

PRECISE ASYMPTOTIC FORMULAS FOR NONLINEAR EIGENVALUE PROBLEMS

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1. Introduction. We consider the following nonlinear two-parameter problem

$$\begin{aligned} -u''(x) + \lambda u(x)^q &= \mu u(x)^p, \quad x \in I = (0, 1), \\ u(x) &> 0, \quad x \in I, \\ u(0) &= u(1) = 0, \end{aligned} \tag{1.1}$$

where $1 < q < p$ and $\lambda, \mu > 0$ are parameters.

The purpose of this paper is to establish the asymptotic formulas for the eigencurve $\mu = \mu(\lambda)$ with the exact second term as $\lambda \rightarrow \infty$ by using a variational method. We also establish the critical relationship between p and q from a point of view of the decaying rate of the second term of $\mu(\lambda)$.

In Shibata [8], by using a standard variational framework (see Section 2), the variational eigencurve $\mu = \mu(\lambda)$ was defined to analyze $S_{\lambda, \mu}$ and the following asymptotic formula for $\mu(\lambda)$ as $\lambda \rightarrow \infty$ was established:

$$\mu(\lambda) = C_1 \lambda^{(p+3)/(2p-q+3)} + o(\lambda^{(p+3)/(2p-q+3)}), \tag{1.2}$$

$$C_1 = \left(\frac{(p+1)(q+3)}{(p+3)(q+1)} \frac{1}{\gamma^{p+1}} \frac{2}{p-q} \sqrt{\frac{\pi(q+1)}{2}} \left(\frac{p+1}{q+1} \right)^{\frac{q+3}{2(p-q)}} \frac{\Gamma\left(\frac{q+3}{2(p-q)}\right)}{\Gamma\left(\frac{p+3}{2(p-q)}\right)} \right)^{\frac{p+q-4}{2p-q+3}},$$

$$\Gamma(r) = \int_0^\infty y^{r-1} e^{-y} dy \quad (r > 0).$$
(1.3)

By this formula, we understood the first term of $\mu(\lambda)$ as $\lambda \rightarrow \infty$. However, the remainder estimate of $\mu(\lambda)$ has not been obtained. The purpose here is to obtain *the exact second term* of $\mu(\lambda)$ as $\lambda \rightarrow \infty$. We emphasize that the second term depends deeply on the relationship between p and q , and the critical case is $p = (3q - 1)/2$. As far as the author knows, this kind of criticality is new for two-parameter problems and great interest by itself. Finally, it should be mentioned that the asymptotic behavior of such eigencurve is also effected by the variational framework (cf. [6, 7]).

2. Main Results. Let $H_0^1(I)$ be the usual real Sobolev space. $\|u\|_r$ denotes the usual L^r -norm. For $u \in H_0^1(I)$

$$E_\lambda(u) := \frac{1}{2} \|u'\|_2^2 + \frac{1}{q+1} \lambda \|u\|_{q+1}^{q+1},$$

$$M_\gamma := \{u \in H_0^1(I) : \|u\|_{p+1} = \gamma\},$$

where $\gamma > 0$ is a *fixed constant*. For a given $\lambda > 0$, we call $\mu(\lambda)$ the variational eigenvalue when the following conditions (2.1)-(2.2) are satisfied:

$$(\lambda, \mu(\lambda), u_\lambda) \in \mathbf{R}_+ \times \mathbf{R}_+ \times M_\gamma \text{ satisfies (1.1).} \quad (2.1)$$

$$E_\lambda(u_\lambda) = \inf_{u \in M_\gamma} E_\lambda(u). \quad (2.2)$$

Then $\mu(\lambda)$ is obtained as a Lagrange multiplier and is represented explicitly as follows:

$$\mu(\lambda) = \frac{\|u'_\lambda\|_2^2 + \lambda \|u_\lambda\|_{q+1}^{q+1}}{\gamma^{p+1}}. \quad (2.3)$$

The existence of $\mu(\lambda)$ for a given $\lambda > 0$ is ensured in [8, Theorem 2.1] and $\mu(\lambda)$ is continuous for $\lambda > 0$ (cf. [8, Theorem 2.2]). Finally, let

$$\begin{aligned} K_1 &:= \left(\sqrt{2} \left(\frac{q+1}{p+1} \right)^{(q-1)/(2(p-q))} \frac{\Gamma\left(\frac{1}{q+1}\right) \Gamma\left(\frac{q-1}{2(q+1)}\right)}{\sqrt{\pi(q+1)}} C_1^{(q-1)/(2(p-q))} \right)^{2(q+1)/(q-1)}, \\ K_2 &:= \frac{1}{2} \int_0^1 \frac{s^{(2p-3q-1)/2} (1-s^{p+1})}{(1-s^{p-q})^{3/2}} ds, \\ K_3 &:= \frac{2^{2(p+2)/(q+1)}}{q+1} \int_0^1 \frac{y^{(2p-2q+2)/(q+1)}}{(1+y)^{2(p+2)/(q+1)} (1-y)^{(2p-2q+2)/(q+1)}} dy, \\ J_0 &= \frac{\sqrt{\pi}}{p-q} \frac{q+3}{p+3} \frac{\Gamma\left(\frac{q+3}{2(p-q)}\right)}{\Gamma\left(\frac{p+3}{2(p-q)}\right)}. \end{aligned}$$

Theorem 2.1. (1) Assume $p > (3q-1)/2$. Then the following asymptotic formula holds as $\lambda \rightarrow \infty$:

$$\mu(\lambda) = C_1 \lambda^{(p+3)/(2p-q+3)} \left\{ 1 + C_2 (1 + o(1)) \lambda^{-2(p+1)(q+1)/((2p-q+3)(q-1))} \right\}, \quad (2.4)$$

where

$$C_2 = K_1 \left(1 - \frac{2(p-q)K_2}{(2p-q+3)J_0} \right).$$

(2) Assume $p < (3q-1)/2$. Then as $\lambda \rightarrow \infty$:

$$\mu(\lambda) = C_1 \lambda^{(p+3)/(2p-q+3)} \left\{ 1 - C_3 (1 + o(1)) \lambda^{-(p+1)/(q-1)} \right\}, \quad (2.5)$$

where

$$C_3 = \frac{2(p-q)}{(2p-q+3)J_0} K_3 K_1^{(2p-q+3)/(2(q+1))}.$$

(3) Assume $p = (3q - 1)/2$. Then as $\lambda \rightarrow \infty$:

$$\mu(\lambda) = C_1 \lambda^{(p+3)/(2p-q+3)} \left\{ 1 - C_4(1 + o(1)) \lambda^{-2(p+1)(q+1)/((2p-q+3)(q-1))} \log \lambda \right\}, \quad (2.6)$$

where

$$C_4 = \frac{2(p-q)(p+1)}{(q-1)(2p-q+3)^2 J_0} K_1.$$

The basic idea of the proof is as follows. Put

$$\begin{aligned} \nu(\lambda) &= \lambda^{\frac{p-1}{2(p-q)}} \mu(\lambda)^{\frac{1-q}{2(p-q)}}, \\ w_\lambda(t) &= \left(\frac{\mu(\lambda)}{\lambda} \right)^{\frac{1}{p-q}} u_\lambda(x), \quad t = \nu(\lambda) \left(x - \frac{1}{2} \right). \end{aligned} \quad (2.7)$$

Then it follows from (1.1) that w_λ satisfies

$$\begin{aligned} -w_\lambda''(t) &= w_\lambda(t)^p - w_\lambda(t)^q, \quad t \in I_{\nu(\lambda)} := \left(-\frac{1}{2}\nu(\lambda), \frac{1}{2}\nu(\lambda) \right), \\ w_\lambda(t) &> 0, \quad t \in I_{\nu(\lambda)}, \\ w_\lambda \left(\pm \frac{1}{2}\nu(\lambda) \right) &= 0. \end{aligned} \quad (2.8)$$

Then by [8, Lemma 5.1],

$$\nu(\lambda) \rightarrow \infty \quad (2.9)$$

as $\lambda \rightarrow \infty$. Put $z_\lambda = w_\lambda / \|w_\lambda\|_\infty$. Then it is easy to see from (2.3) that

$$\begin{aligned} \mu(\lambda) &= \frac{\lambda^{(p+3)/(2(p-q))} \mu(\lambda)^{-(q+3)/(2(p-q))} (\|w'_\lambda\|_2^2 + \|w_\lambda\|_{q+1}^{q+1})}{\gamma^{p+1}} \\ &= \frac{\lambda^{(p+3)/(2(p-q))} \mu(\lambda)^{-(q+3)/(2(p-q))} \|w_\lambda\|_{p+1}^{p+1}}{\gamma^{p+1}} \\ &= \frac{\lambda^{(p+3)/(2(p-q))} \mu(\lambda)^{-(q+3)/(2(p-q))} \|w_\lambda\|_\infty^{p+1} \|z_\lambda\|_{p+1}^{p+1}}{\gamma^{p+1}}. \end{aligned} \quad (2.10)$$

Therefore, it is crucial to study the asymptotic behavior of $\|w_\lambda\|_\infty$ and $\|z_\lambda\|_{p+1}$ as

3. Asymptotic behavior of $\|w_\lambda\|_\infty$. We put

$$\|w_\lambda\|_\infty = \left(\frac{p+1}{q+1} (1 + \epsilon(\lambda)) \right)^{1/(p-q)}. \quad (3.1)$$

Then by [8, (5.10), Lemma 5.2], we know that $\epsilon(\lambda) > 0$ and $\epsilon(\lambda) \rightarrow 0$ as $\lambda \rightarrow \infty$.

Lemma 3.1. *The following equality holds for $\lambda > 0$:*

$$\nu(\lambda) = \sqrt{2(q+1)} \left(\frac{p+1}{q+1} (1 + \epsilon(\lambda)) \right)^{-(q-1)/(2(p-q))} L(\epsilon(\lambda)), \quad (3.2)$$

where

$$L(\epsilon) = \int_0^1 \frac{1}{m(\epsilon, s)} ds, \quad (3.3)$$

$$m(\epsilon, s) = \sqrt{s^{q+1} - s^{p+1} + \epsilon(1 - s^{p+1})} \quad (\epsilon > 0).$$

Proof. Multiply the equation in (2.8) by w'_λ . Then for $t \in I_{\nu(\lambda)}$

$$\frac{d}{dt} \left(\frac{1}{2} (w'_\lambda(t))^2 + \frac{1}{p+1} w_\lambda(t)^{p+1} - \frac{1}{q+1} w_\lambda(t)^{q+1} \right) = 0.$$

We know that $w_\lambda(0) = \|w_\lambda\|_\infty$ and $w'_\lambda(0) = 0$, since $u_\lambda(1/2) = \|u_\lambda\|_\infty$ and $u'_\lambda(1/2) = 0$. Then put $t = 0$ to obtain

$$\frac{1}{2} w'_\lambda(t)^2 + \frac{1}{p+1} w_\lambda(t)^{p+1} - \frac{1}{q+1} w_\lambda(t)^{q+1} \equiv \frac{1}{p+1} \|w_\lambda\|_\infty^{p+1} - \frac{1}{q+1} \|w_\lambda\|_\infty^{q+1}.$$

Note that $w'_\lambda(t) < 0$ for $t \in (0, \nu(\lambda)/2)$, since $u'_\lambda(x) < 0$ for $x \in (1/2, 1)$. Then it follows from this and (3.1) that for $t \in (0, \nu(\lambda)/2)$

$$\begin{aligned} -z'_\lambda(t) &= \|w_\lambda\|_\infty^{(q-1)/2} \sqrt{\frac{2}{q+1}} \sqrt{z_\lambda(t)^{q+1} - z_\lambda(t)^{p+1} + \epsilon(\lambda)(1 - z_\lambda(t)^{p+1})} \\ &= \|w_\lambda\|_\infty^{(q-1)/2} \sqrt{\frac{2}{q+1}} m(\epsilon(\lambda), z_\lambda(t)). \end{aligned} \quad (3.4)$$

Put $s = z_\lambda$. Then (3.1) and (3.4) yield

$$\begin{aligned} \frac{\nu(\lambda)}{2} &= \int_0^{\nu(\lambda)/2} \frac{-z'_\lambda(t)}{\sqrt{\frac{2}{q+1}} \|w_\lambda\|_\infty^{(q-1)/2} m(\epsilon(\lambda), z_\lambda(t))} dt \\ &= \sqrt{\frac{q+1}{2}} \left(\frac{p+1}{q+1} (1 + \epsilon(\lambda)) \right)^{-(q-1)/(2(p-q))} \int_0^1 \frac{1}{m(\epsilon(\lambda), s)} ds. \end{aligned}$$

This implies (3.2). $\square$

Lemma 3.2. For $0 < \epsilon \ll 1$

$$L(\epsilon) = \frac{\Gamma\left(\frac{1}{q+1}\right) \Gamma\left(\frac{q-1}{2(q+1)}\right)}{(q+1)\sqrt{\pi}} \epsilon^{-(q-1)/(2(q+1))} + o(\epsilon^{-(q-1)/(2(q+1))}). \quad (3.5)$$

Proof. Put

$$L_1(\epsilon) := L(\epsilon) - \int_0^1 \frac{1}{\sqrt{s^{q+1} + \epsilon}} ds. \quad (3.6)$$

Put $s = \epsilon^{1/(q+1)} \tan^{2/(q+1)} \theta$. Then

$$\begin{aligned} & \int_0^1 \frac{1}{\sqrt{s^{q+1} + \epsilon}} ds \\ &= \frac{2}{q+1} \epsilon^{-(q-1)/(2(q+1))} \int_0^{\tan^{-1}(1/\sqrt{\epsilon})} \sin^{-(q-1)/(q+1)} \theta \cos^{-2/(q+1)} \theta d\theta \\ &= \frac{2}{q+1} (1 + o(1)) \epsilon^{-(q-1)/(2(q+1))} \int_0^{\pi/2} \sin^{-(q-1)/(q+1)} \theta \cos^{-2/(q+1)} \theta d\theta \\ &= \frac{1}{q+1} (1 + o(1)) \epsilon^{-(q-1)/(2(q+1))} \frac{\Gamma\left(\frac{1}{q+1}\right) \Gamma\left(\frac{q-1}{2(q+1)}\right)}{\sqrt{\pi}}. \end{aligned} \quad (3.7)$$

Next, we calculate $L_1(\epsilon)$. Note that for $0 \leq s \leq 1$

$$m(\epsilon, s) = \sqrt{s^{q+1}(1 - s^{p-q}) + \epsilon(1 - s^{p+1})} \geq \sqrt{(s^{q+1} + \epsilon)(1 - s^{p-q})}. \quad (3.8)$$

By this, we obtain

$$\begin{aligned} & |L_1(\epsilon)| \\ &= \int_0^1 \frac{(1 + \epsilon)s^{p+1}}{m(\epsilon, s)\sqrt{s^{q+1} + \epsilon}(m(\epsilon, s) + \sqrt{s^{q+1} + \epsilon})} ds \\ &\leq \int_0^1 \frac{(1 + \epsilon)s^{p+1}}{\sqrt{(s^{q+1} + \epsilon)(1 - s^{p-q})}\sqrt{s^{q+1} + \epsilon}(\sqrt{(s^{q+1} + \epsilon)(1 - s^{p-q})} + \sqrt{s^{q+1} + \epsilon})} ds \\ &\leq (1 + \epsilon) \int_0^1 \frac{s^{p+1}}{(s^{q+1} + \epsilon)^{3/2} \sqrt{1 - s^{p-q}}(1 + \sqrt{1 - s^{p-q}})} ds \\ &\leq 2 \int_0^1 \frac{s^{p+1}}{(s^{q+1} + \epsilon)^{3/2} \sqrt{1 - s^{p-q}}} ds \\ &= 2 \int_0^\delta \frac{s^{p+1}}{(s^{q+1} + \epsilon)^{3/2} \sqrt{1 - s^{p-q}}} ds + 2 \int_\delta^1 \frac{s^{p+1}}{(s^{q+1} + \epsilon)^{3/2} \sqrt{1 - s^{p-q}}} ds \\ &:= I + II, \end{aligned}$$

where $0 < \delta \ll 1$ is a fixed constant. Let $C_{j,\delta} > 0$ ($j = 1, 2, \dots$) be constants depending only on δ . Put $s = \sin^{2/(p-q)} \theta$. Then

$$\begin{aligned} II &\leq \frac{2}{\delta^{3(q+1)/2}} \int_{\delta}^1 \frac{1}{\sqrt{1-s^{p-q}}} ds \\ &= \frac{2}{\delta^{3(q+1)/2}} \frac{2}{p-q} \int_{\sin^{-1} \delta^{(p-q)/2}}^1 \sin^{(2+q-p)/(p-q)} \theta d\theta \\ &\leq C_{1,\delta}. \end{aligned} \quad (3.10)$$

Moreover, put $s = \epsilon^{1/(q+1)} t$. Then for $0 < \epsilon \ll 1$

$$\begin{aligned} I &\leq \frac{2}{\sqrt{1-\delta^{p-q}}} \int_0^{\delta} \frac{\epsilon^{(p+1)/(q+1)} t^{p+1}}{\epsilon^{3/2} (t^{q+1} + 1)^{3/2}} \epsilon^{1/(q+1)} dt \\ &\leq 2 \frac{\delta^{p+1}}{\sqrt{1-\delta^{p-q}}} \epsilon^{(2p-3q+1)/(2(q+1))} = o(\epsilon^{-(q-1)/(2(q+1))}). \end{aligned} \quad (3.11)$$

By (3.9)–(3.11), we have

$$|L_1(\epsilon)| = o(\epsilon^{-(q-1)/(2(q+1))}).$$

By this, (3.6) and (3.7), we obtain (3.5). $\square$

Lemma 3.3. As $\lambda \rightarrow \infty$

$$\epsilon(\lambda) = K_1(1 + o(1)) \lambda^{-2(p+1)(q+1)/((q-1)(2p-q+3))}. \quad (3.12)$$

Proof. By (1.2) and (2.7), we have

$$\begin{aligned} \nu(\lambda) &= \lambda^{(p-1)/(2(p-q))} \mu(\lambda)^{(1-q)/(2(p-q))} \\ &= C_1^{(1-q)/(2(p-q))} (1 + o(1)) \lambda^{(p+1)/(2p-q+3)}. \end{aligned} \quad (3.13)$$

On the other hand, by Lemmas 3.1–3.2 and Taylor expansion, we have

$$\begin{aligned} \nu(\lambda) &= \sqrt{2(q+1)} \left(\frac{p+1}{q+1} \right)^{-(q-1)/(2(p-q))} (1 + \epsilon(\lambda))^{-(q-1)/(2(p-q))} L(\epsilon(\lambda)) \\ &= \sqrt{2} \left(\frac{p+1}{q+1} \right)^{-(q-1)/(2(p-q))} \frac{\Gamma\left(\frac{1}{q+1}\right) \Gamma\left(\frac{q-1}{2(q+1)}\right)}{\sqrt{\pi(q+1)}} \epsilon(\lambda)^{-(q-1)/(2(q+1))} (1 + o(1)). \end{aligned}$$

By this and (3.13), we obtain (3.12). $\square$

4. Asymptotic behavior of $\|z_\lambda\|_{p+1}$. By (3.4) and putting $s = z_\lambda(t)$, we have

$$\begin{aligned} \|z_\lambda\|_{p+1}^{p+1} &= 2 \int_0^{\nu(\lambda)/2} z_\lambda(t)^{p+1} dt \\ &= 2 \int_0^{\nu(\lambda)/2} z_\lambda(t)^{p+1} \frac{-z'_\lambda(t)}{\|w_\lambda\|_\infty^{(q-1)/2} \sqrt{\frac{2}{q+1} m(\epsilon(\lambda), z_\lambda(t))}} dt \\ &= \frac{\sqrt{2(q+1)}}{\|w_\lambda\|_\infty^{(q-1)/2}} J(\epsilon(\lambda)), \end{aligned} \quad (4.1)$$

where

$$J(\epsilon) := \int_0^1 \frac{s^{p+1}}{m(\epsilon, s)} ds \quad (\epsilon > 0). \quad (4.2)$$

Therefore, we study the precise asymptotics of $J(\epsilon)$ as $\epsilon \rightarrow 0$. Put $s = \sin^{2/(p-q)} \theta$.

Then as $\epsilon \rightarrow 0$

$$\begin{aligned} J(\epsilon) &\rightarrow J(0) = \int_0^1 \frac{s^{(2p-q+1)/2}}{\sqrt{1-s^{p-q}}} ds \\ &= \frac{2}{p-q} \int_0^{\pi/2} \sin^{(p+3)/(p-q)} \theta d\theta \\ &= \frac{\sqrt{\pi}}{p-q} \frac{q+3}{p+3} \frac{\Gamma\left(\frac{q+3}{2(p-q)}\right)}{\Gamma\left(\frac{p+3}{2(p-q)}\right)} \\ &= J_0. \end{aligned} \quad (4.3)$$

We use here the formulas

$$\begin{aligned} \int_0^{\pi/2} \sin^r \theta d\theta &= \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{r+1}{2}\right)}{\Gamma\left(\frac{r}{2}+1\right)} \quad (r > -1), \\ \Gamma(r+1) &= r\Gamma(r). \end{aligned} \quad (4.4)$$

Therefore, put

$$\begin{aligned} J_1(\epsilon) &:= J(\epsilon) - J_0 := -\epsilon J_2(\epsilon), \\ J_2(\epsilon) &:= \int_0^1 \frac{s^{p+1}(1-s^{p+1})}{m(\epsilon, s)m(0, s)(m(\epsilon, s)+m(0, s))} ds. \end{aligned} \quad (4.5)$$

We study the asymptotic behavior of $J_2(\epsilon)$ as $\epsilon \rightarrow 0$.

Lemma 4.1. (1) If $p > (3q - 1)/2$, then $J_2(\epsilon) \rightarrow K_2$ as $\epsilon \rightarrow 0$.

(2) If $p < (3q - 1)/2$, then as $\epsilon \rightarrow 0$

$$J_2(\epsilon) = K_3(1 + o(1))\epsilon^{(2p-3q+1)/(2(q+1))}. \quad (4.6)$$

(3) If $p = (3q - 1)/2$, then as $\epsilon \rightarrow 0$

$$J_2(\epsilon) = -\frac{1}{2(q+1)}(1 + o(1))\log \epsilon. \quad (4.7)$$

Proof. (1) Since $p > (3q - 1)/2$, we have $(2p - 3q - 1)/2 > -1$. Therefore, by Lebesgue's convergence theorem, as $\epsilon \rightarrow 0$

$$J_2(\epsilon) \rightarrow \frac{1}{2} \int_0^1 \frac{s^{(2p-3q-1)/2}(1 - s^{p+1})}{(1 - s^{p-q})^{3/2}} ds = K_2.$$

This completes the proof.

(2) *Step 1.* Assume that $p < (3q - 1)/2$. We introduce $J_3(\epsilon)$ to approximate $J_2(\epsilon)$:

$$\begin{aligned} J_3(\epsilon) &:= \int_0^1 \frac{s^{(2p-q+1)/2}}{\sqrt{s^{q+1} + \epsilon}(s^{(q+1)/2} + \sqrt{s^{q+1} + \epsilon})} ds \\ &= J_4(\epsilon, \delta) + J_5(\epsilon, \delta) \\ &:= \int_0^\delta \frac{s^{(2p-q+1)/2}}{\sqrt{s^{q+1} + \epsilon}(s^{(q+1)/2} + \sqrt{s^{q+1} + \epsilon})} ds \\ &\quad + \int_\delta^1 \frac{s^{(2p-q+1)/2}}{\sqrt{s^{q+1} + \epsilon}(s^{(q+1)/2} + \sqrt{s^{q+1} + \epsilon})} ds, \end{aligned} \quad (4.8)$$

where $0 < \delta \ll 1$ is a fixed small constant. We study the asymptotic behavior of J_3, J_4 and J_5 as $\epsilon \rightarrow 0$. Note that $0 < (2p - 2q + 2)/(q + 1) < 1$, since $p < (3q - 1)/2$. Then put $s = \epsilon^{1/(q+1)} \tan^{2/(q+1)} \theta$ and $y = \tan(\theta/2)$ to obtain

$$\begin{aligned} J_3(\epsilon) &= \frac{2}{q+1} \epsilon^{(2p-3q+1)/(2(q+1))} \int_0^{\tan^{-1} 1/\sqrt{\epsilon}} \frac{\tan^{(2p-2q+2)/(q+1)} \theta}{1 + \sin \theta} d\theta \\ &= K_3(1 + o(1))\epsilon^{(2p-3q+1)/(2(q+1))}. \end{aligned} \quad (4.9)$$

Similarly, we obtain

$$J_4(\epsilon, \delta) = K_3(1 + o(1))\epsilon^{(2p-3q+1)/(2(q+1))}, \quad J_5(\epsilon, \delta) \leq \frac{1}{\delta^{q+1}}. \quad (4.10)$$

Since $p < (3q-1)/2$, this along with (4.9) implies that $J_3(\epsilon)/J_4(\epsilon, \delta) \rightarrow 1$ as $\epsilon \rightarrow 0$ for a fixed δ .

Step 2. We show that as $\epsilon \rightarrow 0$

$$\frac{J_2(\epsilon)}{J_3(\epsilon)} \rightarrow 1. \quad (4.11)$$

Let an arbitrary $0 < \delta \ll 1$ be fixed. Put

$$\begin{aligned} J_2(\epsilon) &= J_6(\epsilon, \delta) + J_7(\epsilon, \delta) \\ &:= \int_0^\delta \frac{s^{p+1}(1-s^{p+1})}{m(\epsilon, s)m(0, s)(m(\epsilon, s) + m(0, s))} ds \\ &\quad + \int_\delta^1 \frac{s^{p+1}(1-s^{p+1})}{m(\epsilon, s)m(0, s)(m(\epsilon, s) + m(0, s))} ds. \end{aligned} \quad (4.12)$$

Then for $0 < \epsilon \ll 1$

$$|J_7(\epsilon, \delta)| \leq C_{2,\delta} \int_\delta^1 \frac{1-s^{p+1}}{(1-s^{p-q})^{3/2}} ds \leq C_{3,\delta}. \quad (4.13)$$

Moreover, by (3.8), we obtain

$$\begin{aligned} (1-\delta^{p+1}) \int_0^\delta \frac{s^{(2p-q+1)/2}}{\sqrt{s^{q+1} + \epsilon}(s^{(q+1)/2} + \sqrt{s^{q+1} + \epsilon})} ds &\leq J_6(\epsilon, \delta) \\ &\leq \frac{1}{(1-\delta^{p-q})^{3/2}} \int_0^\delta \frac{s^{(2p-q+1)/2}}{\sqrt{s^{q+1} + \epsilon}(s^{(q+1)/2} + \sqrt{s^{q+1} + \epsilon})} ds. \end{aligned}$$

This implies

$$(1-\delta^{p+1})J_4(\epsilon, \delta) \leq J_6(\epsilon, \delta) \leq \frac{1}{(1-\delta^{p-q})^{3/2}}J_4(\epsilon, \delta). \quad (4.14)$$

By (4.10), (4.13) and (4.14), we see that $J_7(\epsilon, \delta) = o(J_6(\epsilon, \delta))$ as $\epsilon \rightarrow 0$ for a fixed δ , since $p < (3q-1)/2$. Then by (4.9), (4.10) and (4.12)–(4.14),

$$\begin{aligned} (1-\delta^{p+1}) &\leq \liminf_{\epsilon \rightarrow 0} \frac{J_6(\epsilon, \delta)}{J_4(\epsilon, \delta)} = \liminf_{\epsilon \rightarrow 0} \frac{J_2(\epsilon)}{J_3(\epsilon)} \leq \limsup_{\epsilon \rightarrow 0} \frac{J_2(\epsilon)}{J_3(\epsilon)} \\ &= \limsup_{\epsilon \rightarrow 0} \frac{J_6(\epsilon, \delta)}{J_4(\epsilon, \delta)} \leq \frac{1}{(1-\delta^{p-q})^{3/2}}. \end{aligned} \quad (4.15)$$

By letting $\delta \rightarrow 0$, we obtain (4.11). Then by (4.9) and (4.11), we obtain (4.6).

(3) If $p = (3q - 1)/2$, then by the asymptotic formula

$$\tan^{-1} x = \frac{\pi}{2} - \frac{1}{x} + O\left(\frac{1}{x^3}\right) \quad (x \gg 1),$$

and Taylor expansion of $\tan x$ at $x = \pi/4$ and (4.9), we obtain (4.7) by direct calculation. $\square$

5. Proof of Theorem 2.1. By (2.10), (3.1), (4.1) and (4.5), we have

$$\begin{aligned} \mu(\lambda)^{(2p-q+3)/(2(p-q))} &= \frac{\sqrt{2(q+1)}}{\gamma^{p+1}} \lambda^{(p+3)/(2(p-q))} \|w_\lambda\|_\infty^{(2p-q+3)/2} J(\epsilon(\lambda)) \\ &= \frac{\sqrt{2(q+1)}}{\gamma^{p+1}} \lambda^{(p+3)/(2(p-q))} \left(\frac{p+1}{q+1}\right)^{(2p-q+3)/(2(p-q))} \\ &\quad \times (1 + \epsilon(\lambda))^{(2p-q+3)/(2(p-q))} (J_0 - \epsilon(\lambda) J_2(\epsilon(\lambda))). \end{aligned} \quad (5.1)$$

Moreover, it is easy to check that

$$\left(\frac{\sqrt{2(q+1)}}{\gamma^{p+1}}\right)^{2(p-q)/(2p-q+3)} \frac{p+1}{q+1} J_0^{2(p-q)/(2p-q+3)} = C_1.$$

By this, (5.1) and Taylor expansion, we obtain

$$\begin{aligned} \mu(\lambda) &= C_1 \lambda^{(p+3)/(2p-q+3)} \\ &\quad \times \left(1 + \epsilon(\lambda) - \frac{2(p-q)}{(2p-q+3)J_0} (1 + o(1)) \epsilon(\lambda) J_2(\epsilon(\lambda))\right). \end{aligned} \quad (5.2)$$

Then by Lemma 3.3, Lemma 4.1 and direct calculation, we obtain Theorem 2.1. Thus the proof is complete. $\square$

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