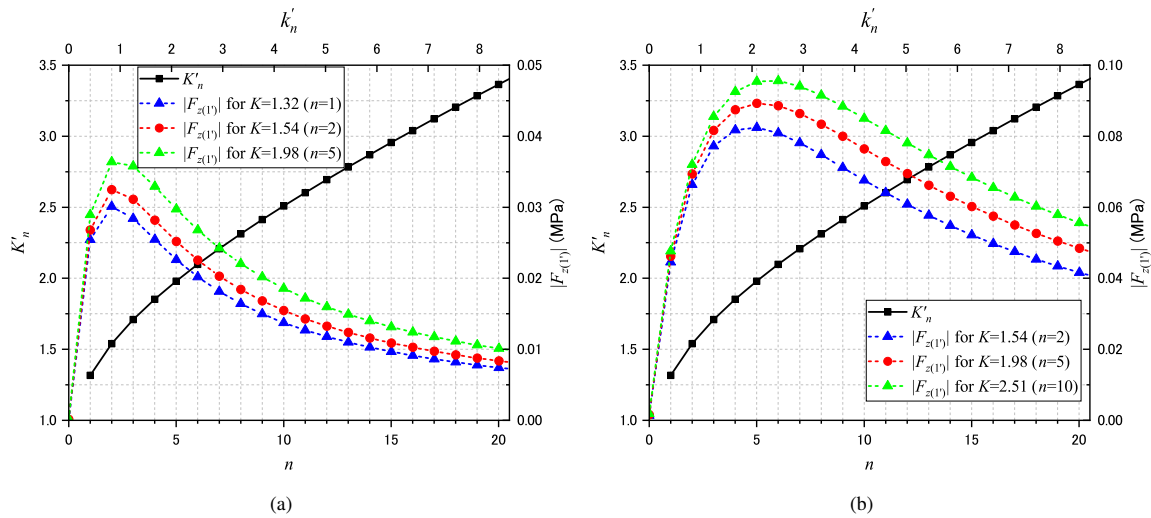


FIGURE 11 Ground stiffness K'_n and Fourier amplitude spectra of $F_{z(1')}$ with varying K for (a) soil and (b) rock. The initial displacement increases monotonically.

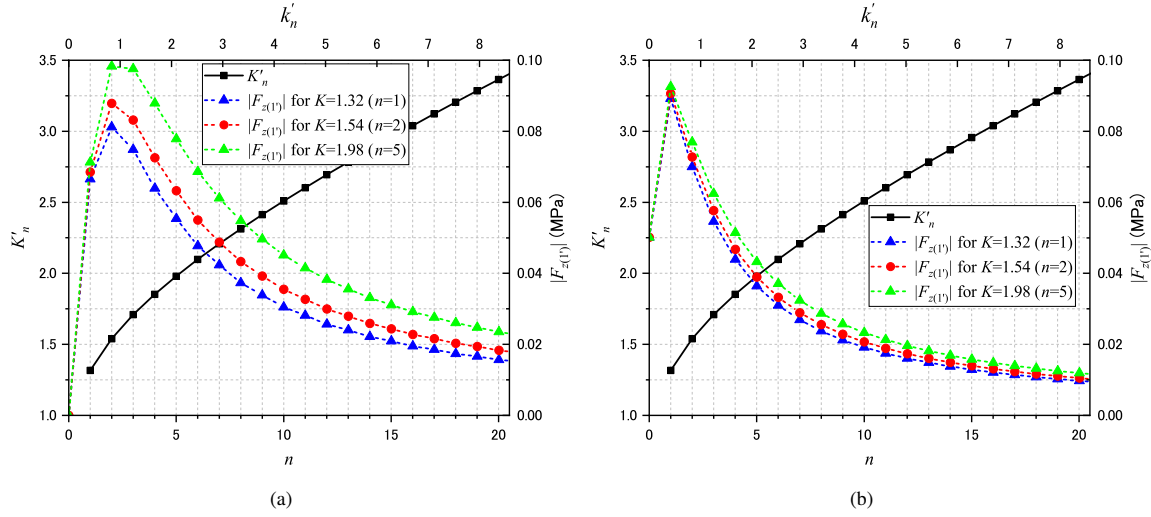


FIGURE 12 Ground stiffness K'_n and Fourier amplitude spectra of $F_{z(1')}$ with varying K for soil. (a) The initial displacement changes abruptly and (b) a 100 MPa pullout axial stress is applied with no initial ground displacement.

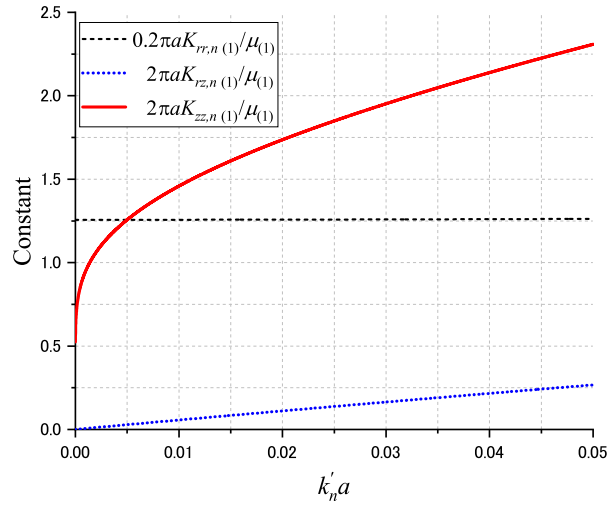


FIGURE 13 Comparison of the stiffness components of the ground.