

# On Some Invariants of Persistence Modules over Posets and Their Applications

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## Abstract

Defining meaningful and computationally efficient invariants is crucial in multi-parameter persistent homology, and more generally, for persistence modules over arbitrary posets. Interval modules form a well-structured subset of indecomposables, facilitating both visualization and practical data analysis. Focusing on interval-based invariants gives us a tractable framework that balances both computational feasibility and theoretical insight.

In this thesis, we study several interval-based invariants of persistence modules over finite posets. We present explicit formulas for two such invariants: the interval multiplicity and the interval rank invariant. Furthermore, we introduce the essential-cover technique, which enables efficient computation of these invariants by translating them to other underlying posets, such as zigzag posets, for which fast decomposition algorithms exist. This approach makes it possible to compute the invariants directly from the filtration level.

As an example of applying the theory, we provide a decomposition method for the bipath persistence by utilizing the essential-cover technique. Moreover, we show the algebraic stability of bipath modules by constructing a covering map and linking to zigzag modules.

Finally, we define an additional invariant called the interval replacement, which preserves all interval ranks while offering a compact and expressive form for distinguishing persistence modules.

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# Contents

|          |                                                                       |            |
|----------|-----------------------------------------------------------------------|------------|
| <b>1</b> | <b>Introduction</b>                                                   | <b>1</b>   |
| 1.1      | Notation conventions . . . . .                                        | 2          |
| 1.2      | Purposes and results . . . . .                                        | 2          |
| 1.3      | Organization . . . . .                                                | 7          |
| <b>2</b> | <b>Preliminaries</b>                                                  | <b>9</b>   |
| 2.1      | Quivers . . . . .                                                     | 9          |
| 2.2      | Posets and categories . . . . .                                       | 10         |
| 2.3      | Incidence algebra and Möbius inversion . . . . .                      | 16         |
| 2.4      | Persistent homology . . . . .                                         | 17         |
| 2.5      | Persistence modules . . . . .                                         | 20         |
| 2.6      | Almost split sequences . . . . .                                      | 27         |
| <b>3</b> | <b>Formulas for interval multiplicities</b>                           | <b>33</b>  |
| 3.1      | The general poset case . . . . .                                      | 34         |
| 3.1.1    | The case where $V_I$ is injective . . . . .                           | 43         |
| 3.1.2    | The case where $V_I$ is non-injective . . . . .                       | 44         |
| 3.2      | The case of 2D-grids . . . . .                                        | 49         |
| 3.2.1    | The case where $V_I$ is injective . . . . .                           | 50         |
| 3.2.2    | The case where $V_I$ is non-injective . . . . .                       | 51         |
| <b>4</b> | <b>Essential-cover technique</b>                                      | <b>55</b>  |
| 4.1      | Definition of essential covers . . . . .                              | 55         |
| 4.2      | Bipath persistence as zigzag persistence . . . . .                    | 64         |
| 4.2.1    | Fast decomposition of bipath persistence . . . . .                    | 64         |
| 4.2.2    | Algebraic stability of bipath persistence . . . . .                   | 70         |
| <b>5</b> | <b>Interval rank invariants</b>                                       | <b>77</b>  |
| 5.1      | Algebraic invariants . . . . .                                        | 77         |
| 5.2      | Compression systems and multiplicities . . . . .                      | 79         |
| 5.3      | Interval rank invariant: a generalization of rank invariant . . . . . | 83         |
| 5.3.1    | Definition of interval rank invariants . . . . .                      | 83         |
| 5.3.2    | Formulas for interval rank invariants . . . . .                       | 84         |
| 5.3.3    | Essential covers relative to compression systems . . . . .            | 89         |
| <b>6</b> | <b>Interval replacements</b>                                          | <b>99</b>  |
| 6.1      | Signed interval multiplicities . . . . .                              | 99         |
| 6.2      | Interval replacement: an additional invariant . . . . .               | 103        |
| 6.3      | Examples . . . . .                                                    | 106        |
| 6.3.1    | Computing the maximal interval-decomposable summand . . . . .         | 106        |
| 6.3.2    | Interval replacement is incomplete . . . . .                          | 106        |
| 6.3.3    | Fibered bar and arc codes versus interval replacements . . . . .      | 108        |
|          | <b>Bibliography</b>                                                   | <b>113</b> |



# Chapter 1

## Introduction

Persistent homology has been invented and well developed in recent decades, regarded as one of the main tools in topological data analysis (TDA for short) [36]. A *filtration* is a functor from an ordered set, viewed as a category, to the category of topological spaces. Applying homology to a *one-parameter* filtration yields a representation of a totally ordered set; this representation is commonly referred to as the *one-dimensional persistence module* in the literature [12, 48]. With the study of *zigzag persistence*, representations of a zigzag partially ordered set (zigzag poset for short) are proposed and called *zigzag modules* [19, 21]. These two types of persistence modules are uniquely determined by all intervals of underlying ordered sets of filtrations [27]. On the other hand, one can distinguish input datasets in the way of examining invariants of associated persistence modules, such as the *multiplicity invariant* (i.e., *persistence diagram* or *barcodes*), or equivalently, the *rank invariant* [22], both of which are defined in terms of intervals.

Many geometric models in TDA, such as the subsampling model [26], the multicover model [16, 47], and the chromatic alpha complexes [32], involve complicated underlying posets of filtrations. This scenario extends one-parameter persistence to a multi-parameter framework, leading to the concept of the multi-dimensional persistence module [22]. Except for a few cases, the category of multi-dimensional persistence modules has infinitely many indecomposables up to isomorphism [13, 42], causing inefficiencies in handling all indecomposable modules in practice. Furthermore, defining meaningful and computationally feasible invariants of multi-parameter persistence remains a challenge.

To address these issues, one may restrict attention to a finite subset of indecomposables and propose invariants based on those selected indecomposables. One tractable approach is choosing interval indecomposables (entirely or partially) because intervals can be interpreted as the lifetime of topological features in the data and can be characterized easily, analogous to the one-parameter case.

Many works contribute to defining invariants for multi-dimensional persistence modules in this spirit. Carlsson and Zomorodian [22] propose the rank invariant also for the multi-dimensional persistence module. Lesnick and Wright [41] introduce the *fibred barcode* and *multigraded Betti numbers* of 2-dimensional persistence modules. Kim and Mémoli [40] introduce the *generalized rank invariant* (GRI for short) and *generalized persistence diagrams* of persistence modules over a locally finite, connected poset. Later, Clause et al. [25] introduce the *zigzag-path-indexed barcode* for  $\mathbb{Z}^2$ -modules and show the relation between GRI with this barcode invariant. Asashiba et al. [5, 6] characterize intervals of 2D-grids and propose an *interval-decomposable replacement* of persistence modules over these grids. Botnan et al. [20] decompose persistence modules in terms of the generalized ranks and propose the *signed barcodes*. Blanchette et al. [14] introduce the *dim-hom invariant* and the concept of *homological invariants*, along with a framework for comparing various invariants. Hiraoka et al. [39] introduce the *(partial) interval approximation*

*invariant* of fully commutative quiver representations to balance the resolution of the approximation against computational cost (the number of examined intervals). Amiot et al. [1] introduce several invariants of persistence modules over finite posets by utilizing order-embedding maps.

In the thesis, we will study persistence theory within the general framework of filtrations indexed by arbitrary finite posets.

## 1.1 Notation conventions

We fix a field  $\mathbb{k}$ . The category of finite-dimensional  $\mathbb{k}$ -vector spaces is denoted by  $\text{mod } \mathbb{k}$ . Let  $\mathbf{P}$  be a poset and let  $\mathbf{I}$  be a nonempty full subposet of  $\mathbf{P}$ . By  $\text{sc}(\mathbf{I})$  (resp.  $\text{sk}(\mathbf{I})$ ) we denote the set of minimal (resp. maximal) elements of  $\mathbf{I}$ . Note that  $\text{sc}(\mathbf{I})$  and  $\text{sk}(\mathbf{I})$  are antichains with respect to the original partial order in  $\mathbf{P}$ . If  $\text{sc}(\mathbf{I})$  is a finite set, we can label elements of  $\text{sc}(\mathbf{I})$  as  $a_1, \dots, a_n$  and endow  $\text{sc}(\mathbf{I})$  with an additional total order  $\preceq$  defined by the natural number ordering of the subscripts. Similarly, for  $\text{sk}(\mathbf{I})$  we label and order elements as  $b_1, \dots, b_m$  if it is a finite set. See Figure 1.1 for an illustration of a nonempty full subposet  $\mathbf{I}$  of  $\mathbf{P}$ , whose minimal elements lie in a lower region enclosed by black dotted lines and whose maximal elements lie in another higher region likewise enclosed by black dotted lines. We artificially set  $\text{sc}(\emptyset) = \text{sk}(\emptyset) = \emptyset$ . An *interval* of the poset  $\mathbf{P}$  is a nonempty connected and convex full subposet of  $\mathbf{P}$ . The set of all intervals of  $\mathbf{P}$  will be denoted by  $\mathbb{I}$ .

In what follows, a filtration indexed by  $\mathbf{P}$  (**P-filtration** for short) is defined as a functor  $\mathcal{F}: \mathbf{P} \rightarrow \mathbf{FinSimp}$ , where  $\mathbf{FinSimp}$  denotes the category of finite simplicial complexes. We define the  $i$ th persistent homology as the composition  $H_i(-; \mathbb{k}) \circ \mathcal{F}$ , where  $H_i(-; \mathbb{k}): \mathbf{FinSimp} \rightarrow \text{mod } \mathbb{k}$  denotes the  $i$ th homology functor with coefficients in  $\mathbb{k}$ .

We study the persistent homology in the representation framework. We call  $M$  a persistence module over  $\mathbf{P}$  if  $M: \mathbb{k}[\mathbf{P}] \rightarrow \text{mod } \mathbb{k}$  is a  $\mathbb{k}$ -linear functor starting from the *incidence category*  $\mathbb{k}[\mathbf{P}]$ . For all  $a, b \in \mathbf{P}$ , we write  $M_{b,a}: M(a) \rightarrow M(b)$  to denote the unique *structure linear map* of  $M$  from  $a$  to  $b$ . By  $\text{mod } \mathbb{k}[\mathbf{P}]$  we denote the category of persistence modules over  $\mathbf{P}$ . By  $V_{\mathbf{I}}$  we denote the interval module supported on  $\mathbf{I}$ . By the Krull–Remak–Schmidt–Azumaya Theorem, every persistence module  $M \in \text{mod } \mathbb{k}[\mathbf{P}]$  is uniquely decomposed into indecomposables up to isomorphism, which gives the multiplicity of each indecomposable  $L$ , denoted by  $d_M(L)$ , in the decomposition of  $M$  (see Theorem 2.5.15).

## 1.2 Purposes and results

In the thesis, we follow the spirit mentioned above and also restrict ourselves to interval modules. We propose a general framework for defining invariants of persistence modules over any finite poset and provide explicit formulas for computing the invariants from the structure linear maps of given persistence modules. In turn, we introduce a potential technique called the *essential cover* for computing those invariants possibly from the filtration level.

We begin by studying the multiplicity invariant of persistence modules: the function that assigns to each indecomposable summand its multiplicity in the modules decomposition. This invariant plays an important role in relating persistence modules to other algebraic invariants. In particular, we derive an explicit formula for the *interval multiplicity*  $d_M(V_I)$ —the multiplicity of each interval module  $V_I$ —in any persistence module  $M$  over finite posets. To state what follows, we now introduce the notations used in the formula.

**Notation 1.2.1.** Suppose  $\mathbf{P}$  is a finite poset and let  $\mathbf{I}$  be an interval of  $\mathbf{P}$ .

- (1) For  $x \in \mathbf{P}$ , we set  $\uparrow x := \{y \in \mathbf{P} \mid x \leq y\}$  and  $\downarrow x := \{y \in \mathbf{P} \mid y \leq x\}$ . In turn, we set

$$\uparrow \mathbf{I} := \bigcup_{x \in \mathbf{I}} \uparrow x \text{ (dually, } \downarrow \mathbf{I} := \bigcup_{x \in \mathbf{I}} \downarrow x), \text{ and } \uparrow \mathbf{I} \setminus \mathbf{I} \text{ (dually, } \downarrow \mathbf{I} \setminus \mathbf{I}).$$

- (2) For any totally ordered set  $\mathbf{T}$ , we denote by  $C_2 \mathbf{T}$  the set of all full subposets of  $\mathbf{T}$  consisting of exactly two elements. For any  $\mathbf{a} \in C_2 \mathbf{T}$ , set  $\underline{\mathbf{a}}$  (resp.  $\bar{\mathbf{a}}$ ) to be the minimum (resp. maximum) element of  $\mathbf{a}$ . Thus  $\mathbf{a} = \{\underline{\mathbf{a}}, \bar{\mathbf{a}}\}$ . For any  $\mathbf{a} \in C_2 \text{sc}(\mathbf{I})$  (resp.  $\mathbf{b} \in C_2 \text{sk}(\mathbf{I})$ ), we set  $\vee' \mathbf{a} := \text{sc}(\uparrow \underline{\mathbf{a}} \cap \uparrow \bar{\mathbf{a}})$  (resp.  $\wedge' \mathbf{b} := \text{sk}(\downarrow \underline{\mathbf{b}} \cap \downarrow \bar{\mathbf{b}})$ ) and call it the *pre-join* (resp. *pre-meet*) of  $\mathbf{a}$  (resp.  $\mathbf{b}$ ). In turn, we set

$$\begin{aligned} \text{sc}_1(\mathbf{I}) &:= \bigsqcup_{\mathbf{a} \in C_2 \text{sc}(\mathbf{I})} \vee' \mathbf{a} = \{ \mathbf{a}_c := (\mathbf{a}, c) \mid \mathbf{a} \in C_2 \text{sc}(\mathbf{I}), c \in \vee' \mathbf{a} \}, \\ \text{and } \text{sk}_1(\mathbf{I}) &:= \bigsqcup_{\mathbf{b} \in C_2 \text{sk}(\mathbf{I})} \wedge' \mathbf{b} = \{ \mathbf{b}_d := (\mathbf{b}, d) \mid \mathbf{b} \in C_2 \text{sk}(\mathbf{I}), d \in \wedge' \mathbf{b} \}. \end{aligned}$$

For example in Figure 1.1, for  $\mathbf{a} := \{a_2, a_3\} \in C_2 \text{sc}(\mathbf{I})$  (with the additional total order  $\preceq$ ) we have  $\underline{\mathbf{a}} = a_2$  and  $\bar{\mathbf{a}} = a_3$ . Moreover, the element  $\{a_2, a_3\}_x$  is minimal in  $\uparrow \underline{\mathbf{a}} \cap \uparrow \bar{\mathbf{a}}$ .

- (3) If  $\text{sc}(\uparrow \mathbf{I}) \neq \emptyset$ , then for each  $a' \in \text{sc}(\uparrow \mathbf{I})$ , we have  $\text{sc}(\mathbf{I}) \cap \downarrow a' \neq \emptyset$  because  $a' \in \uparrow \mathbf{I}$ . Fixing one element in  $\text{sc}(\mathbf{I}) \cap \downarrow a'$  for each  $a' \in \text{sc}(\uparrow \mathbf{I})$  yields a map  $\mathbf{c}: \text{sc}(\uparrow \mathbf{I}) \rightarrow \text{sc}(\mathbf{I})$ . We call such  $\mathbf{c}$  a *choice map*. Dually, if  $\text{sk}(\downarrow \mathbf{I}) \neq \emptyset$ , then for each  $b' \in \text{sk}(\downarrow \mathbf{I})$ , we have  $\text{sk}(\mathbf{I}) \cap \uparrow b' \neq \emptyset$  because  $b' \in \downarrow \mathbf{I}$ . Fixing one element  $b \in \text{sk}(\mathbf{I}) \cap \uparrow b'$  for each  $b' \in \text{sk}(\downarrow \mathbf{I})$  yields another choice map  $\mathbf{d}: \text{sk}(\downarrow \mathbf{I}) \rightarrow \text{sk}(\mathbf{I})$  that sends  $b'$  to  $b$ . See Figure 1.1 for an illustration of such an  $a' \in \text{sc}(\uparrow \mathbf{I})$  and a choice of  $a_1 \in \text{sc}(\mathbf{I})$  such that  $a_1 \leq a'$ .

As a remark in Figure 1.1, there are two elements  $x$  and  $y$  in  $\vee' \{a_2, a_3\}$ , labeled as  $\{a_2, a_3\}_x$  and  $\{a_2, a_3\}_y$  in  $\text{sc}_1(\mathbf{I})$ , respectively. We address that  $\text{sc}_1(\mathbf{I})$  may contain some elements which are not in  $\mathbf{I}$ . For example,  $y$  is not an element of  $\mathbf{I}$  in this illustration. We also note that the element  $w \notin \vee' \{a_2, a_3\}$  since  $w$  is not minimal in  $\uparrow a_2 \cap \uparrow a_3$ , but  $w$  is in both  $\vee' \{a_1, a_2\}$  and  $\vee' \{a_1, a_3\}$ . By this we write  $\{a_1, a_2\}_w$  and  $\{a_1, a_3\}_w$  in  $\text{sc}_1(\mathbf{I})$ , standing for  $w \in \mathbf{I}$ .

With these notations, we are now ready to expose our main results.

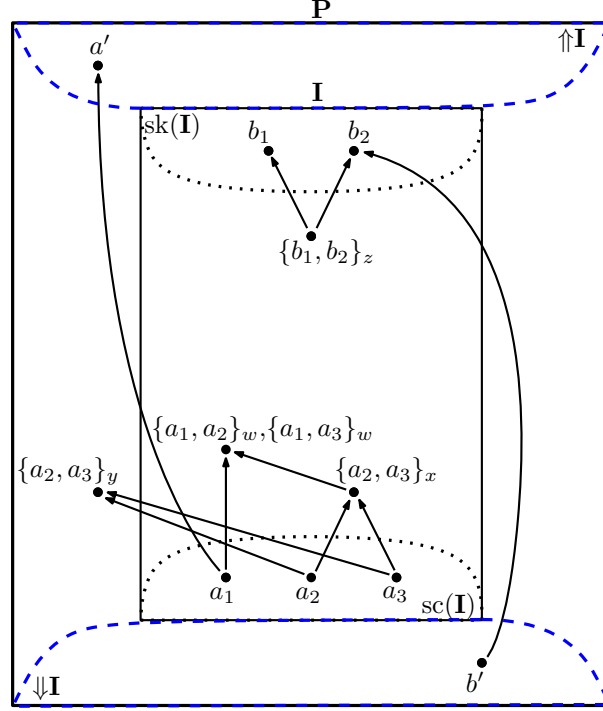


Figure 1.1: Illustration of notations in Notation 1.2.1

**Main result A (Theorem 3.1.26).** *Let  $\mathbf{P}$  be a finite poset,  $M \in \text{mod } \mathbb{k}[\mathbf{P}]$ , and  $\mathbf{I}$  an interval of  $\mathbf{P}$ . Then*

$$d_M(V_{\mathbf{I}}) = \text{rank} \begin{matrix} & \text{sc}(\mathbf{I}) & \text{sk}(\Downarrow \mathbf{I}) & \text{sk}_1(\mathbf{I}) \\ \begin{matrix} \text{sc}_1(\mathbf{I}) \\ \text{sc}(\Uparrow \mathbf{I}) \\ \text{sk}(\mathbf{I}) \end{matrix} & \begin{pmatrix} \mathbf{M}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{M}_2 & \mathbf{0} & \mathbf{0} \\ \mathbf{M}_3 & \mathbf{M}_4 & \mathbf{M}_5 \end{pmatrix} \end{matrix} - \text{rank} \begin{pmatrix} \mathbf{M}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{M}_2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_4 & \mathbf{M}_5 \end{pmatrix} \quad (1.2.1)$$

holds. Here the  $(\mathbf{a}_c, a)$ -entry of  $\mathbf{M}_1$  (resp. the  $(b, \mathbf{b}_d)$ -entry of  $\mathbf{M}_5$ ) is given by

$$(\mathbf{M}_1)_{\mathbf{a}_c, a} := \begin{cases} M_{c,a}, & \text{if } a = \underline{\mathbf{a}}, \\ -M_{c,a}, & \text{if } a = \overline{\mathbf{a}}, \\ 0, & \text{if } a \notin \mathbf{a}, \end{cases} \quad \left( \text{resp. } (\mathbf{M}_5)_{b, \mathbf{b}_d} := \begin{cases} M_{b,d}, & \text{if } b = \underline{\mathbf{b}}, \\ -M_{b,d}, & \text{if } b = \overline{\mathbf{b}}, \\ 0, & \text{if } b \notin \mathbf{b}, \end{cases} \right)$$

and

$$\mathbf{M}_2 := [\delta_{a, \mathbf{c}(a')} M_{a', \mathbf{c}(a')}]_{(a', a) \in \text{sc}(\Uparrow \mathbf{I}) \times \text{sc}(\mathbf{I})}, \quad \mathbf{M}_4 := [\delta_{b, \mathbf{d}(b')} M_{\mathbf{d}(b'), b'}]_{(b, b') \in \text{sk}(\mathbf{I}) \times \text{sk}(\Downarrow \mathbf{I})}.$$

$\mathbf{M}_3$  is given by a choice of pair  $(b_j, a_i) \in \text{sk}(\mathbf{I}) \times \text{sc}(\mathbf{I})$  with  $b_j \geq a_i$ . Namely, the  $(b_j, a_i)$ -entry is the only nonzero entry of  $\mathbf{M}_3$  and it equals to  $M_{b_j, a_i}$ . Index sets of matrices in formula (1.2.1) are allowed to be empty. In this case, we remove rows (columns) of matrices corresponding to empty index sets.

In the one-dimensional persistence module context, every indecomposable persistence module is an interval module, so the interval multiplicity invariant is complete. In multi-dimensional persistence or persistence over the general poset context,

however, there exist indecomposable modules that are not isomorphic to interval modules, and hence the interval multiplicity invariant cannot distinguish these indecomposable modules because the interval multiplicities are all zero. This indicates that the interval multiplicity invariant is incomplete in persistence over the general poset context. To build finer invariants, we introduce the *compression system* for any finite poset. Roughly speaking, a compression system  $\xi$  for  $\mathbf{P}$  assigns each interval  $\mathbf{I}$  of  $\mathbf{P}$  to an order-preserving map  $\xi_{\mathbf{I}}: \mathbf{I}^{\xi} \rightarrow \mathbf{P}$  factoring through the inclusion of  $\mathbf{I}$  into  $\mathbf{P}$  satisfying some conditions on the sets  $\text{sc}(\mathbf{I})$  and  $\text{sk}(\mathbf{I})$ . For example, the family  $\text{tot}$  of the inclusions  $\text{tot}_{\mathbf{I}}: \mathbf{I} \hookrightarrow \mathbf{P}$  for all  $\mathbf{I} \in \mathbb{I}$  turns out to be a compression system, called the *total compression system*.

With the compression system, we define the proposed invariant called *interval rank invariant*. Let  $M \in \text{mod } \mathbb{k}[\mathbf{P}]$  and let  $R_{\mathbf{I}}: \text{mod } \mathbb{k}[\mathbf{P}] \rightarrow \text{mod } \mathbb{k}[\mathbf{I}^{\xi}]$  be the restriction functor induced by  $\xi_{\mathbf{I}}$ . Then the multiplicity of  $R_{\mathbf{I}}(V_{\mathbf{I}})$  in the indecomposable decomposition of  $R_{\mathbf{I}}(M)$  is denoted by  $\text{rank}_{\mathbf{I}}^{\xi} M$  and is called the  $\mathbf{I}$ -rank of  $M$  under  $\xi$ . Collecting  $\mathbf{I}$ -ranks for all  $\mathbf{I} \in \mathbb{I}$  yields the *interval rank invariant* of  $M$  under  $\xi$ , denoted by  $\text{rank}_{\mathbb{I}}^{\xi} M$ .

Notice that the induced restriction functor  $R_{\mathbf{I}}: \text{mod } \mathbb{k}[\mathbf{P}] \rightarrow \text{mod } \mathbb{k}[\mathbf{I}^{\xi}]$  carries persistence modules over  $\mathbf{P}$  to modules over  $\mathbf{I}^{\xi}$ , and recall that the interval rank invariant is defined as the (interval) multiplicity inside  $\text{mod } \mathbb{k}[\mathbf{I}^{\xi}]$ , we obtain an explicit formula for computing each  $\mathbf{I}$ -rank and hence interval rank invariant of  $M$  under  $\xi$ .

**Main result B (Theorem 5.3.7).** *Let  $\mathbf{P}$  be a finite poset,  $\xi = (\xi_{\mathbf{I}}: \mathbf{I}^{\xi} \rightarrow \mathbf{P})_{\mathbf{I} \in \mathbb{I}}$  a compression system for  $\mathbf{P}$ , and  $M \in \text{mod } \mathbb{k}[\mathbf{P}]$ . Then*

$$\text{rank}_{\mathbf{I}}^{\xi} M = \text{rank}_{\substack{\text{sc}(\mathbf{I}^{\xi}) & \text{sk}_1(\mathbf{I}^{\xi}) \\ \text{sc}_1(\mathbf{I}^{\xi}) & \text{sk}(\mathbf{I}^{\xi})}} \begin{pmatrix} \tilde{M}_1 & \mathbf{0} \\ \tilde{M}_3 & \tilde{M}_5 \end{pmatrix} - \text{rank} \begin{pmatrix} \tilde{M}_1 & \mathbf{0} \\ \mathbf{0} & \tilde{M}_5 \end{pmatrix} \quad (1.2.2)$$

holds. Here the  $(\mathbf{a}_c, a)$ -entry of  $\tilde{M}_1$  (resp. the  $(b, \mathbf{b}_d)$ -entry of  $\tilde{M}_5$ ) is given by

$$(\tilde{M}_1)_{\mathbf{a}_c, a} := \begin{cases} M_{\xi_{\mathbf{I}}(c), \xi_{\mathbf{I}}(a)}, & \text{if } a = \underline{\mathbf{a}}, \\ -M_{\xi_{\mathbf{I}}(c), \xi_{\mathbf{I}}(a)}, & \text{if } a = \overline{\mathbf{a}}, \\ \mathbf{0}, & \text{if } a \notin \mathbf{a}. \end{cases} \quad \left( \text{resp. } (\tilde{M}_5)_{b, \mathbf{b}_d} := \begin{cases} M_{\xi_{\mathbf{I}}(b), \xi_{\mathbf{I}}(d)}, & \text{if } b = \underline{\mathbf{b}}, \\ -M_{\xi_{\mathbf{I}}(b), \xi_{\mathbf{I}}(d)}, & \text{if } b = \overline{\mathbf{b}}, \\ \mathbf{0}, & \text{if } b \notin \mathbf{b}. \end{cases} \right)$$

$\tilde{M}_3$  is given by a choice of pair  $(b_j, a_i) \in \text{sk}(\mathbf{I}^{\xi}) \times \text{sc}(\mathbf{I}^{\xi})$  with  $b_j \geq a_i$  in  $\mathbf{I}^{\xi}$ . Namely, the  $(b_j, a_i)$ -entry is the only nonzero entry of  $\tilde{M}_3$  and it is given by  $M_{\xi_{\mathbf{I}}(b_j), \xi_{\mathbf{I}}(a_i)}$ . Similarly, index sets of matrices in formula (1.2.2) are allowed to be empty.

As we saw above, one can compute interval multiplicities and interval rank invariants from the persistent homology (persistence module) obtained. However, computing persistent homology directly from an arbitrary filtration is generally inefficient. To address this, we introduce the *essential-cover* technique, which computes the invariants by focusing on the essential structure of poset  $\mathbf{P}$  and building a new filtration indexed by another poset. This approach subsequently enables the use of efficient computations of standard one-parameter or zigzag persistence if the newly constructed filtration is indexed by the poset of Dynkin type  $\mathbb{A}$ , hence making it possible to compute the invariants starting from the filtration level.

Roughly speaking, the essential cover  $\zeta: \mathbf{Z} \rightarrow \mathbf{P}$  is an order-preserving map, and we say that  $\zeta$  *essentially covers* an interval  $\mathbf{I}$  (resp. *relative to a compression system*

$\xi$ ) if morphisms in  $\mathbb{k}[\mathbf{P}]$  that appear in formula (1.2.1) (resp. formula (1.2.2)) have preimages in  $\mathbb{k}[\mathbf{Z}]$ , subject to some mild technical conditions. Then we have the following.

**Main result C (Theorems 4.1.15 and 5.3.17).** *Let  $\mathbf{P}$  be a finite poset,  $M \in \text{mod } \mathbb{k}[\mathbf{P}]$ , and  $\mathbf{I}$  an interval of  $\mathbf{P}$ . If  $\zeta: \mathbf{Z} \rightarrow \mathbf{P}$  essentially covers  $\mathbf{I}$ , then we have*

$$d_M(V_{\mathbf{I}}) = \bar{d}_{R(M)}(R(V_{\mathbf{I}})), \quad (1.2.3)$$

where  $R$  denotes the restriction functor induced by  $\zeta$ , and  $\bar{d}_N(L)$  denotes the maximal number of copies of  $L$  that can be taken as a direct summand of  $N$  such that no further copies of  $L$  remain in the complement. If  $L$  is indecomposable, then  $\bar{d}_N(L)$  is just the usual multiplicity of  $L$  in  $N$ .

On the other hand, If  $\xi = (\xi_{\mathbf{I}}: \mathbf{I}^{\xi} \rightarrow \mathbf{P})_{\mathbf{I} \in \mathbb{I}}$  is a compression system for  $\mathbf{P}$  and  $\zeta': \mathbf{Z}' \rightarrow \mathbf{P}$  essentially covers  $\mathbf{I}$  relative to  $\xi$ , then we have

$$\text{rank}_{\mathbf{I}}^{\xi} M = \bar{d}_{R'(M)}(R'(V_{\mathbf{I}})), \quad (1.2.4)$$

where  $R'$  denotes the restriction functor induced by  $\zeta'$ .

Equation (1.2.3) (resp. Equation (1.2.4)) provides us with a method to transfer the computation of interval multiplicities (resp. interval rank invariants) over  $\mathbf{P}$  to the computation of corresponding multiplicities over another “essential” poset  $\mathbf{Z}$ . When  $\mathbf{P}$  is a special type of poset, the “essential” poset  $\mathbf{Z}$  can be taken as either a single zigzag poset or a directed tree formed by connecting several zigzag posets. This makes it possible to utilize algorithms designed for computing zigzag persistence (for example, [23, 29, 43]) to compute our proposed invariants. The idea of utilizing the computation of zigzag persistence for computing invariants has also been considered in the literature. For example, Dey and Xin [30] achieve computing the GRI by the *unfolding* process.

We will provide several examples, including but not limited to demonstrating from our perspective that the zigzag poset defined in [31] essentially covers  $\mathbf{I} \in \mathbb{I}$  relative to  $\text{tot}$ . This thereby explains why the computation of GRI can be transformed into computing zigzag persistence, and why the interval rank invariant generalizes GRI.

Interval modules have an outsized role in persistent homology, and we can classify the settings where we always have interval decomposability by a recent result of Aoki et al. [3]: persistence modules over a finite poset always decompose into interval modules if and only if the poset is a totally ordered set, a zigzag poset, or a *bipath poset*, which is defined by the disjoint union of two finite totally ordered sets, with a common minimum and maximum element added. Later in [2], they subsequently investigate the so-called *bipath persistent homology* and propose the *bipath persistence diagrams* by decomposing persistence modules over the bipath poset. They also provide an algorithm for computing the bipath persistence diagrams from the filtration level.

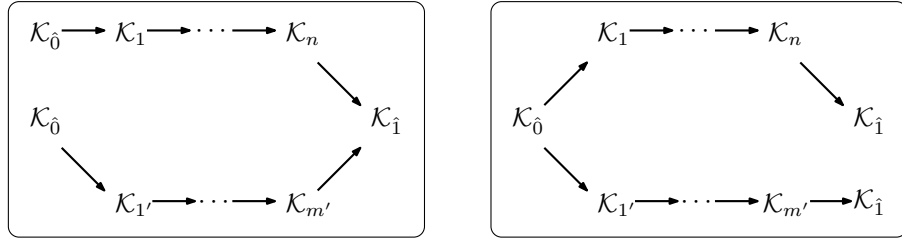
Our next main result provides an alternative way of decomposing bipath persistent homology via the essential-cover technique. In addition, we show that any module over the bipath poset lifts to an infinite zigzag persistence module, viewed as a covering. This covering map enables a direct translation of zigzag persistence

results to the bipath persistence. In particular, it induces a natural interleaving distance—and hence a bottleneck distance on bipath persistence, thus the algebraic stability theorem for zigzag modules immediately yields the algebraic stability for bipath modules.

**Main result D (Propositions 4.2.1 and 4.2.4, and Theorem 4.2.13).** *Given a bipath filtration*

$$\mathcal{F}_{\text{bipath}} := \begin{array}{c} \mathcal{K}_0 \nearrow \mathcal{K}_1 \rightarrow \mathcal{K}_2 \rightarrow \cdots \rightarrow \mathcal{K}_n \searrow \mathcal{K}_{\hat{1}} \\ \mathcal{K}_0 \searrow \mathcal{K}_{1'} \rightarrow \mathcal{K}_{2'} \rightarrow \cdots \rightarrow \mathcal{K}_{m'} \nearrow \mathcal{K}_{\hat{1}} \end{array}$$

*Computing persistence diagrams of the following two zigzag filtrations:*



*gives the bipath persistence diagrams of  $\mathcal{F}_{\text{bipath}}$ . Moreover, if  $M$  and  $N$  are two persistence modules over the bipath poset, then the interleaving distance between  $M$  and  $N$  coincides with the bottleneck distance between their bipath persistence diagrams.*

Finally, we define additional invariants of persistence modules over finite posets. As a natural generalization of the relation between rank and multiplicity invariants, we define the *signed interval multiplicity*  $\delta_M^\xi$  to be the Möbius inversion of the interval rank invariant of  $M$  under the compression system  $\xi$ , which defines the *interval replacement*  $\delta^\xi(M)$  of  $M$  under  $\xi$ . The **I**-rank of the interval replacement  $\delta^\xi(M)$ , denoted by  $\text{rank}_{\mathbf{I}}^\xi \delta^\xi(M)$ , can also be defined in an obvious way. Then our last main result shows that the formation of  $\delta^\xi$  preserves the **I**-ranks, as stated below.

**Main result E (Theorem 6.2.4).** *Let  $\mathbf{P}$  be a finite poset,  $M \in \text{mod } \mathbb{k}[\mathbf{P}]$ , and  $\mathbf{I}$  an interval of  $\mathbf{P}$ . Then we have*

$$\text{rank}_{\mathbf{I}}^\xi \delta^\xi(M) = \text{rank}_{\mathbf{I}}^\xi M.$$

It is worth noticing that **Main result E** holds for all intervals of  $\mathbf{P}$ , which generalizes the result of [39, Theorem 3.29].

## 1.3 Organization

The remainder of this thesis is divided into five chapters. In Chapter 2, we collect background material on quivers, posets, incidence algebras, persistent homology,

and persistence module theory, ending with a brief review of almost split sequences. These notions and notations are used in later chapters.

In Chapter 3, we prove the explicit formula (**Main result A**) for computing the interval multiplicity. We first address the general poset case, then specialize to the 2D-grid case, which is the underlying poset of two-parameter persistence.

In Chapter 4, we introduce the essential-cover technique and show how this technique can be used for computing interval multiplicities from other underlying posets (**Main result C**). In addition, an extended subsection demonstrates the translation between bipath persistence and zigzag persistence, yielding the fast decomposition and stability results for the bipath persistence (**Main result D**).

In Chapter 5, generalizing conventional rank invariants, we define the interval rank invariant in terms of compression systems and derive the explicit formula for its computation (**Main result B**). We then demonstrate how to adapt the essential-cover technique to compute the interval rank invariant.

In Chapter 6, we introduce the signed interval multiplicities and the interval replacement of persistence modules. We prove that interval replacement preserves all interval ranks (**Main result E**), and we provide several applications of the interval replacement.

# Chapter 2

## Preliminaries

Throughout the thesis, we fix a field  $\mathbb{k}$ , and all vector spaces are assumed to be over  $\mathbb{k}$ , and the word “linear” always means “ $\mathbb{k}$ -linear”. The category of finite-dimensional vector spaces is denoted by  $\text{mod } \mathbb{k}$ . Unless stated otherwise, an algebra is defined without requiring the multiplicative identity, and we say an algebra is *unital* if it has the multiplicative identity. For each positive integer  $n$ , we write  $[n]$  the set of  $n$  consecutive integers  $\{1, 2, \dots, n\}$ .

### 2.1 Quivers

A *quiver* is a quadruple  $Q = (Q_0, Q_1, s, t)$  of sets  $Q_0, Q_1$  and maps  $s, t : Q_1 \rightarrow Q_0$ . If we draw each  $x \in Q_0$  as a vertex, and each  $\alpha \in Q_1$  with  $s(\alpha) = x$  and  $t(\alpha) = y$  as an arrow  $x \xrightarrow{\alpha} y$ , then  $Q$  can be expressed as a directed graph. For this, elements of  $Q_1$  are said to be the arrows of  $Q$ , and  $s(\alpha), t(\alpha)$  are said to be the *source* and *target* of the arrow  $\alpha$ . Specially, the quiver  $Q = (Q_0, \emptyset, \emptyset, \emptyset)$  with  $Q_0 \neq \emptyset$  is called the *arrowless quiver*.

**Definition 2.1.1.** Let  $Q = (Q_0, Q_1, s, t)$  be a quiver, and  $x, y \in Q_0$ .

- (1) We say  $Q$  is a quiver *without double arrows* if the map

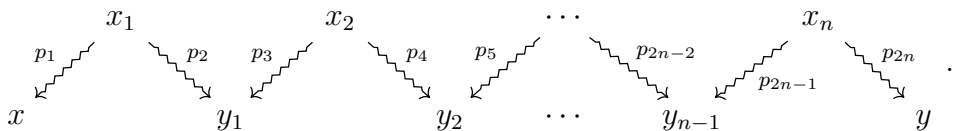
$$(s, t) : Q_1 \rightarrow Q_0 \times Q_0, \alpha \mapsto (s(\alpha), t(\alpha))$$

is injective.

- (2) A *path* from  $x$  to  $y$  of *length*  $n \geq 0$  is a symbol  $p = (y|\alpha_n, \dots, \alpha_1|x)$  consisting of  $\alpha_1, \dots, \alpha_n \in Q_1$  such that  $y = t(\alpha_n)$ ,  $x = s(\alpha_1)$  and  $s(\alpha_{i+1}) = t(\alpha_i)$  for  $i \in [n-1]$ . If  $n = 0$ , we require  $x = y$  and set  $e_x := (x||x)$ . The symbol  $p$  is sometimes expressed by  $p : x \rightsquigarrow y$ . To extend definitions of source and target to paths, we set  $\bar{s}(p) := x$ ,  $\bar{t}(p) := y$  and call them the *source* and *target* of  $p$ .

Furthermore, we write  $Q_n$  to be the set of all paths of length  $n$ , and the set of all paths of  $Q$  is denoted by  $Q_{\geq 0}$ , thus we identify the paths of length 0 and 1 with vertices and arrows, respectively.

- (3) A *walk* between  $x$  and  $y$  in  $Q$  is a sequence  $(p_i)_{i=1}^{2n} \in (Q_{\geq 0})^{2n}$  with  $n \geq 1$  such that  $s(p_1) = s(p_2)$ ,  $s(p_3) = s(p_4)$ ,  $\dots$ ,  $s(p_{2i-1}) = s(p_{2i})$  ( $i \in [n]$ ), and  $x = t(p_1)$ ,  $t(p_2) = t(p_3)$ ,  $\dots$ ,  $t(p_{2i}) = t(p_{2i+1})$ , ( $i \in [n-1]$ ),  $t(p_{2n}) = y$ .



**Remark 2.1.2.** In the definition above, by identifying each path  $(x|\alpha|y)$  of length 1 with the arrow  $\alpha$ , any path  $p = (x|\alpha_1, \dots, \alpha_n|y)$  is expressed as the product  $p = \alpha_1 \cdots \alpha_n$  of arrows.

**Definition 2.1.3.** Let  $Q$  be a quiver.

- (1)  $Q$  is *finite* if both  $Q_0$  and  $Q_1$  are finite.
- (2)  $Q$  is *locally finite* if for all  $x \in Q_0$ , the set  $\{\alpha \in Q_1 \mid s(\alpha) = x \text{ or } t(\alpha) = x\}$  is finite.
- (3) A path  $p \in Q_{\geq 1}$  is an *oriented cycle* if  $\bar{s}(p) = \bar{t}(p)$ .
- (4)  $Q$  is *acyclic* if  $Q$  has no oriented cycles.
- (5)  $Q$  is *connected* if for any pair of vertices  $x, y \in Q_0$ , there exists a walk between  $x$  and  $y$  in  $Q$ , or equivalently,  $Q$  viewed as a directed graph is connected.

## 2.2 Posets and categories

Following the convention given in [28], we denote by  $(\mathbf{P}, \leq)$  the *partially ordered set* (poset for short) where  $\mathbf{P}$  is a set and  $\leq$  is a partial order on  $\mathbf{P}$ . We call a subset  $\mathbf{P}'$  of  $\mathbf{P}$  a *full subposet* if  $\mathbf{P}'$  inherits the partial order from  $(\mathbf{P}, \leq)$ . Namely, given  $x, y \in \mathbf{P}'$ ,  $x \leq y$  in  $\mathbf{P}'$  if and only if  $x \leq y$  in  $\mathbf{P}$ . In what follows, we simply write  $\mathbf{P}$  for the poset instead of  $(\mathbf{P}, \leq)$  when it is not necessary to specify the partial order  $\leq$  explicitly.

**Definition 2.2.1.** Let  $\mathbf{P}$  be a poset, and let  $x \in \mathbf{P}$ .

- (1) The element  $x$  is called a *minimal* (resp. *maximal*) element of  $\mathbf{P}$  if there is no element  $y \in \mathbf{P}$  with  $y < x$  (resp.  $y > x$ ).
- (2) The element  $x$  is called a *minimum* (resp. *maximum*) element of  $\mathbf{P}$  if  $x \leq y$  (resp.  $y \leq x$ ) holds for all  $y \in \mathbf{P}$ .

A minimum is always minimal, and a maximum is always maximal, but not vice versa. For two elements  $x, y \in \mathbf{P}$ , we say that  $x$  and  $y$  are *comparable* if either  $x \leq y$  or  $y \leq x$ .

**Definition 2.2.2.** Let  $(\mathbf{P}, \leq)$  be a poset.

- (1)  $(\mathbf{P}, \leq)$  is called a *chain* or *totally ordered set* if for all  $x, y \in \mathbf{P}$ ,  $x$  and  $y$  are comparable. The order relation  $\leq$  is then called the total order.
- (2)  $(\mathbf{P}, \leq)$  is called a *antichain* if  $x \leq y$  in  $\mathbf{P}$  only if  $x = y$ . In other words, any two distinct elements of  $\mathbf{P}$  are incomparable.

**Definition 2.2.3.** Let  $\mathbf{P}$  be a poset. We say  $\mathbf{P}$  is *connected* if for all  $x, y \in \mathbf{P}$ , there is a finite sequence of elements  $x = z_0, z_1, \dots, z_{n-1}, z_n = y$  in  $\mathbf{P}$  satisfying every two consecutive elements  $z_i$  and  $z_{i+1}$  are comparable.

Throughout the thesis, we mostly assume the connectivity for posets. We now give a few examples of connected posets.

**Example 2.2.4.** (1) For each positive integer  $n$ , we denote by  $\mathbb{A}_n$  the set  $[n]$  with the usual total order  $i < i + 1$ . Similarly, for the set of all real numbers  $\mathbb{R}$ , we always equip it with the usual total order without specifying later. The same for the set of all integers  $\mathbb{Z}$  and the set of all natural numbers  $\mathbb{N}$ . Hence  $\mathbb{A}_n$ ,  $\mathbb{R}$ ,  $\mathbb{Z}$ ,  $\mathbb{N}$  are all chains.

- (2) Given two posets  $\mathbf{P}_1, \mathbf{P}_2$ , we regard the direct product  $\mathbf{P}_1 \times \mathbf{P}_2$  as the poset with the partial order defined by  $(x, y) \leq (x', y')$  if and only if  $x \leq x'$  and  $y \leq y'$  for all  $(x, y), (x', y') \in \mathbf{P}_1 \times \mathbf{P}_2$ . In particular, we set  $\mathbf{G}_{m,n} := \mathbb{A}_m \times \mathbb{A}_n$  ( $m, n \geq 2$ ) and call it a *2D-grid*. A 2D-grid  $\mathbf{G}_{m,n}$  with either  $m = 2$  or  $n = 2$  is called the *commutative ladder*.

**Definition 2.2.5.** Let  $\mathbf{P}_1$  and  $\mathbf{P}_2$  be two posets. A map  $f: \mathbf{P}_1 \rightarrow \mathbf{P}_2$  is called the *order-preserving map* if  $x \leq y$  in  $\mathbf{P}_1$  implies  $f(x) \leq f(y)$  in  $\mathbf{P}_2$ . Furthermore,  $f$  is called the *order-embedding map* if  $x \leq y$  in  $\mathbf{P}_1$  if and only if  $f(x) \leq f(y)$  in  $\mathbf{P}_2$ .

Note that if  $\mathbf{P}_1$  is a full subposet of  $\mathbf{P}_2$ , then the inclusion map  $\iota: \mathbf{P}_1 \hookrightarrow \mathbf{P}_2$ ,  $x \mapsto x$  is the order-embedding map. We say that a poset  $\mathbf{P}'$  is a *subposet* of  $\mathbf{P}$  if  $\mathbf{P}' \subseteq \mathbf{P}$  and the inclusion map  $\iota: \mathbf{P}' \hookrightarrow \mathbf{P}$  is merely order-preserving.

From now on, we set  $[\leq]_{\mathbf{P}} := \{(x, y) \in \mathbf{P} \times \mathbf{P} \mid x \leq y\}$  for convenience. In most of our cases, we may assume finiteness on posets.

**Definition 2.2.6.** Let  $\mathbf{P}$  be a poset.

- (1) For any  $(x, y) \in [\leq]_{\mathbf{P}}$ , we set  $[x, y] := \{z \in \mathbf{P} \mid x \leq z \leq y\}$ , and call it the *segment* from  $x$  to  $y$  in  $\mathbf{P}$ . The set of all segments in  $\mathbf{P}$  is denoted by  $\text{Seg}(\mathbf{P})$ .
- (2) The poset  $\mathbf{P}$  is said to be *finite* if it has only finitely many elements, and it is said to be *locally finite* if segments  $[x, y]$  are finite for all  $(x, y) \in [\leq]_{\mathbf{P}}$ .

The following concept is necessary to introduce the diagram associated with a poset.

**Definition 2.2.7.** Let  $\mathbf{P}$  be a poset and let  $(x, y) \in [\leq]_{\mathbf{P}}$ . We say  $y$  *covers*  $x$  if  $x < y$ , and  $x \leq z < y$  implies  $z = x$ .

In other words,  $y$  covers  $x$  means that there is no intermediate element between  $x$  and  $y$ .

**Definition 2.2.8.** Let  $\mathbf{P}$  be a locally finite poset. Then the *Hasse quiver*  $H(\mathbf{P})$  of  $\mathbf{P}$  is a quiver defined as follows. The set  $H(\mathbf{P})_0$  of vertices is equal to  $\mathbf{P}$ , the set  $H(\mathbf{P})_1$  of arrows is given by the set  $\{(x, y) \in [\leq]_{\mathbf{P}} \mid y \text{ covers } x\}$ , and the source and the target of  $(x, y)$  are  $x$  and  $y$ , respectively.

**Remark 2.2.9.** (1) Hasse quivers of posets are acyclic quivers without double arrows. Conversely, every (locally) finite acyclic quiver  $Q$  without double arrows uniquely determines a (locally) finite poset whose underlying set is  $Q_0$ , equipped with the partial order defined by  $x \leq y$  if and only if there exists a path in  $Q$  from  $x$  to  $y$ . By this, we identify (locally) finite acyclic quivers without double arrows and (locally) finite posets.

- (2) If the poset  $\mathbf{P}$  is finite, then  $\mathbf{P}$  is connected if and only if its Hasse quiver  $H(\mathbf{P})$  is connected.
- (3) We note for the reader that the above definition applies even to posets that are not locally finite. For example, the Hasse quiver  $H(\mathbb{R}) = (\mathbb{R}, \emptyset, \emptyset, \emptyset)$  is the arrowless quiver. However, in order to avoid issues related to triality, we reserve the term Hasse quiver exclusively for locally finite posets throughout the thesis.

**Example 2.2.10.** (1) The commutative ladder  $\mathbf{G}_{5,2}$  has the following Hasse quiver:

$$\begin{array}{ccccccccc} (1, 2) & \longrightarrow & (2, 2) & \longrightarrow & (3, 2) & \longrightarrow & (4, 2) & \longrightarrow & (5, 2) \\ \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ (1, 1) & \longrightarrow & (2, 1) & \longrightarrow & (3, 1) & \longrightarrow & (4, 1) & \longrightarrow & (5, 1) \end{array} .$$

- (2) A poset  $\mathbf{P}$  of cardinality  $n$  is called a *finite zigzag poset* if  $n \geq 3$  and the Hasse quiver of  $\mathbf{P}$  is of the form:

$$1 \longleftrightarrow 2 \longleftrightarrow \cdots \longleftrightarrow n ,$$

where each bidirectional arrow indicates that the arrow is oriented either left-to-right or right-to-left, and at least one arrow has a distinct orientation. In contrast, if all arrows have the same left-to-right orientation, then  $\mathbf{P} = \mathbb{A}_n$ .

Given a poset  $\mathbf{P}$ , one can obtain a new poset  $\mathbf{P}^{\text{op}}$ , the dual of  $\mathbf{P}$ , by defining  $x \leq y$  to hold in  $\mathbf{P}^{\text{op}}$  if and only if  $y \leq x$  holds in  $\mathbf{P}$ .

**Example 2.2.11.** (1) Let  $\mathbb{Z}^{\text{op}}$  be the dual poset of  $\mathbb{Z}$ . The *standard infinite zigzag poset*  $\mathbb{Z}\mathbb{Z}$  is the full subposet of  $\mathbb{Z}^{\text{op}} \times \mathbb{Z}$  given by

$$\{ (a, b) \in \mathbb{Z}^{\text{op}} \times \mathbb{Z} \mid a = b \text{ or } a = b + 1 \}.$$

- (2) A poset  $\mathbf{P}$  of cardinality  $n$  is called a *finite poset of Dynkin type  $\mathbb{A}$*  if  $\mathbf{P}$  is either  $\mathbb{A}_n$ , its dual  $\mathbb{A}_n^{\text{op}}$ , or the finite zigzag poset.

**Definition 2.2.12.** Let  $x \in \mathbf{P}$ , and let  $\mathbf{U}$  be a subset of  $\mathbf{P}$ . Then we set

$$\begin{aligned} \uparrow_{\mathbf{P}} x &:= \{y \in \mathbf{P} \mid y \geq x\}, \text{ and } \downarrow_{\mathbf{P}} x := \{y \in \mathbf{P} \mid y \leq x\}, \\ \uparrow_{\mathbf{P}} \mathbf{U} &:= \bigcup_{x \in \mathbf{U}} \uparrow_{\mathbf{P}} x, \text{ and } \downarrow_{\mathbf{P}} \mathbf{U} := \bigcup_{x \in \mathbf{U}} \downarrow_{\mathbf{P}} x. \end{aligned}$$

We say that  $\mathbf{U}$  is an *up-set* (resp. a *down-set*) of  $\mathbf{P}$  if  $\mathbf{U} = \uparrow_{\mathbf{P}} \mathbf{U}$  (resp.  $\mathbf{U} = \downarrow_{\mathbf{P}} \mathbf{U}$ ). Up-sets and down-sets are naturally considered as full subposets of  $\mathbf{P}$  without specifying in the sequel.

**Remark 2.2.13.** (1) If  $\mathbf{P}$  is finite, then any up-set (resp. down-set)  $\mathbf{U}$  of  $\mathbf{P}$  can be written as  $\mathbf{U} = \uparrow_{\mathbf{P}} \text{sc}(\mathbf{U}) = \bigcup_{x \in \text{sc}(\mathbf{U})} \uparrow_{\mathbf{P}} x$  (resp.  $\mathbf{U} = \downarrow_{\mathbf{P}} \text{sk}(\mathbf{U}) = \bigcup_{x \in \text{sk}(\mathbf{U})} \downarrow_{\mathbf{P}} x$ ).

- (2) If  $\mathbf{X} = \uparrow_{\mathbf{P}} \mathbf{U}$  (resp.  $\mathbf{X} = \downarrow_{\mathbf{P}} \mathbf{U}$ ) and  $\mathbf{X}$  is a finite poset, then  $\text{sc}(\mathbf{X}) = \text{sc}(\mathbf{U})$  (resp.  $\text{sk}(\mathbf{X}) = \text{sk}(\mathbf{U})$ ).
- (3) It is easy to see that  $\uparrow_{\mathbf{P}} \uparrow_{\mathbf{P}} \mathbf{U} = \uparrow_{\mathbf{P}} \mathbf{U}$  and  $\downarrow_{\mathbf{P}} \downarrow_{\mathbf{P}} \mathbf{U} = \downarrow_{\mathbf{P}} \mathbf{U}$ .
- (4) If  $\mathbf{U}$  is an up-set (resp. down-set) of  $\mathbf{P}$  and  $x$  is any element of  $\mathbf{U}$ , then  $\uparrow_{\mathbf{P}} x = \uparrow_{\mathbf{U}} x$  (resp.  $\downarrow_{\mathbf{P}} x = \downarrow_{\mathbf{U}} x$ ).

We simply write  $\uparrow, \downarrow$  for  $\uparrow_{\mathbf{P}}, \downarrow_{\mathbf{P}}$ , respectively, if there seems to be no confusion.

**Definition 2.2.14.** Let  $\mathbf{P}$  be a poset and let  $\mathbf{U}$  be a subset of  $\mathbf{P}$ .

- (1) An element  $x \in \mathbf{P}$  is said to be an *upper bound* (resp. a *lower bound*) of  $\mathbf{U}$  if  $u \leq x$  (resp.  $u \geq x$ ) holds for all  $u \in \mathbf{U}$ .
- (2) If the set of all upper bounds (resp. lower bounds) of  $\mathbf{U}$  has a least (resp. greatest) element  $y$ , then  $y$  is called the *join* (resp. *meet*) of  $\mathbf{U}$  and is denoted by  $y := \bigvee \mathbf{U}$  (resp.  $y := \bigwedge \mathbf{U}$ ).

**Definition 2.2.15.** Let  $\mathbf{P}$  be a poset and  $\mathbf{I}$  a nonempty full subposet of  $\mathbf{P}$ .

- (1) For any  $(x, y) \in [\leq]_{\mathbf{P}}$ , we set  $[x, y] := \{z \in \mathbf{P} \mid x \leq z \leq y\}$  and call it the *segment* from  $x$  to  $y$  in  $\mathbf{P}$ . An illustration is shown on the left in Figure 2.2. The set of all segments in  $\mathbf{P}$  is denoted by  $\text{Seg}(\mathbf{P})$ .
- (2) A *source* (resp. *sink*) of  $\mathbf{I}$  is nothing but a minimal (resp. maximal) element of  $\mathbf{I}$ . The set of all sources (sinks) in  $\mathbf{I}$  is denoted by  $\text{sc}(\mathbf{I})$  (resp.  $\text{sk}(\mathbf{I})$ ). If  $\mathbf{I}$  has the maximum (resp. minimum) element, then it is denoted by  $\max(\mathbf{I})$  (resp.  $\min(\mathbf{I})$ ). Artificially, we set  $\text{sc}(\emptyset) := \emptyset$  and  $\text{sk}(\emptyset) := \emptyset$ .
- (3)  $\mathbf{I}$  is said to be *convex* if for all  $(x, y) \in [\leq]_{\mathbf{I}}$ , we have  $[x, y] \subseteq \mathbf{I}$ .
- (4) If  $\mathbf{P}$  is finite, then the *convex hull* of  $\mathbf{I}$ , denoted by  $\text{conv}(\mathbf{I})$ , is defined as the smallest (with respect to the inclusion) convex subposet of  $\mathbf{P}$  that contains  $\mathbf{I}$ . Equivalently,  $\text{conv}(\mathbf{I})$  is the union of all segments  $[x, y] \in \text{Seg}(\mathbf{P})$  with  $x, y \in \mathbf{I}$ .
- (5)  $\mathbf{I}$  is called an *interval* if  $\mathbf{I}$  is connected and convex.
- (6) The set of all intervals of  $\mathbf{P}$  is denoted by  $\mathbb{I}(\mathbf{P})$ , or simply by  $\mathbb{I}$ . We regard  $\mathbb{I}$  as a poset  $\mathbb{I} = (\mathbb{I}, \leq)$  with the inclusion relation:  $\mathbf{I} \leq \mathbf{J} \Leftrightarrow \mathbf{I} \subseteq \mathbf{J}$  for all  $\mathbf{I}, \mathbf{J} \in \mathbb{I}$ .

**Example 2.2.16.** (1) If  $\mathbf{P}$  is the finite poset of Dynkin type  $\mathbb{A}$  and  $\mathbf{I}$  is an interval of  $\mathbf{P}$ , then the Hasse quiver of  $\mathbf{I}$  is

$$s \longleftrightarrow s+1 \longleftrightarrow \cdots \longleftrightarrow t,$$

where the orientation of each bidirectional arrow is determined by the order relation in  $\mathbf{P}$ . If  $\mathbf{P}$  is either  $\mathbb{N}$  or  $\mathbb{Z}$ , then  $\mathbb{I}(\mathbf{P}) = \{\mathbf{P}\} \cup \text{Seg}(\mathbf{P}) \cup \{\uparrow_{\mathbf{P}} s \mid s \in \mathbf{P}\} \cup \{\downarrow_{\mathbf{P}} t \mid t \in \mathbf{P}\}$ .

- (2) Assume  $\mathbf{P} = \mathbb{R}$ . We introduce the following angle-bracket notations.

- For  $s \in \mathbb{R}$ , we set  $\langle s, s \rangle := [s, s]$ .  $\langle s, +\infty \rangle$  denotes two types of intervals:  $[s, +\infty)$  or  $(s, +\infty)$ .  $\langle -\infty, s \rangle$  denotes two types of intervals:  $(-\infty, s]$  or  $(-\infty, s)$ .

- For  $s, t \in \mathbb{R}$  with  $s < t$ ,  $\langle s, t \rangle$  denotes four types of intervals:  $[s, t]$ ,  $(s, t)$ ,  $[s, t)$ ,  $(s, t]$ .

Hence  $\mathbb{I}(\mathbf{P}) = \{\mathbb{R}\} \cup \{\langle s, s \rangle, \langle s, +\infty \rangle, \langle -\infty, s \rangle, \langle s, t \rangle \mid s, t \in \mathbb{R} \text{ with } s < t\}$ .

- (3) If  $\mathbf{P} = \mathbb{ZZ}$ , then every interval of  $\mathbf{P}$  can be written as one of the following forms by adopting the convention introduced in [10] for  $a, b \in \mathbb{Z} \cup \{\pm\infty\}$ :

$$\begin{aligned} [a, b]_{\mathbb{ZZ}} &:= \{(c, d) \in \mathbb{ZZ} \mid c \leq b, d \geq a\}, & [a, b)_{\mathbb{ZZ}} &:= \{(c, d) \in \mathbb{ZZ} \mid a \leq d < b\}, \\ (a, b]_{\mathbb{ZZ}} &:= \{(c, d) \in \mathbb{ZZ} \mid a < c \leq b\}, & (a, b)_{\mathbb{ZZ}} &:= \{(c, d) \in \mathbb{ZZ} \mid c > a, d < b\}. \end{aligned}$$

See Figure 2.1 for an illustration.

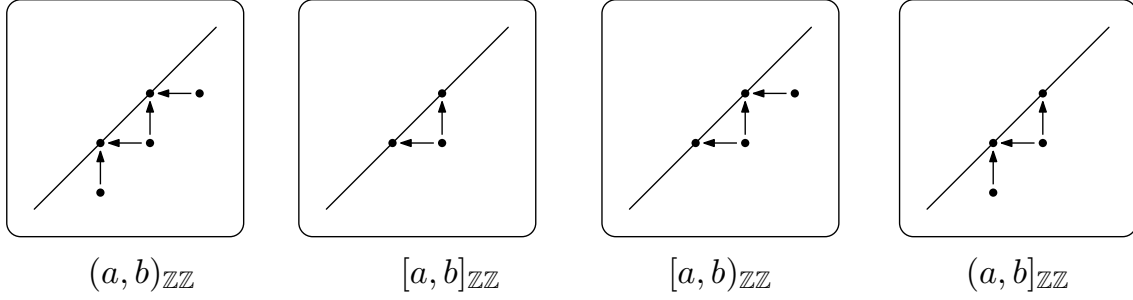


Figure 2.1: Types of intervals of  $\mathbb{ZZ}$

- (4) If  $\mathbf{P} = \mathbf{G}_{m,n}$ , then each interval  $\mathbf{I}$  can be written as:

$$\mathbf{I} = \bigsqcup_{s \leq i \leq t} [b_i, d_i]_i$$

for some  $s, t \in [n]$  with  $s \leq t$ . Here  $1 \leq b_{i+1} \leq b_i \leq d_{i+1} \leq d_i \leq m$ , and  $[b_i, d_i]_i := \{(x, i) \in \mathbb{A}_m \times \{i\} \mid b_i \leq x \leq d_i\}$  for  $s \leq i \leq t$ . In short, the intervals of 2D-grids exhibit staircase shapes (Figure 2.2).

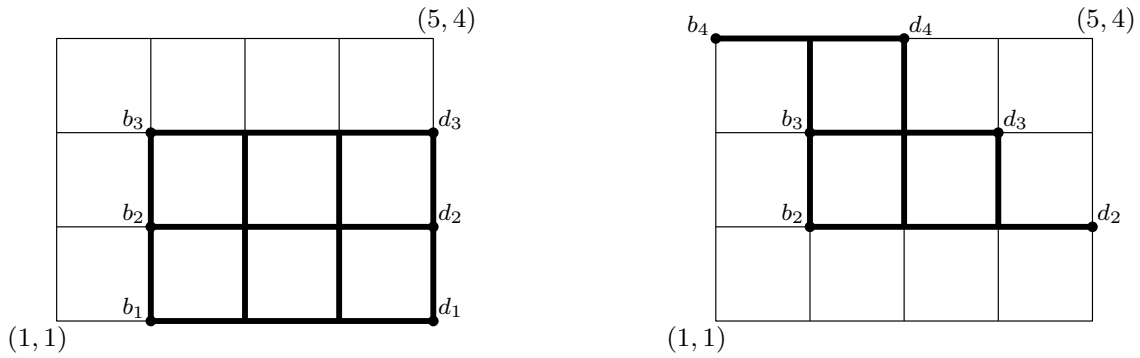


Figure 2.2: An illustration of two intervals of  $\mathbf{G}_{5,4}$

Posets can be viewed as categories in the following sense.

**Definition 2.2.17.** Let  $\mathbf{P}$  be a poset. The poset  $\mathbf{P}$  is regarded as a *category* as follows.

- The set  $\mathbf{P}_0$  of objects is defined by  $\mathbf{P}_0 := \mathbf{P}$ .
- For each pair  $(x, y) \in \mathbf{P} \times \mathbf{P}$ , the set  $\mathbf{P}(x, y)$  of morphisms from  $x$  to  $y$  is defined by  $\mathbf{P}(x, y) := \{p_{y,x}\}$  if  $x \leq y$ , and  $\mathbf{P}(x, y) := \emptyset$  otherwise. Here we write  $p_{y,x} := (x, y) \in [\leq]_{\mathbf{P}}$  for short.
- The composition is defined by  $p_{z,y}p_{y,x} = p_{z,x}$  for all  $x, y, z \in \mathbf{P}$  with  $x \leq y \leq z$ .
- The identity  $\mathbb{1}_x$  at an object  $x \in \mathbf{P}$  is given by  $\mathbb{1}_x = p_{x,x}$ .

We abuse the same notation  $\mathbf{P}$  for this category if there seems to be no ambiguity.

In the literature, the functor  $M: \mathbf{P} \rightarrow \text{mod } \mathbb{k}$  is called the  $\mathbf{P}$ -indexed persistence module or simply the  $\mathbf{P}$ -persistence module. However, to smoothly apply some techniques in the representation theory, we shall consider the linear category instead, leading to the following definition.

**Definition 2.2.18.** The *incidence category*  $\mathbb{k}[\mathbf{P}]$  of the poset  $\mathbf{P}$  is defined as the  $\mathbb{k}$ -linearization of the category  $\mathbf{P}$ . Namely, it is a  $\mathbb{k}$ -linear category defined as follows.

- The set of objects  $\mathbb{k}[\mathbf{P}]_0$  is equal to  $\mathbf{P}$ ,
- For each pair  $(x, y) \in \mathbf{P} \times \mathbf{P}$ , the set of morphisms  $\mathbb{k}[\mathbf{P}](x, y)$  is the vector space with basis  $\mathbf{P}(x, y)$ ; thus it is a one-dimensional vector space  $\mathbb{k}p_{y,x}$  if  $x \leq y$ , or zero otherwise.  $\mathbb{k}[\mathbf{P}](x, y)$  is called the *Hom space* of the pair  $(x, y) \in \mathbf{P} \times \mathbf{P}$ . The set of all morphisms is denoted by  $\mathbb{k}[\mathbf{P}]_1$ .
- The composition is defined as the  $\mathbb{k}$ -bilinear extension of that of  $\mathbf{P}$ .
- The identity  $\mathbb{1}_x$  at an object  $x \in \mathbf{P}$  is given by  $\mathbb{1}_x = p_{x,x}$ .

A category is said to be *small* if both the objects and the morphisms form sets, not proper classes. A  $\mathbb{k}$ -linear category is said to be *finite* if it has only finitely many objects, and every Hom space is finite-dimensional. Note that  $\mathbb{k}[\mathbf{P}]$  is a small category, and it is a finite  $\mathbb{k}$ -linear category if  $\mathbf{P}$  is finite.

The  $\mathbb{k}$ -linearization of posets is functorial. More concretely, if  $f: \mathbf{P}_1 \rightarrow \mathbf{P}_2$  is the order-preserving map, then  $\mathbb{k}[f]: \mathbb{k}[\mathbf{P}_1] \rightarrow \mathbb{k}[\mathbf{P}_2]$  is the linear functor between incidence categories given by

$$\mathbb{k}[f](x) := f(x), \quad (2.2.1)$$

for all  $x \in \mathbb{k}[\mathbf{P}_1]_0$ , and

$$\mathbb{k}[f](0) := 0, \text{ and } \mathbb{k}[f](kp_{y,x}) := kp_{f(y),f(x)} \text{ } (k \in \mathbb{k}), \quad (2.2.2)$$

where  $p_{y,x}$  is the unique basis of  $\mathbb{k}[\mathbf{P}_1](x, y)$  whenever  $(x, y) \in [\leq]_{\mathbf{P}_1}$ , and  $p_{f(y),f(x)}$  is the unique basis of  $\mathbb{k}[\mathbf{P}_2](f(x), f(y))$ .

## 2.3 Incidence algebra and Möbius inversion

We may associate each poset with a non-unital associative algebra. In particular, when the poset is finite, such a non-unital algebra becomes unital.

**Definition 2.3.1.** Let  $\mathbf{P}$  be a poset. We define the *incidence algebra* (may not have the multiplicative identity)  $\mathbb{k}\mathbf{P}$  of  $\mathbf{P}$  by using the incidence category  $\mathbb{k}[\mathbf{P}]$  as

$$\mathbb{k}\mathbf{P} := \bigoplus_{(x,y) \in \mathbf{P} \times \mathbf{P}} \mathbb{k}[\mathbf{P}](x,y) = \bigoplus_{(x,y) \in [\leq]_{\mathbf{P}}} \mathbb{k}p_{y,x},$$

with the multiplication

$$\begin{aligned} \sum_{(y,z) \in [\leq]_{\mathbf{P}}} k_{z,y} p_{z,y} \cdot \sum_{(w,x) \in [\leq]_{\mathbf{P}}} k'_{x,w} p_{x,w} &:= \sum_{(y,z) \in [\leq]_{\mathbf{P}}} \sum_{(w,x) \in [\leq]_{\mathbf{P}}} \delta_{x,y} k_{z,y} k'_{x,w} p_{z,w} \\ &= \sum_{\substack{w,x,z \in \mathbf{P} \\ w \leq x \leq z}} k_{z,x} k'_{x,w} p_{z,w}, \end{aligned} \quad (2.3.3)$$

where coefficients  $k_{z,y}$  and  $k'_{x,w}$  are elements of  $\mathbb{k}$ .

We remark that the multiplication (2.3.3) is well-defined because  $\mathbb{k}\mathbf{P}$  is defined as a direct sum, so in the summations all but finitely many coefficients  $k_{z,y}$  and  $k'_{x,w}$  are zero.

Again, we caution the reader that the incidence algebra  $\mathbb{k}\mathbf{P}$  defined in the thesis is unital if  $\mathbf{P}$  is finite. Namely,  $\mathbb{k}\mathbf{P}$  has the identity  $1_{\mathbb{k}\mathbf{P}} := \sum_{x \in \mathbf{P}} \mathbb{1}_x$  whenever  $\mathbf{P}$  is finite.

**Remark 2.3.2.** In the definition above, we note the following remarks. For brevity, we assume  $\mathbf{P}$  is finite.

- (1) (As a matrix algebra with blocks  $\mathbb{k}$  or  $\mathbf{0}$ ) To express each element of  $\mathbb{k}\mathbf{P}$  as a matrix, we fix a total order on  $\mathbf{P}$  extending the original partial order. By regarding the isomorphism  $\mathbb{k} \rightarrow \mathbb{k}p_{y,x}$  sending 1 to  $p_{y,x}$  as the identity map, we can regard  $\mathbb{k}\mathbf{P}$  as a matrix algebra over  $\mathbb{k}$  with the set of  $(y,x)$ -entries  $\mathbb{k}$  if  $x \leq y$ , and  $\mathbf{0}$  otherwise.
- (2) (As a set of functions from  $\text{Seg}(\mathbf{P})$  to  $\mathbb{k}$ ) Note that we have a bijection  $\text{Seg}(\mathbf{P}) \rightarrow \{p_{y,x} \mid x, y \in \mathbf{P}, x \leq y\}$  sending  $[x,y]$  to  $p_{y,x}$ , and that each element  $m$  of  $\mathbb{k}\mathbf{P}$  can be regarded as a function  $\{p_{y,x} \mid x, y \in \mathbf{P}, x \leq y\} \rightarrow \mathbb{k}$  sending each  $p_{y,x}$  to the  $(y,x)$ -entry  $m_{y,x}$  of  $m$ . By combining these we can also regard  $\mathbb{k}\mathbf{P}$  as the set  $\mathbb{k}^{\text{Seg}(\mathbf{P})}$  of functions  $\text{Seg}(\mathbf{P}) \rightarrow \mathbb{k}$ , namely, by identifying an element  $m \in \mathbb{k}\mathbf{P}$  with the function sending each segment  $[x,y]$  to the  $(y,x)$ -entry  $m_{y,x}$  of  $m$ .
- (3) (Right action on  $\mathbb{k}\mathbf{P}$ ) Let  $\mathbb{k}\mathbf{P}$  be the vector space of all  $\mathbb{k}$ -valued functions with the domain  $\mathbf{P}$ . Then  $\mathbb{k}\mathbf{P}$  has a right  $\mathbb{k}\mathbf{P}$ -module structure, the explicit definition of which is given as follows: Let  $f \in \mathbb{k}\mathbf{P}$  and  $(x,y) \in [\leq]_{\mathbf{P}}$ . Then

$$(f \cdot p_{y,x})(z) := \delta_{x,z} f(y) \quad (2.3.4)$$

for all  $z \in \mathbf{P}$ .

**Definition 2.3.3** (Zeta and Möbius functions). Let  $\mathbf{P}$  be a finite poset. We set

$$\zeta := \sum_{x \leq y} p_{y,x} \in \bigoplus_{x \leq y} \mathbb{k} p_{y,x} = \mathbb{k}\mathbf{P} \cong \mathbb{k}^{\text{Seg}(\mathbf{P})}$$

(see Remark 2.3.2 (2)), and call it the *zeta function* (on  $\mathbf{P}$ ). We note that  $\zeta$  is expressed as a lower triangular matrix with all diagonal entries 1 in  $\mathbb{k}\mathbf{P}$  as a matrix algebra (see Remark 2.3.2 (1)). Thus it is invertible in  $\mathbb{k}\mathbf{P}$ , the inverse is given by the adjoint matrix of  $\zeta$ , which is denoted by  $\mu$ , and called the *Möbius function* (on  $\mathbf{P}$ ).

Note that for any  $f \in \mathbb{k}\mathbf{P}$ , we have

$$(f \cdot \zeta)(z) = \sum_{x \leq y} \delta_{x,z} f(y) = \sum_{z \leq y} f(y) \quad (2.3.5)$$

for all  $z \in \mathbf{P}$  by (2.3.4).

**Theorem 2.3.4** (Möbius Inversion Formula). *Let  $\mathbf{P}$  be a finite poset. For any  $f, g \in \mathbb{k}\mathbf{P}$  and  $x \in \mathbf{P}$ , the following statements are equivalent:*

- (1)  $f(x) = \sum_{x \leq y \in \mathbf{P}} g(y)$ ; and
- (2)  $g(x) = \sum_{x \leq y \in \mathbf{P}} f(y) \mu([x, y])$ .

*Proof.* Since  $\mu = \sum_{x \leq y} \mu([x, y]) p_{y,x}$ , we have

$$(f \cdot \mu)(z) = \sum_{x \leq y} \delta_{x,z} f(y) \mu([x, y]) = \sum_{z \leq y} f(y) \mu([z, y]).$$

By this together with (2.3.5), the equivalence follows from the fact that  $f = g \cdot \zeta$  if and only if  $f \cdot \mu = g$ .  $\square$

## 2.4 Persistent homology

We formalize the concept of persistent homology in the following categorical sense.

**Definition 2.4.1.** Let  $\mathbf{P}$  be a poset, and  $\mathbf{FinSimp}$  the category of finite simplicial complexes. The  $\mathbf{P}$ -filtration is a functor  $\mathcal{F}: \mathbf{P} \rightarrow \mathbf{FinSimp}$ .

Hereafter, we shall simply call  $\mathcal{F}$  the filtration without adding the prefix  $\mathbf{P}$  if the underlying poset is evident from the context.

**Example 2.4.2.** (1) Let  $\mathcal{X}$  be a finite set of points in the  $d$ -dimensional Euclidean space  $\mathbb{R}^d$ , equipped with the usual Euclidean metric, and let  $r \in \mathbb{R}_{\geq 0}$ . Define the abstract simplicial complex indexed by  $r$  as follows:

$$\check{C}_r(\mathcal{X}) := \left\{ \sigma \subseteq \mathcal{X} \mid \bigcap_{x \in \sigma} B_x(r) \neq \emptyset \right\}.$$

Here and subsequently,  $B_x(r)$  denotes the closed ball with radius  $r$  centered at  $x$ . The complex  $\check{C}_r(\mathcal{X})$  is called the *Čech complex at radius  $r$* . It is evident that the assignment

$$\mathbb{R}_{\geq 0} \rightarrow \mathbf{FinSimp}, \quad r \mapsto \check{C}_r(\mathcal{X}),$$

satisfying  $\check{C}_r(\mathcal{X}) \subseteq \check{C}_{r'}(\mathcal{X})$  whenever  $r \leq r'$ , is thus an  $\mathbb{R}_{\geq 0}$ -filtration. We call this  $\mathbb{R}_{\geq 0}$ -filtration the *Čech filtration of  $\mathcal{X}$* .

- (2) Let  $\mathcal{X}$  and  $r$  be as above. Define the abstract simplicial complex indexed by  $r$  as follows:

$$R_r(\mathcal{X}) := \{ \sigma \subseteq \mathcal{X} \mid B_x(r) \cap B_y(r) \neq \emptyset \text{ for all } x, y \in \sigma \}.$$

The complex  $R_r(\mathcal{X})$  is called the *Vietoris–Rips complex at radius  $r$* . Again, it is evident that the assignment

$$\mathbb{R}_{\geq 0} \rightarrow \mathbf{FinSimp}, \quad r \mapsto R_r(\mathcal{X}),$$

satisfying  $R_r(\mathcal{X}) \subseteq R_{r'}(\mathcal{X})$  whenever  $r \leq r'$ , is thus an  $\mathbb{R}_{\geq 0}$ -filtration. We call this  $\mathbb{R}_{\geq 0}$ -filtration the *Vietoris–Rips filtration of  $\mathcal{X}$* .

- (3) We review the barycentric decomposition of filtrations in [47]. Let  $\mathcal{K}$  be a simplicial complex. A *flag* of  $\mathcal{K}$  is an ordered subset of simplices  $\{\sigma_1, \dots, \sigma_t\} \subseteq \mathcal{K}$  satisfying  $\sigma_1 \subseteq \dots \subseteq \sigma_t$ . The set of all flags of  $\mathcal{K}$  is denoted by  $\text{Bary}(\mathcal{K})$  and called the *barycentric decomposition* of  $\mathcal{K}$ . For every  $k \in \mathbb{N}$ , we set

$$k\text{-Bary}(\mathcal{K}) := \left\{ \gamma \in \text{Bary}(\mathcal{K}) \mid \min_{\sigma \in \gamma} |\sigma| \geq k \right\}.$$

The barycentric decomposition of a  $\mathbf{P}$ -filtration naturally gives us a  $(\mathbf{P} \times \mathbb{N}^{\text{op}})$ -filtration. For example, if we let the filtration  $\mathcal{F}$  be the Čech filtration of  $\mathcal{X}$  defined above, then the assignment

$$\mathbb{R}_{\geq 0} \times \mathbb{N}^{\text{op}} \rightarrow \mathbf{FinSimp}, \quad (r, k) \mapsto k\text{-Bary}(\check{C}_r(\mathcal{X})),$$

is an  $(\mathbb{R}_{\geq 0} \times \mathbb{N}^{\text{op}})$ -filtration.

We denote by  $H_i(-; \mathbb{k})$  the  $i$ th homology functor with the coefficient field  $\mathbb{k}$ . Now we are in the position to define the persistent homology.

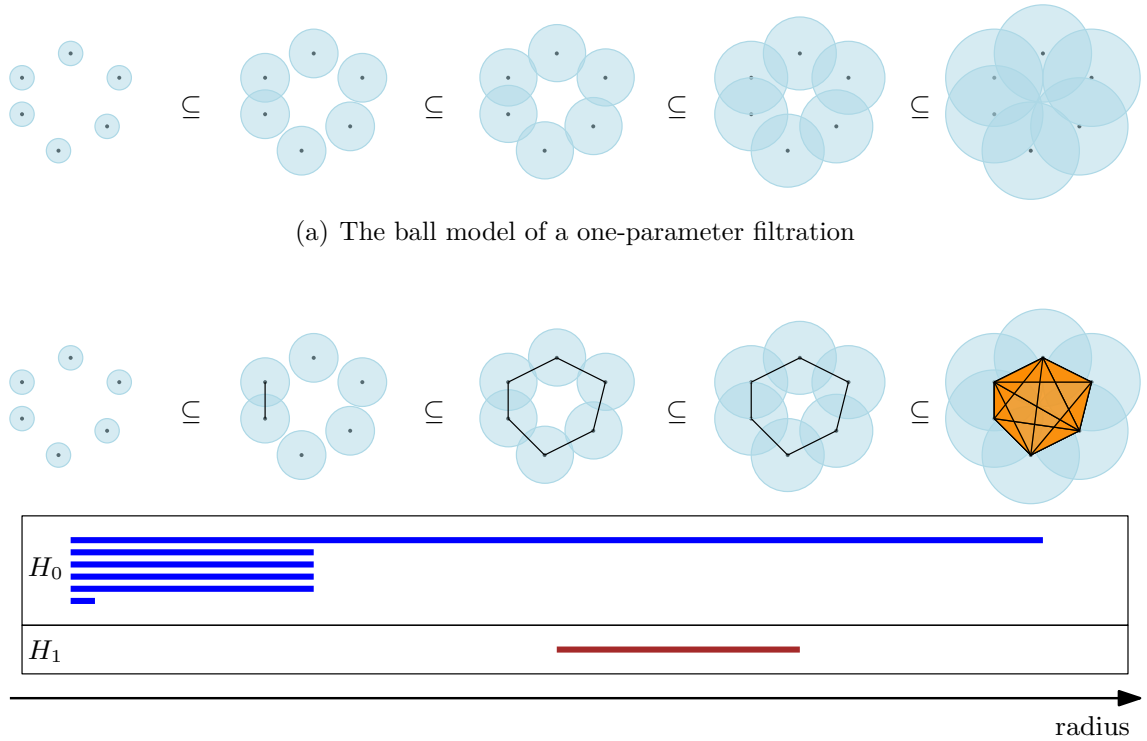
**Definition 2.4.3.** Let  $\mathcal{F}$  be a  $\mathbf{P}$ -filtration. Then the composition  $H_i(-; \mathbb{k}) \circ \mathcal{F}: \mathbf{P} \rightarrow \text{mod } \mathbb{k}$  is called the  *$i$ th persistent homology (over  $\mathbf{P}$ )*.

**Example 2.4.4.** (1) If  $\mathbf{P}$  is either  $\mathbb{A}_n$ ,  $\mathbb{N}$ ,  $\mathbb{Z}$ , or  $\mathbb{R}$ , then persistent homology over  $\mathbf{P}$  is referred to as the *one-parameter* (or *single-parameter*) *persistent homology*. Hence the induced persistent homology from the Čech filtration or Vietoris–Rips filtration is a one-parameter persistent homology. This is reasonable because the radius  $r$  is the only parameter in both filtrations.

- (2) Let  $k \geq 2$ . If  $\mathbf{P}$  is either  $\prod_{i \in [k]} \mathbb{A}_{n_i}$  ( $n_i \geq 2$ ),  $\mathbb{N}^k$ ,  $\mathbb{Z}^k$ , or  $\mathbb{R}^k$ , then persistent homology over  $\mathbf{P}$  is referred to as the  *$k$ -parameter persistent homology*.

- (3) Persistent homology over  $\mathbf{P}$  is referred to as the *zigzag persistent homology* if  $\mathbf{P}$  is either the finite zigzag poset (Example 2.2.10 (2)) or  $\mathbb{Z}\mathbb{Z}$  (Example 2.2.11).

To illustrate persistent homology concretely, consider the ech filtration defined previously. Given a finite set  $\mathcal{X}$  of points in  $\mathbb{R}^2$  as the input dataset, the Čech filtration can be constructed by taking unions of closed balls (called the ball model) of radius  $r$  centered at each point. By the nerve theorem (see [35, Chapter III.2]), this ball model—considered as a topological space—is homotopy equivalent to its nerve complex, which, by definition, is precisely the Čech complex at radius  $r$  (Example 2.4.2 (1)). Figure 2.3 visually demonstrates this construction, depicting the progression from isolated points (0-simplices) through the formation of edges (1-simplices) and higher-dimensional simplices as the balls intersect. Persistent homology then tracks how homological features, such as connected components and loops, appear and disappear as  $r$  grows. This yields intervals of the underlying poset (in this case,  $\mathbb{R}$  for instance) capturing the “lifespan” of these topological features, which encode meaningful geometric information inherent to the input dataset.



(b) The filtration of the nerve complexes and barcodes of persistent homology

Figure 2.3: An example of one-parameter persistent homology

Figure 2.4 provides another example of filtration over a 2D-grid that is not the finite poset of Dynkin type  $\mathbb{A}$ , giving the two-parameter persistent homology. In this illustration, the filtration is parameterized horizontally by the radius of balls and vertically by the cardinality (or density) of datasets.

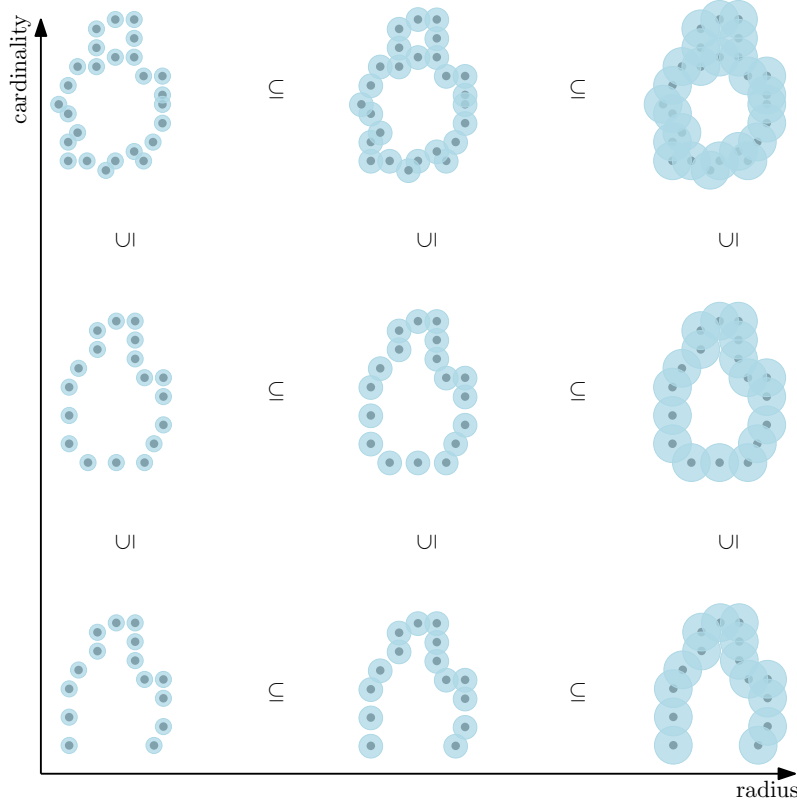


Figure 2.4: An example of two-parameter filtration

## 2.5 Persistence modules

It is natural to carry the persistent homology theory to the algebraic framework. This way of generalization allows one to study the persistent homology from the viewpoint of representation theory.

Notice that each functor  $M: \mathbf{P} \rightarrow \text{mod } \mathbb{k}$  is uniquely extended to a linear functor  $\bar{M}: \mathbb{k}[\mathbf{P}] \rightarrow \text{mod } \mathbb{k}$ . Therefore, we identify  $M$  with  $\bar{M}$ , and denote it simply by  $M$ . Throughout the thesis, we will treat the  $\mathbf{P}$ -persistence module as the  $\mathbb{k}$ -linear functor.

**Definition 2.5.1** (Persistence modules). Let  $\mathbf{P}$  be a poset. A  $\mathbf{P}$ -persistence module is a (covariant)  $\mathbb{k}$ -linear functor  $\mathbb{k}[\mathbf{P}] \rightarrow \text{mod } \mathbb{k}$ . Specifically, the zero functor is called the *zero* (or *trivial*) *module* and is denoted by  $0$ . All  $\mathbf{P}$ -persistence modules together with natural transformations between them as morphisms form a  $\mathbb{k}$ -linear category, which is denoted by  $\text{mod } \mathbb{k}[\mathbf{P}]$ .

In particular, if  $\mathbf{P}$  is either  $\prod_{i \in [k]} \mathbb{A}_{n_i}$  ( $n_i \geq 2$ ),  $\mathbb{N}^k$ ,  $\mathbb{Z}^k$ , or  $\mathbb{R}^k$ , then  $\mathbf{P}$ -persistence modules are called *k-dimensional persistence modules*. If  $\mathbf{P}$  is either the finite zigzag poset or  $\mathbb{Z}\mathbb{Z}$ , then  $\mathbf{P}$ -persistence modules are called *zigzag modules*.

**Remark 2.5.2.** (1) Note that  $\text{mod } \mathbb{k}[\mathbf{P}]$  is the abelian category. To better distinguish notations for Hom spaces in two categories  $\mathbb{k}[\mathbf{P}]$  and  $\text{mod } \mathbb{k}[\mathbf{P}]$ , by  $\mathbb{k}[\mathbf{P}](x, y)$  we denote the Hom space from  $x$  to  $y$ , and by  $\text{Hom}_{\mathbb{k}[\mathbf{P}]}(M, N)$  we denote the Hom space from  $M$  to  $N$ .

- (2) Given a  $\mathbf{P}$ -persistence module  $M$ , we call linear maps  $M(p_{y,x}): M(x) \rightarrow M(y)$  the *structure linear maps of  $M$*  for all  $(x, y) \in [\leq]_{\mathbf{P}}$ . To shorten the notation,  $M(p_{y,x})$  is also written as  $M_{y,x}$ . Clearly, for all  $x \in \mathbf{P}$ ,  $M_{x,x}$  is the identity map.
- (3) In the sequel, we simply refer to natural transformations between  $\mathbf{P}$ -persistence modules  $M$  and  $N$  as morphisms, and we adopt the simpler notation  $f: M \rightarrow N$  instead of  $f: M \Rightarrow N$ . Sometimes we also write  $(f_x)_{x \in \mathbf{P}}$  to express  $f$ . By the definition of natural transformation, the following diagram

$$\begin{array}{ccc} M(x) & \xrightarrow{M(p_{y,x})} & M(y) \\ f_x \downarrow & & \downarrow f_y \\ N(x) & \xrightarrow{N(p_{y,x})} & N(y) \end{array}$$

commutes for all  $(x, y) \in [\leq]_{\mathbf{P}}$ .

- (4) In fact, the persistence modules defined in this thesis are known as *point-wise finite-dimensional persistence modules* [19, 27]. By Definition 2.4.3, the persistent homology over  $\mathbf{P}$  is a  $\mathbf{P}$ -persistence module. In particular, the  $k$ -parameter persistent homology is the  $k$ -dimensional persistence module.

As a direct example, we take the 0th persistent homology of the  $\mathbb{A}_5$ -filtration  $\mathcal{F}$  provided in Figure 2.3(b). The associated  $\mathbb{A}_5$ -persistence module  $M := H_0(-; \mathbb{k}) \circ \mathcal{F}$  can be expressed by

$$\mathbb{k}^6 \xrightarrow{M_{2,1}} \mathbb{k}^5 \xrightarrow{M_{3,2}} \mathbb{k} \xrightarrow{M_{4,3}} \mathbb{k} \xrightarrow{M_{5,4}} \mathbb{k},$$

where

$$M_{2,1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad M_{3,2} = [1, 0, 0, 0, 0], \quad M_{4,3} = M_{5,4} = 1 \quad (1 := \mathbb{1}_{\mathbb{k}}).$$

In what follows, we will sometimes call  $\mathbf{P}$ -persistence modules “persistence modules over  $\mathbf{P}$ ”. The  $\mathbf{P}$ -persistence modules induce left  $\mathbb{k}\mathbf{P}$ -modules, where we recall that  $\mathbb{k}\mathbf{P}$  is the incidence algebra of  $\mathbf{P}$  (see Definition 2.3.1). Indeed, given a  $\mathbf{P}$ -persistence module  $M$ , we let  $\prod_{x \in \mathbf{P}} M(x)$  be the  $\mathbb{k}$ -vector space and define the left  $\mathbb{k}\mathbf{P}$  multiplication on the vector space by setting

$$\begin{aligned} \sum_{(x,y) \in [\leq]_{\mathbf{P}}} k_{y,x} p_{y,x} \cdot (m_z)_{z \in \mathbf{P}} &:= \left( \sum_{w \in \downarrow z} k_{z,w} p_{z,w} \cdot m_w \right)_{z \in \mathbf{P}} \\ &:= \left( \sum_{w \in \downarrow z} k_{z,w} M(p_{z,w})(m_w) \right)_{z \in \mathbf{P}}. \end{aligned} \quad (2.5.6)$$

The above multiplication is well-defined since only finitely many coefficients  $k_{y,x}$  are nonzero by the definition of  $\mathbb{k}\mathbf{P}$ . Particularly, we address the definition of the following left multiplication

$$p_{z,w} \cdot m_w := M(p_{z,w})(m_w), \quad (2.5.7)$$

where  $M \in \text{mod } \mathbb{k}[\mathbf{P}]$ ,  $p_{z,w} \in \mathbb{k}[\mathbf{P}](w, z)$ , and  $m_w \in M(w)$ .

**Definition 2.5.3.** Let  $\mathbf{P}$  be a poset and let  $M \in \text{mod } \mathbb{k}[\mathbf{P}]$ . Then the *dimension vector* of  $M$ , denoted by  $\underline{\dim} M$ , is a function

$$\underline{\dim} M: \mathbf{P} \rightarrow \mathbb{Z}, x \mapsto \dim M(x)$$

where  $\dim M(x)$  denotes the dimension of vector space  $M(x)$ . If  $\mathbf{P}$  is finite, then we call  $\dim M := \sum_{x \in \mathbf{P}} \dim M(x)$  the *dimension* of  $M$ .

**Example 2.5.4.** Let  $\mathbf{P} = \mathbf{G}_{3,3}$ . It is straightforward to verify the following assignment:

$$\begin{array}{ccccc} \mathbb{k} & \xrightarrow{1} & \mathbb{k} & \longrightarrow & 0 \\ \uparrow & & \uparrow [0,1] & & \uparrow \\ \mathbb{k} & \xrightarrow{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} & \mathbb{k}^2 & \xrightarrow{[1,1]} & \mathbb{k} \\ \uparrow & & \uparrow \begin{bmatrix} 1 \\ 0 \end{bmatrix} & & \uparrow 1 \\ 0 & \longrightarrow & \mathbb{k} & \xrightarrow{1} & \mathbb{k} \end{array}$$

is a persistence module over  $\mathbf{P}$ , whose dimension vector is visualized by  $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$  and dimension is 8.

An order-preserving map between posets induces a functor between the associated categories of persistence modules. This functor will be frequently used later, thus we give the following definition.

**Definition 2.5.5.** Let  $f: \mathbf{P}_1 \rightarrow \mathbf{P}_2$  be an order-preserving map and  $\mathbb{k}[f]$  the  $\mathbb{k}$ -linearization of  $f$  defined in (2.2.1) and (2.2.2). Then the *restriction functor*, denoted by  $R_f$ , is the  $\mathbb{k}$ -linear functor defined by

$$R_f: \text{mod } \mathbb{k}[\mathbf{P}_2] \rightarrow \text{mod } \mathbb{k}[\mathbf{P}_1], M \mapsto M \circ \mathbb{k}[f]. \quad (2.5.8)$$

From now on, we shall simply call a persistence module without adding the prefix  $\mathbf{P}$  if the underlying poset is fixed and unique in the statement.

**Definition 2.5.6.** Let  $M \in \text{mod } \mathbb{k}[\mathbf{P}]$ . Then the vector space  $\text{Hom}_{\mathbb{k}[\mathbf{P}]}(M, M)$  endowed with the multiplication  $*$  given by

$$f * g := g \circ f,$$

where  $\circ$  denotes the usual composition of morphisms (from right to left), is called the *endomorphism algebra* of  $M$ . We denote  $\text{Hom}_{\mathbb{k}[\mathbf{P}]}(M, M)$  shortly by  $\text{End}_{\mathbb{k}[\mathbf{P}]} M$ . Morphisms in  $\text{End}_{\mathbb{k}[\mathbf{P}]} M$  are called *endomorphisms* of  $M$ .

**Definition 2.5.7.** Let  $M \in \text{mod } \mathbb{k}[\mathbf{P}]$ , and let  $\text{End}_{\mathbb{k}[\mathbf{P}]} M$  be the endomorphism algebra of  $M$ . Then  $\text{End}_{\mathbb{k}[\mathbf{P}]} M$  is said to be *local* if for each  $f \in \text{End}_{\mathbb{k}[\mathbf{P}]} M$ , either  $f$  or  $\mathbb{1}_M - f$  is invertible in terms of the multiplication  $*$ .

Note that the endomorphism algebra  $\text{End}_{\mathbb{k}[\mathbf{P}]} M$  is unital because the identity  $\mathbb{1}_M$  is the multiplicative identity of  $\text{End}_{\mathbb{k}[\mathbf{P}]} M$ . Moreover,  $M$  induces a right  $\text{End}_{\mathbb{k}[\mathbf{P}]} M$ -module. Indeed, we let  $\prod_{x \in \mathbf{P}} M(x)$  be the  $\mathbb{k}$ -vector space and set

$$(m_x)_{x \in \mathbf{P}} \cdot f := (f_x(m_x))_{x \in \mathbf{P}}$$

for  $f \in \text{End}_{\mathbb{k}[\mathbf{P}]} M$  and  $(m_x)_{x \in \mathbf{P}} \in \prod_{x \in \mathbf{P}} M(x)$ . In the sequel, we sometimes omit  $\circ$  and simply write  $gf$  for the usual composition of morphisms.

**Definition 2.5.8.** Let  $f: M \rightarrow N$  be a morphism between  $\mathbf{P}$ -persistence modules.

- (1) The morphism  $f$  is called a *monomorphism* (resp. *epimorphism*) if for all  $x \in \mathbf{P}$ , the linear map  $f_x: M(x) \rightarrow N(x)$  is injective (resp. surjective).  $f$  is called an *isomorphism* if  $f$  is both monomorphism and epimorphism. In this case, we write  $M \cong N$  and say that  $M$  and  $N$  are isomorphic.
- (2) The morphism  $f$  is called a *section* (resp. *retraction*) if there exists a morphism  $g: N \rightarrow M$  such that  $gf = \mathbb{1}_M$  (resp.  $fg = \mathbb{1}_N$ ).

**Definition 2.5.9** (Submodules). Let  $M, N \in \text{mod } \mathbb{k}[\mathbf{P}]$ . Then  $M$  is called a *submodule* of  $N$ , denoted by  $M \subseteq N$ , if there is a morphism  $f: M \rightarrow N$  such that for every  $x \in \mathbf{P}$ ,  $M(x)$  is a linear subspace of  $N(x)$  and the linear map  $f_x$  is the inclusion. Moreover, the submodule  $M$  is called *proper* if  $M \neq N$ .

We remark that the morphism  $f: M \rightarrow N$  in the definition above is a monomorphism, and it will sometimes be written as  $f: M \hookrightarrow N$  hereafter. Moreover, the submodule relation  $\subseteq$  gives a partial order on the set of submodules of  $N$ .

Given a  $\mathbf{P}$ -persistence module and its submodule, one can define the quotient.

**Definition 2.5.10** (Factor modules). Let  $M, N \in \text{mod } \mathbb{k}[\mathbf{P}]$  with  $M \subseteq N$ . The quotient  $N/M$ , called the *factor module*, is again a  $\mathbf{P}$ -persistence module defined by  $(N/M)(x) := N(x)/M(x) \in \text{mod } \mathbb{k}$  for  $x \in \mathbf{P}$ , and structure linear maps of  $N/M$  are given by the universal property of cokernel in  $\text{mod } \mathbb{k}$ .

We note that there exists an epimorphism, called the *canonical projection*, from  $N$  to  $N/M$ , which is given by surjective linear maps  $N(x) \rightarrow (N/M)(x)$  for all  $x \in \mathbf{P}$ . The canonical projection will sometimes be expressed as  $N \xrightarrow{\text{can.}} N/M$  in the sequel.

**Definition 2.5.11.** Let  $M_1, M_2 \in \text{mod } \mathbb{k}[\mathbf{P}]$ . Their *direct sum*  $M_1 \oplus M_2$  is again a persistence module defined as follows. For each pair  $(x, y) \in \mathbf{P} \times \mathbf{P}$  and each  $p \in \mathbb{k}[\mathbf{P}](x, y)$ ,

$$(M_1 \oplus M_2)(p) := \begin{bmatrix} M_1(p) & \mathbf{0} \\ \mathbf{0} & M_2(p) \end{bmatrix}.$$

The modules  $M_1$  and  $M_2$  are called *direct summands* of  $M_1 \oplus M_2$ .

**Definition 2.5.12.** A short exact sequence  $0 \rightarrow L \xrightarrow{f} M \xrightarrow{g} N \rightarrow 0$  in  $\text{mod } \mathbb{k}[\mathbf{P}]$  is said to be *split* if  $f$  is a section, or equivalently,  $g$  is a retraction. Hence  $M \cong L \oplus N$ .

**Definition 2.5.13** (Indecomposable modules). A nonzero  $\mathbf{P}$ -persistence module  $M$  is *indecomposable* if  $M \cong M_1 \oplus M_2$  then either  $M_1 = 0$  or  $M_2 = 0$ , we also say that  $M$  admits only trivial decompositions in this case. Otherwise,  $M$  is said to be *decomposable* if  $M$  admits a nontrivial decomposition.

For later use, we state the next useful and well-known proposition ([7, Corollary 4.8]).

**Proposition 2.5.14.** Let  $M \in \text{mod } \mathbb{k}[\mathbf{P}]$ . If its endomorphism algebra  $\text{End}_{\mathbb{k}[\mathbf{P}]} M$  is local, then  $M$  is indecomposable. Conversely, if  $M$  is a finite-dimensional indecomposable module, then  $\text{End}_{\mathbb{k}[\mathbf{P}]} M$  is local.

Note that if  $\mathbf{P}$  is finite, then  $M$  is indecomposable if and only if  $\text{End}_{\mathbb{k}[\mathbf{P}]} M$  is local.

Determining whether every  $\mathbf{P}$ -persistence module admits a unique decomposition into indecomposable modules is a fundamental question in practical analysis, as uniqueness in this decomposition ensures a consistent and canonical representation of data. According to the result by Botnan and Crawley-Boevey [18, Theorem 1.1], any functor  $M \in \text{Fun}(\mathcal{C}, \text{mod } \mathbb{k})$  defined on a small category  $\mathcal{C}$  is a direct sum of indecomposable functors with local endomorphism ring in  $\text{Fun}(\mathcal{C}, \text{mod } \mathbb{k})$ . Recalling that the incidence category  $\mathbb{k}[\mathbf{P}]$  is small and persistence modules are functors in  $\text{Fun}(\mathbb{k}[\mathbf{P}], \text{mod } \mathbb{k})$ , it then follows that every  $\mathbf{P}$ -persistence module admits a decomposition into indecomposable modules, each possessing a local endomorphism ring.

On the other hand, by the result of Azumaya [9] (also known as the Krull–Remak–Schmidt–Azumaya theorem), such an indecomposable decomposition is unique up to reordering and isomorphism. Subsequently we summarize the above consequences as the unique decomposition theorem for persistence modules.

**Theorem 2.5.15** (Unique decomposition). *Let  $\mathbf{P}$  be a poset and  $\mathbb{k}[\mathbf{P}]$  the incidence category. Fix a complete set  $\mathcal{L}$  of representatives of isoclasses of indecomposable modules in  $\text{mod } \mathbb{k}[\mathbf{P}]$ . Then every  $\mathbf{P}$ -persistence module  $M$  is isomorphic to the direct sum  $\bigoplus_{L \in \mathcal{L}} L^{d_M(L)}$  for some unique function  $d_M: \mathcal{L} \rightarrow \mathbb{Z}_{\geq 0}$ . Therefore another  $\mathbf{P}$ -persistence module  $N$  is isomorphic to  $M$  if and only if  $d_M = d_N$ .*

We will use the notation  $d_M(L)$  to denote the *multiplicity* of an indecomposable direct summand  $L$  of a module  $M$  in its indecomposable decomposition.

Studying all indecomposable modules in  $\text{mod } A$  is a wild problem, thus we shall restrict our attention to a smaller collection of indecomposable modules.

**Definition 2.5.16** (Simple modules). Let  $\mathbf{P}$  be a poset. A  $\mathbf{P}$ -persistence module  $S$  is said to be *simple* if  $S \neq 0$  and it has no nontrivial proper submodule. Equivalently,  $S$  is simple if  $S$  is isomorphic to the following  $\mathbf{P}$ -persistence module  $S_x$  for some element  $x \in \mathbf{P}$ :

$$S_x(y) := \begin{cases} \mathbb{k}, & \text{if } y = x, \\ 0, & \text{otherwise,} \end{cases}$$

and for  $p \in \mathbb{k}[\mathbf{P}]_1$ ,

$$S_x(p) := \begin{cases} k \mathbb{1}_x, & \text{if } p := k \mathbb{1}_x \in \mathbb{k}[\mathbf{P}](x, x), \\ 0, & \text{otherwise.} \end{cases}$$

It is evident that simple modules are indecomposable.

In what follows, we identify  $\mathbb{k}[\mathbf{P}^{\text{op}}]$  with  $(\mathbb{k}[\mathbf{P}])^{\text{op}}$  in an obvious way. By  $D$  we denote the standard  $\mathbb{k}$ -duality  $\text{Hom}_{\mathbb{k}}(-, \mathbb{k})$ , which induces the duality functors  $\text{mod } \mathbb{k}[\mathbf{P}] \rightarrow \text{mod } \mathbb{k}[\mathbf{P}^{\text{op}}]$  and  $\text{mod } \mathbb{k}[\mathbf{P}^{\text{op}}] \rightarrow \text{mod } \mathbb{k}[\mathbf{P}]$ . Note that  $D$  is an equivalence of categories and thus it is an exact functor.

**Definition 2.5.17** (Projective and injective modules). Let  $\mathbf{P}$  be a poset and  $M$  a  $\mathbf{P}$ -persistence module.

- (1)  $M$  is said to be *projective* if for any epimorphism  $f: L \rightarrow N$ , the induced map of vector spaces

$$\text{Hom}_{\mathbb{k}[\mathbf{P}]}(M, f): \text{Hom}_{\mathbb{k}[\mathbf{P}]}(M, L) \rightarrow \text{Hom}_{\mathbb{k}[\mathbf{P}]}(M, N), \quad g \mapsto fg,$$

is surjective. In particular, for each  $x \in \mathbf{P}$ ,  $\mathbb{k}[\mathbf{P}](x, -)$  is projective and indecomposable. We denote it by  $P_x$  and call it the *projective indecomposable module at  $x$* .

- (2)  $M$  is said to be *injective* if for any monomorphism  $f: L \rightarrow N$ , the induced map of vector spaces

$$\text{Hom}_{\mathbb{k}[\mathbf{P}]}(f, M): \text{Hom}_{\mathbb{k}[\mathbf{P}]}(N, M) \rightarrow \text{Hom}_{\mathbb{k}[\mathbf{P}]}(L, M), \quad g \mapsto gf,$$

is injective. In particular, for each  $x \in \mathbf{P}$ ,  $D(\mathbb{k}[\mathbf{P}](-, x))$  is injective and indecomposable. We denote it by  $I_x$  and call it the *injective indecomposable module at  $x$* .

It is straightforward to show that direct summands of projective modules are again projective modules.

**Notation 2.5.18.** We set  $P'_x := \mathbb{k}[\mathbf{P}^{\text{op}}](x, -)$  to be the projective indecomposable  $\mathbb{k}[\mathbf{P}^{\text{op}}]$ -module at  $x$ , and  $I'_x := D(\mathbb{k}[\mathbf{P}^{\text{op}}](-, x))$  to be the injective indecomposable  $\mathbb{k}[\mathbf{P}^{\text{op}}]$ -module at  $x$ .

The following wider collection of indecomposables will play the main roles later.

**Definition 2.5.19** (Interval modules). Let  $\mathbf{I}$  be an interval of  $\mathbf{P}$ .

- (1) A  $\mathbf{P}$ -persistence module associated with  $\mathbf{I}$ , denoted by  $V_{\mathbf{I}}$ , is defined as follows. For  $x \in \mathbf{P}$ ,

$$V_{\mathbf{I}}(x) := \begin{cases} \mathbb{k}, & \text{if } x \in \mathbf{I}, \\ 0, & \text{otherwise,} \end{cases}$$

and for  $p \in \mathbb{k}[\mathbf{P}](x, y)$ ,

$$V_{\mathbf{I}}(p) := \begin{cases} k\mathbb{1}_{\mathbb{k}}, & \text{if } (x, y) \in [\leq]_{\mathbf{I}} \text{ and } p := kp_{y,x}, \\ 0, & \text{otherwise.} \end{cases}$$

It is obvious that  $V_{\mathbf{I}}$  is an indecomposable persistence module. We sometimes call  $V_{\mathbf{I}}$  is supported on  $\mathbf{I}$ .

- (2) A  $\mathbf{P}$ -persistence module isomorphic to  $V_{\mathbf{I}}$  for some  $\mathbf{I} \in \mathbb{I}$  is called an *interval module*. We thus call  $d_M(L)$  the *interval multiplicity* if  $L$  is an interval module.
- (3) A  $\mathbf{P}$ -persistence module is said to be *interval-decomposable* if it is isomorphic to a direct sum of interval modules. Thus the zero  $\mathbf{P}$ -persistence module is trivially interval-decomposable.

**Remark 2.5.20.** Simple modules, projective and injective indecomposable modules are interval modules because they are isomorphic to  $V_{\{x\}}$ ,  $V_{\uparrow y}$ , and  $V_{\downarrow z}$  for some  $x, y$ , and  $z \in \mathbf{P}$ , respectively. It is evident that  $V_{\mathbf{I}}$  is uniquely determined by its dimension vector, thus we sometimes identify  $V_{\mathbf{I}}$  with its dimension vector, and will write  $V_{\mathbf{I}}$  by its dimension vector hereafter.

Let us briefly review the standard one-parameter (resp. zigzag) persistent homology theory from the perspective of representation theory. By the unique decomposition theorem (Theorem 2.5.15) and [27, Theorem 1.1], every  $i$ th persistent homology is uniquely determined by the set of all intervals of underlying totally ordered (resp. zigzag) posets  $\mathbf{P}$ . In other words, every one-dimensional persistence (resp. zigzag) module  $M$  is interval-decomposable and it is completely characterized by the *multiset*  $(\mathbb{I}(\mathbf{P}), d_M)$ . The multiset  $(\mathbb{I}(\mathbf{P}), d_M)$  is called the *persistence diagram* of  $M$  (denoted by  $\mathcal{D}(M)$ ) or *barcodes* of  $M$  (denoted by  $\mathcal{B}(M)$ ). See Figure 2.3(b) for an illustration of barcodes of  $H_0(-; \mathbb{k}) \circ \mathcal{F}$  and  $H_1(-; \mathbb{k}) \circ \mathcal{F}$ . Again, we highlight that intervals capture the “lifespan” of the topological features of input data. For example, the decomposition of the 0th persistent homology of the  $\mathbb{A}_5$ -filtration in Figure 2.3(b) can be written as

$$H_0(-; \mathbb{k}) \circ \mathcal{F} \cong V_{[1,5]} \oplus V_{[1,2]}^4 \oplus V_{[1,1]}.$$

The longest interval  $[1, 5]$  in the decomposition above corresponds to the unique connected component born at the first filtration step that persists until the last step, giving it the maximum lifespan. In contrast, there are four same intervals  $[1, 2]$  in the middle—each tracking a connected component born at step 1 and dying at step 2—and one interval  $[1, 1]$ , for the single connected component born and killed immediately at step 1.

The fast computation of one-parameter and zigzag persistent homology has been a central focus in applied topology. For standard persistence, efficient matrix reduction algorithms were introduced by Edelsbrunner et al. [36], and further optimized in software packages like Ripser [11]. For zigzag persistence, the seminal work of Carlsson and De Silva [21] extended these ideas using reflection functors and similar algebraic techniques, enabling efficient computation over finite filtrations. More recently, Dey and Hou [29] proposed a faster algorithm for computing zigzag persistence, improving the practical efficiency of the original framework.

We end this section with a classical correspondence given by the Yoneda lemma. Let  $M \in \text{mod } \mathbb{k}[\mathbf{P}]$ , and let  $x, y \in \mathbf{P}$ . By the Yoneda lemma (see [7, Theorem 6.1] for example), we have an isomorphism

$$M(x) \rightarrow \text{Hom}_{\mathbb{k}[\mathbf{P}]}(P_x, M), \quad m \mapsto \rho_m^M \quad (m \in M(x)),$$

where  $\rho_m^M: P_x \rightarrow M$  is a morphism  $(\rho_{m,y}^M: P_x(y) \rightarrow M(y))_{y \in \mathbf{P}}$  defined by

$$\rho_{m,y}^M(p) := p \cdot m = M(p)(m) \tag{2.5.9}$$

for all  $y \in \mathbf{P}$  and  $p \in P_x(y) = \mathbb{k}[\mathbf{P}](x, y)$ . Note that the second equality in (2.5.9) is given by (2.5.7). Sometimes we just write  $\rho_m^M(p) := M(p)(m)$  by omitting  $y$ . Dually, we have an isomorphism

$$N(x) \rightarrow \text{Hom}_{\mathbb{k}[\mathbf{P}^{\text{op}}]}(P'_x, N), \quad m \mapsto \lambda_m^N \quad (m \in N(x)),$$

where the notation  $P'_x$  is provided in Notation 2.5.18 and  $\lambda_m^N: P'_x \rightarrow N$  is defined by

$$\lambda_m^N(p) := m \cdot p = N(p)(m). \tag{2.5.10}$$

Again, note that the second equality in (2.5.10) is given by (2.5.7) together with the duality.

To shorten formulas later, we introduce the following notation.

**Notation 2.5.21.** (1) The structure linear map  $P_z(p_{y,x})$  of the projective indecomposable  $P_z$  is given by

$$P_z(p_{y,x}): P_z(x) \rightarrow P_z(y), \quad m \mapsto p_{y,x}m \quad (m \in P_z(x)).$$

- (2) Since  $p_{y,x} \in \mathbb{k}[\mathbf{P}](x, y) = P_x(y)$ , we can set  $\mathbf{P}_{y,x} := \rho_{p_{y,x}} := \rho_{p_{y,x}}^{P_x}: P_y \rightarrow P_x$ . Furthermore, if  $x \leq y \leq z$ , then

$$(\mathbf{P}_{y,x})_z: P_y(z) \rightarrow P_x(z), \quad p_{z,y} \mapsto p_{z,y}p_{y,x} = p_{z,x}. \quad (2.5.11)$$

We note here that  $\mathbf{P}_{y,x} = 0$  if  $x \not\leq y$  in  $\mathbf{P}$ . Similarly, we set  $p_{x,y}^{\text{op}} := p_{y,x} \in \mathbb{k}[\mathbf{P}^{\text{op}}](y, x) = \mathbb{k}[\mathbf{P}](x, y)$  for all  $(x, y) \in [\leq]_{\mathbf{P}}$ . It induces a morphism  $\mathbf{P}'_{x,y} := \rho_{p_{x,y}^{\text{op}}}: P'_x \rightarrow P'_y$  in  $\text{mod } \mathbb{k}[\mathbf{P}^{\text{op}}]$ .

## 2.6 Almost split sequences

To make our exposition self-contained, we collect some necessary materials in [7, Chapter IV]. Throughout this section, the poset  $\mathbf{P}$  is assumed to be finite, and let  $A := \mathbb{k}[\mathbf{P}]$  for short, so the category of persistence modules is denoted by  $\text{mod } A$ . It is worth noting that  $\text{mod } A$  is an abelian  $\mathbb{k}$ -linear category.

We begin with reviewing the socle and radical of persistence modules. A persistence module isomorphic to a finite direct sum of simple modules is called the *semisimple module*. Clearly, except for the linear maps  $M_{x,x}$  ( $x \in \mathbf{P}$ ), all structure linear maps of a semisimple module  $M$  are zero by Definitions 2.5.11 and 2.5.16.

**Definition 2.6.1.** Let  $L, M \in \text{mod } A$  with  $L \subseteq M$ .

- (1)  $L$  is called the *socle* of  $M$  if  $L$  is the maximum semisimple submodule of  $M$ . We denote this by  $L = \text{soc } M$ .
- (2)  $L$  is called the *radical* of  $M$  if  $L$  is the intersection of all maximal submodules of  $M$ . We denote this by  $L = \text{rad } M$ .
- (3) The factor module  $M/\text{rad } M$  is called the *top* of  $M$ . We denote this module by  $\text{top } M$ .

**Remark 2.6.2.** (1) We remark that for  $x \in \mathbf{P}$ ,  $\text{soc } I_x \cong \text{top } P_x \cong S_x$ .

- (2) The socle (resp. radical, top) preserves the finite direct sum. Namely,

$$\text{soc}(M_1 \oplus \cdots \oplus M_n) \cong \text{soc } M_1 \oplus \cdots \oplus \text{soc } M_n.$$

The isomorphism holds similarly for the radical and top.

**Definition 2.6.3.** We have the following definitions in  $\text{mod } A$ .

- (1) An epimorphism (resp. monomorphism)  $f: M \rightarrow N$  is called *minimal* if for any morphism  $u: L \rightarrow M$  (resp.  $v: N \rightarrow L'$ ), the composition  $fu$  is an epimorphism (resp. the composition  $vf$  is a monomorphism) if and only if  $u$  is an epimorphism (resp.  $v$  is a monomorphism).

- (2) Moreover, a minimal epimorphism  $f: P \rightarrow N$  is said to be a *projective cover* of  $N$  if  $P$  is projective. Dually, a minimal monomorphism  $g: M \rightarrow I$  is said to be an *injective envelope* of  $M$  if  $I$  is injective.
- (3) An exact sequence  $P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} N \rightarrow 0$  is said to be a *projective presentation* of  $N$  if both  $P_0$  and  $P_1$  are projective. Moreover, such an exact sequence is said to be a *minimal projective presentation* of  $N$  if both morphisms  $f_0: P_0 \rightarrow N$  and  $f_1: P_1 \rightarrow \text{Im } f_1$  are projective covers.

Dually, An exact sequence  $0 \rightarrow M \xrightarrow{g_0} I_0 \xrightarrow{g_1} I_1$  is said to be an *injective presentation* of  $M$  if both  $I_0$  and  $I_1$  are injective. Moreover, such an exact sequence is said to be a *minimal injective presentation* of  $M$  if both morphisms  $g_0: M \rightarrow I_0$  and  $g_1: \text{Im } g_1 \rightarrow I_1$  are injective envelopes.

**Definition 2.6.4.** We have the following definitions in  $\text{mod } A$ .

- (1) A morphism  $g: M \rightarrow N$  is said to be *right minimal* if for all endomorphisms  $\varphi: M \rightarrow M$  satisfying  $g\varphi = g$  are automorphisms. Dually, a morphism  $f: L \rightarrow M$  is said to be *left minimal* if for all endomorphisms  $\varphi: M \rightarrow M$  satisfying  $\varphi f = f$  are automorphisms.
- (2) A morphism  $g: M \rightarrow N$  is said to be *right almost split* if  $g$  is not a retraction, and for any non-retraction morphism  $g': M' \rightarrow N$ , there is a morphism  $\psi: M' \rightarrow M$  such that  $g\psi = g'$ . Dually, a morphism  $f: L \rightarrow M$  is said to be *left almost split* if  $f$  is not a section, and for any non-section morphism  $f': L \rightarrow M'$ , there is a morphism  $\phi: M \rightarrow M'$  such that  $\phi f = f'$ .
- (3) A short exact sequence  $0 \rightarrow L \xrightarrow{f} M \xrightarrow{g} N \rightarrow 0$  is said to be an *almost split sequence* if  $f$  is left almost split and left minimal, and  $g$  is right almost split and right minimal.

The following lemma establishes a connection between the minimality and the right minimality for epimorphisms with projective domains, and it will be useful in the sequel.

**Lemma 2.6.5.** *Let  $f: P \rightarrow M$  be an epimorphism in  $\text{mod } A$ , where  $P$  is a projective module. Then the following statements are equivalent:*

- (1)  $f$  is minimal, hence  $f$  is a projective cover of  $M$ .
- (2)  $f$  is right minimal.
- (3) For any section  $s: X \rightarrow P$ ,  $fs = 0$  implies that  $s = 0$ .

*Proof.* (1) implies (3):

Assume that (1) holds. For any section  $s: X \rightarrow P$  with  $fs = 0$ , we have  $\text{Im } s \subseteq \text{Ker } f$ . Thus the morphism  $\Phi: P/\text{Im } s \rightarrow P/\text{Ker } f$ ,  $x + \text{Im } s \mapsto x + \text{Ker } f$  is a well-defined epimorphism. By  $\pi_f$  (resp.  $\pi_s$ ) we denote the canonical projection  $P \rightarrow P/\text{Ker } f$  (resp.  $P \rightarrow P/\text{Im } s$ ). Apparently,  $\Phi\pi_s = \pi_f$ . We claim that  $\pi_s$  is minimal. Indeed, for any morphism  $u: L \rightarrow P$  satisfying that the composition

$$L \xrightarrow{u} P \xrightarrow{\pi_s} P/\text{Im } s$$

is an epimorphism, the following composition

$$L \xrightarrow{u} P \xrightarrow{\pi_s} P/\text{Im } s \xrightarrow{\Phi} P/\text{Ker } f \xrightarrow{\sim} M$$

is again an epimorphism. By the minimality of  $f: P \rightarrow M$ ,  $u$  is an epimorphism. This shows the minimality of  $\pi_s$ . Now, because  $s$  is a section,  $\pi_s$  is a retraction and hence there exists a section  $v: P/\text{Im } s \rightarrow P$  such that  $\pi_s v = \mathbb{1}_{P/\text{Im } s}$ . The minimality of  $\pi_s$  yields that  $v$  is an epimorphism and thus is an isomorphism. It follows that  $\text{Im } s = 0$ . Therefore, (3) holds.

(3) implies (2):

Assume that (3) holds. By [8, Chapter I, Theorem 2.2], there is a decomposition  $P = P_1 \oplus P_2$  such that  $f|_{P_1}$  is right minimal and  $f|_{P_2} = 0$ , where  $f|_{P_i}$  denotes the restriction of  $f$  to  $P_i$  ( $i = 1, 2$ ). By (3) we have  $P_2 = 0$  and  $f|_{P_1} = f$ . Therefore, (2) holds.

(2) implies (1):

Assume that (2) holds. To show the minimality of  $f$ , take any morphism  $u: L \rightarrow P$  such that  $fu: L \rightarrow M$  is an epimorphism. Since  $P$  is projective, there is a morphism  $h: P \rightarrow L$  such that  $(fu)h = f(uh) = f$ . By (2), the right minimality of  $f$  implies that  $uh$  is an automorphism. Therefore,  $h$  is an epimorphism and (1) holds.

Hence, the three statements are equivalent.  $\square$

Lemma 2.6.5 together with [8, Chapter I, Theorem 2.2] immediately shows the following.

**Corollary 2.6.6.** *Let  $f: P \rightarrow M$  be an epimorphism in  $\text{mod } A$ , where  $P$  is a projective module. Then there is a decomposition  $P = P_1 \oplus P_2$  such that  $f|_{P_1}$  is a projective cover of  $M$  and  $f|_{P_2} = 0$ . In other words, the epimorphism  $f$  can be written as  $f = [f_1, \mathbf{0}] := P_1 \oplus P_2 \rightarrow M$ , where  $f_1: P_1 \rightarrow M$  is a projective cover of  $M$ .*

In short, an epimorphism with a projective domain includes a projective cover as the component.

It has been known that if  $0 \rightarrow L \xrightarrow{f} M \xrightarrow{g} N \rightarrow 0$  is an almost split sequence, then both  $L$  and  $N$  are indecomposable (see [7, Chapter IV, Lemma 1.3]). Conversely, if  $N$  (resp.  $L$ ) is an indecomposable module of certain types, one can always obtain an almost split sequence ending at  $N$  (resp. starting from  $L$ ). We would like to address the converse direction later, which is known as the existence theorem of almost split sequences.

The incidence category  $A$  itself can be viewed as a  $\mathbf{P}$ -persistence module  $A(?, -)$ . Indeed, we might treat  $A(?, -)$  as  $\bigoplus_{w \in \mathbf{P}} A(w, -) = \bigoplus_{w \in \mathbf{P}} P_w$ .

**Definition 2.6.7.** We recall the  $\mathbb{k}$ -duality functor  $D := \text{Hom}_{\mathbb{k}}(-, \mathbb{k})$ .

(1) We denote by  $(\star)^t$  the following contravariant functors

$$\begin{aligned} \text{Hom}_A(\star, A(?, -)): \text{mod } A &\rightarrow \text{mod } A^{\text{op}}, \\ M &\mapsto \text{Hom}_A(M, A(?, -)), \text{ and} \\ \text{Hom}_{A^{\text{op}}}(\star, A^{\text{op}}(-, ?)): \text{mod } A^{\text{op}} &\rightarrow \text{mod } A, \\ M &\mapsto \text{Hom}_{A^{\text{op}}}(M, A(?, -)). \end{aligned}$$

Recall the notation  $P'_x$  provided in Notation 2.5.18. Then by the Yoneda lemma, we have

$$P_x^t = \text{Hom}_A(A(x, -), A(?, -)) \cong A(?, x) = A^{\text{op}}(x, ?) = P'_x$$

for all  $x \in \mathbf{P}$ . By this natural isomorphism, we usually identify  $P'_x$  with  $P_x^t$ , and  $P'_{x,y}$  with  $(P_{y,x})^t$  for all  $x, y \in \mathbf{P}$ . For this reason, we write  $P^t$  instead of  $P'$  in the sequel if there is no confusion. The contravariant functors  $(\star)^t$  are called *A-duality functors*. In what follows, we sometimes write  $\text{Hom}_A(\star, A)$  for short.

(2) The composition  $\nu := D \circ (\star)^t$  is called the *Nakayama functor*.

**Remark 2.6.8.** (1) The *A-duality functors* are dualities between  $\text{prj } A$  and  $\text{prj } A^{\text{op}}$ , where  $\text{prj } A$  denotes the full subcategory of  $\text{mod } A$  consisting of all projective modules. Concretely, the isomorphism  $P^{tt} \cong P$  holds for all projective modules  $P$ .

(2) The Nakayama functor  $\nu$  is right exact, and the isomorphisms  $\nu P_x \cong I_x$ ,  $\nu I_x \cong P_x$  hold.

Let  $V \in \text{mod } A$ . Then we take a minimal projective presentation of  $V$  in  $\text{mod } A$  as follows:

$$P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} V \rightarrow 0.$$

Applying the *A-duality functor*  $(\star)^t$  on the minimal projective presentation yields the following exact sequence in  $\text{mod } A^{\text{op}}$ :

$$0 \rightarrow V^t \xrightarrow{f_0^t} P_0^t \xrightarrow{f_1^t} P_1^t \rightarrow \text{Coker } f_1^t \rightarrow 0. \quad (2.6.12)$$

The cokernel  $\text{Coker } f_1^t$  is called the *transpose of V* and denoted by  $\text{Tr } V$ .

Now we are in the position to define the *Auslander–Reiten translations* of persistence modules, and the existence theorem of almost split sequences.

**Definition 2.6.9.** Let  $V \in \text{mod } A$ . Then compositing first  $\text{Tr}$  and then  $D$  yields  $D \circ \text{Tr}(V) \in \text{mod } A$ , denoted by  $\tau V$ . On the otherhand, compositing first  $D$  and then  $\text{Tr}$  yields  $\text{Tr} \circ D(V) \in \text{mod } A$ , denoted by  $\tau^{-1}V$ . Both  $\tau V$  and  $\tau^{-1}V$  are called the *Auslander–Reiten translation of V*.

**Theorem 2.6.10** ([7]). *Let  $V \in \text{mod } A$ .*

(1) *If  $V$  is a non-injective indecomposable module, then there exists an almost split sequence starting from  $V$ :*

$$0 \rightarrow V \rightarrow E \rightarrow \tau^{-1}V \rightarrow 0. \quad (2.6.13)$$

(2) *If  $V$  is a non-projective indecomposable module, then there exists an almost split sequence ending at  $V$ :*

$$0 \rightarrow \tau V \rightarrow F \rightarrow V \rightarrow 0.$$

**Remark 2.6.11.** (1) If  $V$  is an injective indecomposable module, then the projection  $V \rightarrow V/\text{soc } V$  is the right almost split and right minimal morphism.

- (2) If  $V$  is a projective indecomposable module, then the inclusion  $\text{rad } V \hookrightarrow V$  is the left almost split and left minimal morphism.

One application of the almost split sequence is to compute multiplicities of indecomposable modules. We cite [4, Theorem 3] as the following theorem (see also [34, Corollary 2.3]).

**Theorem 2.6.12** ([4]). *Let  $M, V \in \text{mod } A$  and assume that  $V$  is indecomposable. Then we have the following formulas.*

- (1) *If  $V$  is injective, then*

$$d_M(V) = \dim \text{Hom}_A(V, M) - \dim \text{Hom}_A(V/\text{soc } V, M). \quad (2.6.14)$$

- (2) *If  $V$  is non-injective, then there is an almost split sequence (2.6.13) starting from  $V$  by Theorem 2.6.10. Moreover,*

$$d_M(V) = \dim \text{Hom}_A(V, M) - \dim \text{Hom}_A(E, M) + \dim \text{Hom}_A(\tau^{-1}V, M). \quad (2.6.15)$$



# Chapter 3

## Formulas for interval multiplicities

In the standard one-parameter persistent homology, the multiplicity of an interval can be computed by taking ranks (also known as persistent Betti numbers) along some larger intervals and then operating an alternating sum of the ranks by the inclusion-exclusion principle. This computation is well-known as the formula of the persistent Betti numbers and the multiplicity (see [35, Chapter VII]).

More precisely, if we let poset  $\mathbf{P} = \mathbb{A}_n$ ,  $\mathcal{F}$  a  $\mathbf{P}$ -filtration, and  $M := H_i(-; \mathbb{K}) \circ \mathcal{F}$  the  $i$ th persistent homology, then the multiplicity of each interval  $\mathbf{I} := [s, t] \subseteq \mathbf{P}$  appearing in the  $i$ th persistence barcodes of  $M$  is equal to the alternating sum:

$$\text{rank } M_{t,s} - \text{rank } M_{t,s-1} - \text{rank } M_{t+1,s} + \text{rank } M_{t+1,s-1},$$

where we recall that  $M_{t,s}$  denotes the linear map  $M_{t,s}: M(s) \rightarrow M(t)$ . As a demonstration, in Figure 3.1 we consider the multiplicity of interval  $[3, 4]$ . The intervals that appear in the alternating sum are illustrated in violet. It is straightforward to see that  $\text{rank } M_{4,3} = 2$ ,  $\text{rank } M_{4,2} = \text{rank } M_{5,3} = 1$ , and  $\text{rank } M_{5,2} = 0$ , thus the multiplicity of  $[3, 4]$  is zero. Similarly, one can compute that multiplicities of intervals  $[2, 4]$  and  $[3, 5]$  are both 1, and other intervals have zero multiplicities.

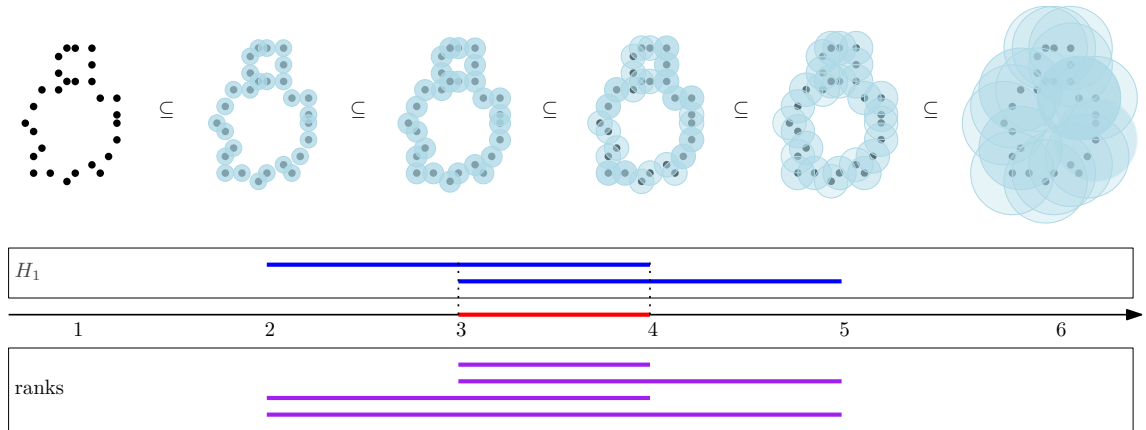


Figure 3.1: An illustration of the formula relating the persistent Betti numbers and multiplicities in one-parameter persistent homology. The barcodes of the 1st persistent homology are shown in blue. The red bar  $[3, 4]$  is the interval that we would like to demonstrate the computation of its multiplicity. The violet bars are intervals along which we take the ranks to recover the multiplicity.

We address that the meanings of the multiplicity and the rank are different. The multiplicity of an interval  $[s, t]$  in persistence barcodes indicates the number of generators of homology that are newly born at  $s$  (do not exist before  $s$ ) and die at  $t$  (do not exist after  $t$ ), persisting from  $s$  to  $t$ . In contrast, the rank computed over

$[s, t]$  only counts the number of generators of homology persisting throughout the interval, regardless of whether these generators are newly created or vanish precisely at the endpoints. In other words, the rank (or persistent Betti number) along an interval  $[s, t]$  depends solely on information contained within that interval itself, but determining multiplicity requires additional information beyond  $[s, t]$ , specifically information recorded within a larger interval, such as  $[s - 1, t + 1]$  in this case.

In multi-parameter persistence or persistence over general posets context, the persistent homology obtained is generally not interval-decomposable. Therefore, we need a formula for computing the interval multiplicity in general cases. The purpose of this chapter is to compute the interval multiplicity  $d_M(V_{\mathbf{I}})$ . We will first present our result in the general poset case in Section 3.1, which will be specialized to the case of 2D-grids in Section 3.2.

Throughout this chapter, we always assume that the poset  $\mathbf{P}$  is finite,  $\mathbf{I}$  is an interval of  $\mathbf{P}$  and  $M \in \text{mod } A$ , where  $A := \mathbb{k}[\mathbf{P}]$ .

### 3.1 The general poset case

Without loss of generality, we may assume that the poset  $\mathbf{P}$  is connected.

**Notation 3.1.1.** Let  $M \in \text{mod } A$ , and

$$\mu: \bigoplus_{j \in [n]} P_{y_j} \rightarrow \bigoplus_{i \in [m]} P_{x_i}$$

a morphism between projective modules of the form  $\mu := [a_{ji} \mathbf{P}_{y_j, x_i}]_{(i,j) \in [m] \times [n]}$  with  $a_{ji} \in \mathbb{k}$ ,  $((i, j) \in [m] \times [n])$  for some  $x_1, x_2, \dots, x_m, y_1, y_2, \dots, y_n \in \mathbf{P}$ . Then we set  $M(\mu)$  to be the matrix

$$M(\mu) := [a_{ji} M_{y_j, x_i}]_{(j,i) \in [n] \times [m]}.$$

The right-hand side is obtained by first replacing  $\mathbf{P}$  with  $M$ , and then making the transpose.

For any  $C, M \in \text{mod } A$ , the following lemma makes it possible to compute the dimension of  $\text{Hom}_A(C, M)$  by using a projective presentation of  $C$  and the module structure of  $M$ . Later we will mainly apply this to the case where  $C$  is a term of the almost split sequence starting from an interval module  $V_{\mathbf{I}}$  for some  $\mathbf{I} \in \mathbb{I}$ .

**Lemma 3.1.2.** *Let  $C, M \in \text{mod } A$ . Assume that  $C$  has a projective presentation*

$$\bigoplus_{j \in [n]} P_{y_j} \xrightarrow{\mu} \bigoplus_{i \in [m]} P_{x_i} \xrightarrow{\varepsilon} C \rightarrow 0$$

for some  $x_1, x_2, \dots, x_m, y_1, y_2, \dots, y_n \in \mathbf{P}$ , and  $\mu := [a_{ji} \mathbf{P}_{y_j, x_i}]_{(i,j) \in [m] \times [n]}$  with  $a_{ji} \in \mathbb{k}$ ,  $((i, j) \in [m] \times [n])$ . Then we have

$$\dim \text{Hom}_A(C, M) = \sum_{i=1}^m \dim M(x_i) - \text{rank } M(\mu). \quad (3.1.1)$$

*Proof.* Set  $Y := \bigoplus_{j \in [n]} P_{y_j}$ ,  $X := \bigoplus_{i \in [m]} P_{x_i}$  for short. Then we have an exact sequence  $Y \xrightarrow{\mu} X \xrightarrow{\varepsilon} C \rightarrow 0$ , which yields an exact sequence

$$0 \rightarrow \text{Hom}_A(C, M) \rightarrow \text{Hom}_A(X, M) \xrightarrow{\text{Hom}_A(\mu, M)} \text{Hom}_A(Y, M).$$

Hence  $\text{Hom}_A(C, M) \cong \text{Ker } \text{Hom}_A(\mu, M)$ . Now we have

$$\begin{aligned} \text{Ker } \text{Hom}_A(\mu, M) &= \{f \in \text{Hom}_A(X, M) \mid f\mu = 0\} \\ &\cong \left\{ [f_1, \dots, f_m] \in \bigoplus_{i \in [m]} \text{Hom}_A(P_{x_i}, M) \mid [f_1, \dots, f_m] \cdot [a_{ji} P_{y_j, x_i}]_{(i,j)} = 0 \right\} \\ &= \left\{ [f_1, \dots, f_m] \in \bigoplus_{i \in [m]} \text{Hom}_A(P_{x_i}, M) \mid \left[ \sum_{i \in [m]} a_{ji} f_i P_{y_j, x_i} \right]_{j \in [n]} = 0 \right\} \\ &\cong \left\{ \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} \in \bigoplus_{i \in [m]} M(x_i) \mid \left[ \sum_{i \in [m]} a_{ji} M_{y_j, x_i}(b_i) \right]_{j \in [n]} = 0 \right\} \\ &= \left\{ \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} \in \bigoplus_{i \in [m]} M(x_i) \mid {}^t([a_{ji} M_{y_j, x_i}]_{(i,j)}) \cdot \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} = 0 \right\} \\ &= \text{Ker} \left( M(\mu): \bigoplus_{i \in [m]} M(x_i) \rightarrow \bigoplus_{j \in [n]} M(y_j) \right). \end{aligned}$$

Hence  $\dim \text{Hom}_A(C, M) = \sum_{i \in [m]} \dim M(x_i) - \text{rank } M(\mu)$ .  $\square$

**Notation 3.1.3.** For a totally ordered set  $\mathbf{T}$ , we set  $C_2 \mathbf{T}$  to be the set of all full subposets of  $\mathbf{T}$  consisting of exactly two elements, and for any  $\mathbf{a} \in C_2 \mathbf{T}$ , we set  $\underline{\mathbf{a}} := \min \mathbf{a}$  and  $\bar{\mathbf{a}} := \max \mathbf{a}$ . Thus  $\mathbf{a} = \{\underline{\mathbf{a}}, \bar{\mathbf{a}}\}$ . Let  $\mathbf{U}$  be a full subposet of  $\mathbf{P}$ .

(1) Suppose  $|\text{sc}(\mathbf{U})| = n$  (resp.  $|\text{sk}(\mathbf{U})| = m$ ). We give a total order on the set  $\text{sc}(\mathbf{U})$  (resp.  $\text{sk}(\mathbf{U})$ ) by giving a poset isomorphism  $a: [n] \rightarrow \text{sc}(\mathbf{U})$ ,  $i \mapsto a_i$  (resp.  $b: [m] \rightarrow \text{sk}(\mathbf{U})$ ,  $i \mapsto b_i$ ), and apply the notation above for  $\mathbf{T} = \text{sc}(\mathbf{U})$  and  $\text{sk}(\mathbf{U})$ .

(2) For any  $\mathbf{a} \in C_2 \text{sc}(\mathbf{U})$  (resp.  $\mathbf{b} \in C_2 \text{sk}(\mathbf{U})$ ), we set

$$\vee' \mathbf{a} := \text{sc}(\uparrow_{\mathbf{P}} \underline{\mathbf{a}} \cap \uparrow_{\mathbf{P}} \bar{\mathbf{a}}) \quad (\text{resp. } \wedge' \mathbf{b} := \text{sk}(\downarrow_{\mathbf{P}} \underline{\mathbf{b}} \cap \downarrow_{\mathbf{P}} \bar{\mathbf{b}})).$$

Note that if  $\mathbf{U}$  is an up-set (resp. a down-set), then  $\vee' \mathbf{a} \subseteq \mathbf{U}$  (resp.  $\wedge' \mathbf{b} \subseteq \mathbf{U}$ ) by the definition of up-sets (resp. down-sets). In particular, if  $|\text{sc}(\uparrow_{\mathbf{P}} \underline{\mathbf{a}} \cap \uparrow_{\mathbf{P}} \bar{\mathbf{a}})| = 1$  (resp.  $|\text{sk}(\downarrow_{\mathbf{P}} \underline{\mathbf{b}} \cap \downarrow_{\mathbf{P}} \bar{\mathbf{b}})| = 1$ ), then the unique element is the join (resp. meet) of  $\mathbf{a}$  (resp.  $\mathbf{b}$ ) and is denoted by  $\vee \mathbf{a}$  (resp.  $\wedge \mathbf{b}$ ). See Definition 2.2.14. For this reason, we call  $\vee' \mathbf{a}$  (resp.  $\wedge' \mathbf{b}$ ) the *pre-join* (resp. *pre-meet*) of  $\mathbf{a}$  (resp.  $\mathbf{b}$ ).

(3) We set  $\text{sc}_1(\mathbf{U})$  (resp.  $\text{sk}_1(\mathbf{U})$ ) to be the disjoint union of the pre-joins (resp. pre-meets) of the two-element subsets of  $\text{sc}(\mathbf{U})$  (resp.  $\text{sk}(\mathbf{U})$ ):

$$\begin{aligned} \text{sc}_1(\mathbf{U}) &:= \bigsqcup_{\mathbf{a} \in C_2 \text{sc}(\mathbf{U})} \vee' \mathbf{a} = \{\mathbf{a}_c := (\mathbf{a}, c) \mid \mathbf{a} \in C_2 \text{sc}(\mathbf{U}), c \in \vee' \mathbf{a}\}, \\ \text{sk}_1(\mathbf{U}) &:= \bigsqcup_{\mathbf{b} \in C_2 \text{sk}(\mathbf{U})} \wedge' \mathbf{b} = \{\mathbf{b}_d := (\mathbf{b}, d) \mid \mathbf{b} \in C_2 \text{sk}(\mathbf{U}), d \in \wedge' \mathbf{b}\}. \end{aligned}$$

Note here that the family  $(\vee' \mathbf{a})_{\mathbf{a} \in C_2 \text{sc}(\mathbf{U})}$  (resp.  $(\wedge' \mathbf{b})_{\mathbf{b} \in C_2 \text{sk}(\mathbf{U})}$ ) does not need to be disjoint. By the definition above, we see that if  $\text{sc}(\mathbf{W}) = \text{sc}(\mathbf{U})$  (resp.  $\text{sk}(\mathbf{W}) = \text{sk}(\mathbf{U})$ ) for a  $\mathbf{W} \subseteq \mathbf{P}$ , then  $\text{sc}_1(\mathbf{W}) = \text{sc}_1(\mathbf{U})$  (resp.  $\text{sk}_1(\mathbf{W}) = \text{sk}_1(\mathbf{U})$ ).

Furthermore, we equip  $\text{sc}_1(\mathbf{U})$  with another total order  $\preceq_{\text{lex}}$ , defined by  $\mathbf{a}_c \preceq_{\text{lex}} \mathbf{a}'_c$  if and only if  $(\underline{\mathbf{a}}, \overline{\mathbf{a}}, c) \leq_{\text{lex}} (\underline{\mathbf{a}'}, \overline{\mathbf{a}'}, c')$ , where  $\leq_{\text{lex}}$  denotes the lexicographic order from left to right. Similarly, we give a total order to  $\text{sk}_1(\mathbf{W})$ . These total orders will be used to express matrices having  $\text{sc}_1(\mathbf{U})$  or  $\text{sk}_1(\mathbf{W})$  as an index set later.

- (4) For any nonempty subset  $X$  of  $\mathbf{P}$ , or a disjoint union  $X = \bigsqcup_{s \in S} X_s := \{s_x \mid s \in S, x \in X_s\}$  of nonempty subsets  $X_s$  of  $\mathbf{P}$  with nonempty index set  $S$ , we set  $P_X := \bigoplus_{t \in X} P_t$  and  $P'_X := \bigoplus_{t \in X} P'_t$ , where  $P_t := P_x$ ,  $P'_t := P'_x$  if  $t = s_x \in X = \bigsqcup_{s \in S} X_s$  with  $s \in S$  and  $x \in X_s$ . In addition, we set  $P_X$  and  $P'_X$  to be the zero modules if  $X = \emptyset$ .

The following lemmas are necessary in the sequel.

**Lemma 3.1.4.** *Let  $\mathbf{U}$  be either an up-set or a down-set of  $\mathbf{P}$ . Then  $\mathbf{U}$  is convex in  $\mathbf{P}$ . Moreover, if  $\mathbf{U}$  is connected, then  $\mathbf{U}$  is an interval.*  $\square$

**Lemma 3.1.5.** *Let  $\mathbf{I}$  be a connected subposet of  $\mathbf{P}$ . Then both  $\uparrow \mathbf{I}$  and  $\downarrow \mathbf{I}$  are again connected.*  $\square$

**Lemma 3.1.6.** *Let  $\mathbf{I}$  be an interval of  $\mathbf{P}$ . Then there exist up-sets  $\mathbf{U}, \mathbf{U}'$  and down-sets  $\mathbf{W}, \mathbf{W}'$  of  $\mathbf{P}$  such that  $\mathbf{I} = \mathbf{U} \setminus \mathbf{U}' = \mathbf{W} \setminus \mathbf{W}'$ , which are given by*

$$\mathbf{U} := \uparrow \mathbf{I}, \mathbf{U}' := \uparrow \mathbf{I} \setminus \mathbf{I}; \text{ and } \mathbf{W} := \downarrow \mathbf{I}, \mathbf{W}' := \downarrow \mathbf{I} \setminus \mathbf{I}.$$

*Proof.* We only need to show that  $\uparrow \mathbf{I} \setminus \mathbf{I}$  is an up-set of  $\mathbf{P}$ . Take any  $u \in \uparrow \mathbf{I}$  and  $x \in \mathbf{P}$  with  $u \leq x$ . Then there exists  $y \in \mathbf{I}$  with  $y \leq u$  since  $u \in \uparrow \mathbf{I}$ . Hence  $\mathbf{I} \ni y \leq u \leq x$ , which shows that  $x \in \uparrow \mathbf{I}$ . If  $x \in \mathbf{I}$ , then the convexity of  $\mathbf{I}$  shows that  $u \in \mathbf{I}$ , a contradiction. Thus  $x \notin \mathbf{I}$ , and hence  $x \in \uparrow \mathbf{I} \setminus \mathbf{I}$ . The proof for  $\downarrow \mathbf{I} \setminus \mathbf{I}$  to be a down-set of  $\mathbf{P}$  is similar.  $\square$

**Remark 3.1.7.** We note here that  $\uparrow \mathbf{I}$  (resp.  $\downarrow \mathbf{I}$ ) is connected, thus an interval by the previous lemmas. However,  $\uparrow \mathbf{I} \setminus \mathbf{I}$  (resp.  $\downarrow \mathbf{I} \setminus \mathbf{I}$ ) may not be connected in general.

Let  $\mathbf{U}$  be a connected up-set, and  $\mathbf{W}$  a connected down-set of  $\mathbf{P}$ . Note that  $V_{\mathbf{U}}$  is projective (resp.  $V_{\mathbf{W}}$  is injective) if and only if  $|\text{sc}(\mathbf{U})| = 1$  (resp.  $|\text{sk}(\mathbf{W})| = 1$ ) because  $V_{\mathbf{U}}$  and  $V_{\mathbf{W}}$  are indecomposable modules and  $\dim \text{top } V_{\mathbf{U}} = |\text{sc}(\mathbf{U})|$  (resp.  $\dim \text{soc } V_{\mathbf{W}} = |\text{sk}(\mathbf{W})|$ ). To show that the set  $\text{sc}_1(\mathbf{U})$  (resp.  $\text{sk}_1(\mathbf{W})$ ) is not empty if  $V_{\mathbf{U}}$  is not projective (resp.  $V_{\mathbf{W}}$  is not injective), we review a fundamental property of finite posets.

**Definition 3.1.8.** Let  $\mathbf{S}$  be a finite poset. A topology on  $\mathbf{S}$  is defined by setting the set of up-sets to be the open sets of  $\mathbf{S}$ , which is called the *Alexandrov topology* on  $\mathbf{S}$ . It is easy to see that it has a basis  $\{\uparrow_{\mathbf{S}} x \mid x \in \mathbf{S}\}$ .

The following lemmas are easy to show and the proofs are left to the reader.

**Lemma 3.1.9.** *Let  $\mathbf{S}$  be a finite poset considered as a topological space by the Alexandrov topology on  $\mathbf{S}$ . Then  $\mathbf{S}$  is a connected space if and only if  $\mathbf{S}$  is a connected poset.*  $\square$

**Lemma 3.1.10.** *Let  $\mathbf{S}$  be a finite poset considered as a topological space by the Alexandrov topology on  $\mathbf{S}$ . Then  $\mathbf{S}$  is a locally connected space.*  $\square$

Under the preparation above, we prove the following.

**Proposition 3.1.11.** *Let  $\mathbf{U}$  be a connected up-set,  $\mathbf{W}$  a connected down-set of  $\mathbf{P}$ .*

- (1)  $|\text{sc}(\mathbf{U})| \geq 2$  if and only if  $\text{sc}_1(\mathbf{U}) \neq \emptyset$ .
- (2)  $|\text{sk}(\mathbf{W})| \geq 2$  if and only if  $\text{sk}_1(\mathbf{W}) \neq \emptyset$ .

*Proof.* (1) Since the implication  $(\Leftarrow)$  is trivial, we show the implication  $(\Rightarrow)$ . Set  $\text{sc}(\mathbf{U}) = \{a_1, \dots, a_n\}$  and assume that  $n \geq 2$ . By Remark 2.2.13 (1), we have

$$\mathbf{U} = \uparrow a_1 \cup (\uparrow a_2 \cup \dots \cup \uparrow a_n). \quad (3.1.2)$$

Now suppose that  $\text{sc}_1(\mathbf{U}) = \emptyset$ . Then for any  $\{i, j\} \in C_2[n]$ , we have  $\text{sc}(\uparrow a_i \cap \uparrow a_j) = \emptyset$ , and hence  $\uparrow a_i \cap \uparrow a_j = \emptyset$ . This shows that

$$\uparrow a_1 \cap (\uparrow a_2 \cup \dots \cup \uparrow a_n) = \emptyset. \quad (3.1.3)$$

The equalities (3.1.2) and (3.1.3) show that the topological space  $\mathbf{U}$  with Alexandrov topology is not connected (also by noticing Remark 2.2.13 (4)). Hence  $\mathbf{U}$  is not a connected poset by Lemma 3.1.9, a contradiction. As a consequence,  $\text{sc}_1(\mathbf{U}) \neq \emptyset$ .

(2) This is shown similarly.  $\square$

With the above preliminaries, we first give a projective presentation of  $V_{\mathbf{U}} \in \text{mod } A$ , where  $\mathbf{U}$  is a connected up-set of  $\mathbf{P}$ .

**Proposition 3.1.12.** *Let  $\mathbf{U}$  be a connected up-set of  $\mathbf{P}$ . Then  $V_{\mathbf{U}}$  has the following (not necessarily minimal) projective presentation:*

$$P_{\text{sc}_1(\mathbf{U})} \xrightarrow{\varepsilon_1^{\mathbf{U}}} P_{\text{sc}(\mathbf{U})} \xrightarrow{\varepsilon_0^{\mathbf{U}}} V_{\mathbf{U}} \rightarrow 0, \quad (3.1.4)$$

where  $\varepsilon_0^{\mathbf{U}} = [\rho_{1_a}^{V_{\mathbf{U}}}]_{a \in \text{sc}(\mathbf{U})}$ , and we set  $1_u := 1_{\mathbf{k}} \in \mathbf{k} = V_{\mathbf{U}}(u)$  for all  $u \in \mathbf{U}$ , and

$$\varepsilon_1^{\mathbf{U}} := [\tilde{P}_{a, \mathbf{a}_c}]_{(a, \mathbf{a}_c) \in \text{sc}(\mathbf{U}) \times \text{sc}_1(\mathbf{U})} \quad (3.1.5)$$

with the entries given by

$$\tilde{P}_{a, \mathbf{a}_c} := \begin{cases} P_{c, a}, & \text{if } a = \underline{\mathbf{a}}, \\ -P_{c, a}, & \text{if } a = \overline{\mathbf{a}}, \\ 0, & \text{if } a \notin \mathbf{a}, \end{cases} \quad (3.1.6)$$

for all  $\mathbf{a}_c \in \text{sc}_1(\mathbf{U})$  and  $a \in \text{sc}(\mathbf{U})$ . Here and subsequently, we write the matrices following the lexicographic order  $\preceq_{\text{lex}}$  (see Notation 3.1.3 (3)) of indices.

*Proof.* Set  $\text{sc}(\mathbf{U}) = \{a_1, \dots, a_n\}$ . If  $n = 1$ , then  $V_{\mathbf{U}} = P_{\text{sc}(\mathbf{U})}$  and  $P_{\text{sc}_1(\mathbf{U})} = 0$  by Proposition 3.1.11 and Notation 3.1.3 (4). Hence the sequence (3.1.4) is trivially a projective presentation of  $V_{\mathbf{U}}$ . Now we assume  $n \geq 2$  and verify the exactness of the sequence (3.1.4) in the following steps:

- (a) Show that  $\varepsilon_0^{\mathbf{U}}$  is an epimorphism.
- (b) Show that  $\varepsilon_0^{\mathbf{U}} \varepsilon_1^{\mathbf{U}} = 0$ .
- (c) Show that  $\dim \text{Im } \varepsilon_1^{\mathbf{U}} \geq \dim \text{Ker } \varepsilon_0^{\mathbf{U}}$ .

(a) It is enough to show that for all  $b \in \mathbf{P}$ ,

$$(\varepsilon_0^{\mathbf{U}})_b = [(\rho_{1_a}^{V_{\mathbf{U}}})_b]_{a \in \text{sc}(\mathbf{U})} : \bigoplus_{i \in [n]} P_{a_i}(b) \rightarrow V_{\mathbf{U}}(b)$$

is surjective. If  $b \notin \mathbf{U}$ , then  $V_{\mathbf{U}}(b) = 0$  and the assertion trivially holds. Otherwise,  $V_{\mathbf{U}}(b) = \mathbb{k}$  with the unique basis  $1_b$ , hence it suffices to show that  $1_b$  has a preimage in  $\bigoplus_{i \in [n]} P_{a_i}(b)$ . In this case, it is evident that there is  $a_j \in \text{sc}(\mathbf{U})$  such that  $a_j \leq b$ . Set  $(m_i)_{i \in [n]} \in \bigoplus_{i \in [n]} P_{a_i}(b)$  with all entries zero except for the  $j$ -th entry  $m_j := p_{b, a_j}$ . Then

$$(\varepsilon_0^{\mathbf{U}})_b \left( (m_i)_{i \in [n]} \right) = \sum_{i \in [n]} \left( \rho_{1_{a_i}}^{V_{\mathbf{U}}} \right)_b (m_i) = \left( \rho_{1_{a_j}}^{V_{\mathbf{U}}} \right)_b (m_j) = 1_b,$$

where the last equality holds by the Yoneda lemma and noticing (2.5.9):

$$\left( \rho_{1_{a_j}}^{V_{\mathbf{U}}} \right)_b : P_{a_j}(b) \rightarrow V_{\mathbf{U}}(b), \quad p_{b, a_j} \mapsto p_{b, a_j} \cdot 1_{a_j} = V_{\mathbf{U}}(p_{b, a_j})(1_{a_j}) = 1_b.$$

Therefore for all  $b \in \mathbf{U}$ ,  $1_b$  has the preimage and thus  $\varepsilon_0^{\mathbf{U}}$  is an epimorphism.

(b) It suffices to show that the composition

$$P_{\mathbf{a}_c} \xrightarrow{\varepsilon_{1,c}^{\mathbf{U}}} \bigoplus_{i \in [n]} P_{a_i} \xrightarrow{\varepsilon_0^{\mathbf{U}}} V_{\mathbf{U}} \quad (3.1.7)$$

is zero for all  $\mathbf{a}_c \in \text{sc}_1(\mathbf{U})$ , where  $\varepsilon_{1,c}^{\mathbf{U}}$  is the  $\mathbf{a}_c$ -th column vector of the matrix  $\varepsilon_1^{\mathbf{U}}$  having the form:

$$\varepsilon_{1,c}^{\mathbf{U}} := \begin{matrix} & \mathbf{a}_c \\ a_1 & \left( \begin{array}{c} \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \mathbf{P}_{c, \mathbf{a}} \\ \vdots \\ \mathbf{0} \\ \mathbf{0} \end{array} \right) \\ \vdots & \\ \mathbf{a} & \\ \vdots & \\ \bar{\mathbf{a}} & \left( \begin{array}{c} \mathbf{0} \\ -\mathbf{P}_{c, \bar{\mathbf{a}}} \\ \vdots \\ \mathbf{0} \end{array} \right) \\ \vdots & \\ a_n & \left( \begin{array}{c} \mathbf{0} \\ \vdots \\ \mathbf{0} \end{array} \right) \end{matrix}.$$

We verify pointwise. For  $t \in \mathbf{P}$ , the composition is trivially zero when  $t \notin \uparrow_{\mathbf{P}} c$ , thus we consider the case where  $t \in \uparrow_{\mathbf{P}} c$ . Note that  $\mathbf{U}$  is the up-set of  $\mathbf{P}$ ,  $c \in \text{sc}_1(\mathbf{U}) \subseteq \mathbf{U}$  and hence  $t \in \mathbf{U}$ . Then

$$\begin{aligned} (\varepsilon_0^{\mathbf{U}} \varepsilon_{1,c}^{\mathbf{U}})(p_{t,c}) &= (\rho_{1_{\underline{\mathbf{a}}}}^{V_{\mathbf{U}}} P_{c,\underline{\mathbf{a}}})(p_{t,c}) - (\rho_{1_{\bar{\mathbf{a}}}}^{V_{\mathbf{U}}} P_{c,\bar{\mathbf{a}}})(p_{t,c}) \\ &= \rho_{1_{\underline{\mathbf{a}}}}^{V_{\mathbf{U}}}(p_{t,\underline{\mathbf{a}}}) - \rho_{1_{\bar{\mathbf{a}}}}^{V_{\mathbf{U}}}(p_{t,\bar{\mathbf{a}}}) \\ &= V_{\mathbf{U}}(p_{t,\underline{\mathbf{a}}})(1_{\underline{\mathbf{a}}}) - V_{\mathbf{U}}(p_{t,\bar{\mathbf{a}}})(1_{\bar{\mathbf{a}}}) \\ &= 1_t - 1_t = 0. \end{aligned}$$

(c) It suffices to show that  $\dim \text{Im}(\varepsilon_1^{\mathbf{U}})_x \geq \dim \text{Ker}(\varepsilon_0^{\mathbf{U}})_x$  for all  $x \in \mathbf{U}$ . Fix  $x \in \mathbf{U}$ , and let  $w := |\text{sc}(\mathbf{U}) \cap \downarrow_{\mathbf{P}} x|$ . Then  $w \geq 1$ . We may write  $\downarrow x$  for short subsequently and set  $\text{sc}(\mathbf{U}) \cap \downarrow x := \{a_1, \dots, a_w\}$  without loss of generality.

**Case 1.** Consider the case where  $w = 1$ . If  $P_{\text{sc}_1(\mathbf{U})}(x) \neq 0$ , then there exists some  $\mathbf{a}_c \in \text{sc}_1(\mathbf{U})$  such that  $P_{\mathbf{a}_c}(x) \neq 0$ , which shows that  $\underline{\mathbf{a}} \leq c$ ,  $\bar{\mathbf{a}} \leq c$ , and  $c \leq x$ . Thus  $w \geq |\mathbf{a}| = 2$ , a contradiction. Therefore in this case,  $P_{\text{sc}_1(\mathbf{U})}(x) = 0$ . Hence the evaluation of (3.1.4) at  $x$  becomes

$$0 \rightarrow \mathbb{k}p_{x,a_1} \xrightarrow{\varphi} \mathbb{k} \rightarrow 0,$$

where  $\varphi$  is an isomorphism defined by  $\varphi(p_{x,a_1}) := 1_{\mathbb{k}}$ . The claim follows since  $\text{Im}(\varepsilon_1^{\mathbf{U}})_x = 0 = \text{Ker}(\varepsilon_0^{\mathbf{U}})_x$ .

**Case 2.** Consider the case where  $w \geq 2$ . By Notation 3.1.3,  $P_{\text{sc}(\mathbf{U})}(x) = \bigoplus_{a \in \text{sc}(\mathbf{U})} P_a(x) = \bigoplus_{a \in \text{sc}(\mathbf{U}) \cap \downarrow x} \mathbb{k}p_{x,a}$ , and

$$P_{\text{sc}_1(\mathbf{U})}(x) = \bigoplus_{\mathbf{a}_c \in \text{sc}_1(\mathbf{U})} P_{\mathbf{a}_c}(x) = \bigoplus_{\substack{\mathbf{a} \in C_2 \text{sc}(\mathbf{U}) \\ c \in V'\mathbf{a} \cap \downarrow x}} P_c(x) = \bigoplus_{\substack{\mathbf{a} \in C_2 \text{sc}(\mathbf{U}) \\ c \in V'\mathbf{a} \cap \downarrow x}} \mathbb{k}p_{x,c}.$$

Then the evaluation of (3.1.4) at  $x$  becomes the first row of the commutative diagram

$$\begin{array}{ccccc} \bigoplus_{\substack{\mathbf{a} \in C_2 \text{sc}(\mathbf{U}) \\ c \in V'\mathbf{a} \cap \downarrow x}} \mathbb{k}p_{x,c} & \xrightarrow{(\varepsilon_1^{\mathbf{U}})_x} & \bigoplus_{a \in \text{sc}(\mathbf{U}) \cap \downarrow x} \mathbb{k}p_{x,a} & \xrightarrow{(\varepsilon_0^{\mathbf{U}})_x} & \mathbb{k} \longrightarrow 0 \\ \uparrow \alpha & & \uparrow \beta & & \parallel \\ \bigoplus_{\substack{\mathbf{a} \in C_2 \text{sc}(\mathbf{U}) \\ c \in V'\mathbf{a} \cap \downarrow x}} \mathbb{k} & \xrightarrow{\varepsilon_1^{\mathbf{U}}(x)} & \bigoplus_{a \in \text{sc}(\mathbf{U}) \cap \downarrow x} \mathbb{k} & \xrightarrow{\varepsilon_0^{\mathbf{U}}(x)} & \mathbb{k} \end{array},$$

where  $\alpha = \bigoplus_{\mathbf{a}_c} \alpha_c$  (resp.  $\beta = \bigoplus_a \beta_a$ ) is the isomorphism defined by  $\alpha_c(1_{\mathbb{k}}) := p_{x,c}$  (resp.  $\beta_a(1_{\mathbb{k}}) := p_{x,a}$ ) for all  $\mathbf{a} \in C_2 \text{sc}(\mathbf{U})$ ,  $c \in V'\mathbf{a} \cap \downarrow x$  (resp.  $a \in \text{sc}(\mathbf{U}) \cap \downarrow x$ ). The commutativity of the left square follows from the fact that  $(P_{a,b})_x(p_{x,a}) = p_{x,b}$  for  $b \leq a \leq x$  (recall Notation 2.5.21 and Equation (2.5.11)). The remaining commutativity is trivial.

We note here that

$$\text{sc}(\uparrow_{\mathbf{P}} \underline{\mathbf{a}} \cap \uparrow_{\mathbf{P}} \bar{\mathbf{a}} \cap \downarrow x) \subseteq V'\mathbf{a} \cap \downarrow x \quad (3.1.8)$$

holds for all  $\mathbf{a} \in C_2 \text{sc}(\mathbf{U})$ . Indeed, assume that  $y$  is an element of the left-hand side. Let  $z$  be any element of  $\mathbf{U}$  satisfying  $z < y$ . Then by the minimality of  $y$ ,

$z \notin \uparrow_{\mathbf{P}\underline{\mathbf{a}}} \cap \uparrow_{\mathbf{P}\bar{\mathbf{a}}} \cap \downarrow x$ . Because  $z < y \leq x$ , we have  $z \in \downarrow x$ . Thus  $z \notin \uparrow_{\mathbf{P}\underline{\mathbf{a}}} \cap \uparrow_{\mathbf{P}\bar{\mathbf{a}}}$ . Hence  $y$  is an element of the right-hand side.

Since  $w \geq 2$ ,  $C_2(\text{sc}(\mathbf{U}) \cap \downarrow x) \neq \emptyset$ . Note that for each  $\mathbf{a} \in C_2(\text{sc}(\mathbf{U}) \cap \downarrow x)$ , we have  $\uparrow_{\mathbf{P}\underline{\mathbf{a}}} \cap \uparrow_{\mathbf{P}\bar{\mathbf{a}}} \cap \downarrow x \neq \emptyset$  because the left-hand side contains  $x$ . Hence  $\text{sc}(\uparrow_{\mathbf{P}\underline{\mathbf{a}}} \cap \uparrow_{\mathbf{P}\bar{\mathbf{a}}} \cap \downarrow x) \neq \emptyset$ . Take any  $c_{\mathbf{a},x}$  from this set. Then by (3.1.8), we have  $c_{\mathbf{a},x} \in \vee' \mathbf{a} \cap \downarrow x$ . The pair  $\mathbf{a}_{c_{\mathbf{a},x}} = (\mathbf{a}, c_{\mathbf{a},x})$  is denoted by  $\mathbf{a}_{c_x}$  for short. Therefore, the matrix  $\varepsilon_1^{\mathbf{U}}(x)$  has a  $w \times \binom{w}{2}$  submatrix  $\tilde{\varepsilon}_1^{\mathbf{U}}(x)$  given by

$$\begin{matrix} & \{a_1, a_2\}_{c_x} & \{a_1, a_3\}_{c_x} & \cdots & \{a_1, a_w\}_{c_x} & \{a_2, a_3\}_{c_x} & \cdots & \{a_2, a_w\}_{c_x} & \cdots & \{a_{w-1}, a_w\}_{c_x} \\ \begin{matrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{w-1} \\ a_w \end{matrix} & \left( \begin{array}{cccccccccc} 1 & 1 & \cdots & 1 & 0 & \cdots & 0 & \cdots & 0 \\ -1 & 0 & \cdots & 0 & 1 & \cdots & 1 & \cdots & 0 \\ 0 & -1 & \cdots & 0 & -1 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 1 \\ 0 & 0 & \cdots & -1 & 0 & \cdots & -1 & \cdots & -1 \end{array} \right), \end{matrix} \quad (3.1.9)$$

and the matrix  $\varepsilon_0^{\mathbf{U}}(x)$  is given by  $(\underbrace{1, 1, \dots, 1}_w)$ . It is clear that  $\text{rank } \varepsilon_0^{\mathbf{U}}(x) = 1$ . For the

matrix  $\tilde{\varepsilon}_1^{\mathbf{U}}(x)$ , note that the last  $w-1$  rows (from  $a_2$  to  $a_w$ ) are linearly independent, and that the sum of all rows is a zero row vector. This shows that  $\text{rank } \tilde{\varepsilon}_1^{\mathbf{U}}(x) = w-1$ . Thus  $\dim \text{Im } \varepsilon_1^{\mathbf{U}}(x) = \text{rank } \varepsilon_1^{\mathbf{U}}(x) \geq \text{rank } \tilde{\varepsilon}_1^{\mathbf{U}}(x) = w-1 = \dim \text{Ker } \varepsilon_0^{\mathbf{U}}(x)$ .  $\square$

Given an up-set  $\mathbf{U}$  of  $\mathbf{P}$  ( $\mathbf{U}$  might be nonconnected), we consider its decomposition into connected components and apply Proposition 3.1.12 on each connected component. Following this spirit, we let  $\mathbf{U} := \mathbf{U}_1 \sqcup \cdots \sqcup \mathbf{U}_k$ . By Lemma 3.1.10 and [45, Theorem 25.3], each component is again an open set, thus an up-set of  $\mathbf{P}$ . Then  $V_{\mathbf{U}} \cong V_{\mathbf{U}_1} \oplus \cdots \oplus V_{\mathbf{U}_k}$ , and the following is easy to show.

**Proposition 3.1.13.** *Let  $\mathbf{P}$  be a finite poset, and  $\mathbf{U} = \mathbf{U}_1 \sqcup \cdots \sqcup \mathbf{U}_k$  an up-set of  $\mathbf{P}$  with  $k$  connected components ( $k \geq 1$ ). Then  $V_{\mathbf{U}}$  has the following (not necessarily minimal) projective presentation:*

$$P_{\text{sc}_1(\mathbf{U})} \xrightarrow{\varepsilon_1^{\mathbf{U}}} P_{\text{sc}(\mathbf{U})} \xrightarrow{\varepsilon_0^{\mathbf{U}}} V_{\mathbf{U}} \rightarrow 0,$$

where  $\varepsilon_i^{\mathbf{U}} := \varepsilon_i^{\mathbf{U}_1} \oplus \cdots \oplus \varepsilon_i^{\mathbf{U}_k}$  ( $i = 0, 1$ ).  $\square$

The following lemma will be frequently used later.

**Lemma 3.1.14.** *Let  $M, M' \in \text{mod } A$ . Assume that*

- (1)  *$M'$  is a submodule of  $M$ , i.e., we have a short exact sequence*

$$0 \rightarrow M' \xrightarrow{\iota} M \xrightarrow{\pi} M/M' \rightarrow 0;$$

- (2)  *$M$  has a projective presentation (not necessarily a minimal one)*

$$P_1^M \xrightarrow{\varepsilon_1^M} P_0^M \xrightarrow{\varepsilon_0^M} M \rightarrow 0; \text{ and}$$

(3)  $M'$  has an epimorphism  $\varepsilon_0^{M'}: P_0^{M'} \rightarrow M'$  with  $P_0^{M'}$  projective.

Then the factor module  $M/M'$  has a projective presentation (not necessarily a minimal one)

$$P_1^M \oplus P_0^{M'} \xrightarrow{\eta_1} P_0^M \xrightarrow{\eta_0} M/M' \rightarrow 0.$$

Here  $\eta_0 := \pi \varepsilon_0^M$  and  $\eta_1 := [\varepsilon_1^M, \eta_{11}]$ , where  $\eta_{11}: P_0^{M'} \rightarrow P_0^M$  is a lift of  $\iota \varepsilon_0^{M'}$  along  $\varepsilon_0^M$ , i.e., a morphism satisfying the equality  $\varepsilon_0^M \eta_{11} = \iota \varepsilon_0^{M'}$ , the existence of which is guaranteed by the projectivity of  $P_0^{M'}$ .

*Proof.* Express  $\varepsilon_1^M$  as the composite  $\varepsilon_1^M = \iota_0 \tau_1: P_1^M \xrightarrow{\tau_1} \text{Im } \varepsilon_1^M \xrightarrow{\iota_0} P_0^M$ , where  $\iota_0$  is the inclusion and  $\tau_1$  is an epimorphism obtained from  $\varepsilon_1^M$  by restricting the codomain. Consider the following commutative diagram of solid arrows with exact rows surrounded by dashed lines:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & P_1^M & \xrightarrow{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} & P_1^M \oplus P_0^{M'} & \xrightarrow{[0, 1]} & P_0^{M'} \longrightarrow 0 \\
 & & \downarrow \tau_1 & & \downarrow \eta_1 & \swarrow \eta_{11} & \downarrow \iota \varepsilon_0^{M'} \\
 0 & \longrightarrow & \text{Im } \varepsilon_1^M & \xrightarrow{\iota_0} & P_0^M & \xleftarrow{\varepsilon_0^M} & M \longrightarrow 0 \\
 & & \downarrow & & \downarrow \text{coker } \eta_1 & & \downarrow \pi \\
 & & 0 & \longrightarrow & \text{Coker } \eta_1 & \xrightarrow{\overline{\varepsilon_0^M}} & M/M' \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
 \end{array} \tag{3.1.10}$$

Notice that the right-most column

$$P_0^{M'} \xrightarrow{\iota \varepsilon_0^{M'}} M \xrightarrow{\pi} M/M' \rightarrow 0$$

is exact since  $\text{Im } \iota \varepsilon_0^{M'} = \text{Im } \iota = M' = \text{Ker } \pi$ . By applying the snake lemma to this diagram, we obtain that  $\overline{\varepsilon_0^M}: \text{Coker } \eta_1 \rightarrow M/M'$  is an isomorphism. Since  $\overline{\varepsilon_0^M} \circ \text{coker } \eta_1 = \pi \varepsilon_0^M = \eta_0$ , the central column yields the exact sequence

$$P_0^{M'} \oplus P_1^M \xrightarrow{\eta_1} P_0^M \xrightarrow{\eta_0} M/M' \rightarrow 0. \quad \square$$

**Notation 3.1.15.** Let  $\mathbf{I}$  be an interval of  $\mathbf{P}$ .

(1) Note that  $\text{sc}(\mathbf{I}) = \text{sc}(\uparrow \mathbf{I})$ , and hence also  $\text{sc}_1(\mathbf{I}) = \text{sc}_1(\uparrow \mathbf{I})$ . Therefore,  $\varepsilon_1^{\uparrow \mathbf{I}}: P_{\text{sc}_1(\uparrow \mathbf{I})} \rightarrow P_{\text{sc}(\uparrow \mathbf{I})}$  is denoted by  $\varepsilon_1^{\uparrow \mathbf{I}}: P_{\text{sc}_1(\mathbf{I})} \rightarrow P_{\text{sc}(\mathbf{I})}$ , where the definition of  $\varepsilon_1^{\uparrow \mathbf{I}}$  is given by (3.1.5) for  $\mathbf{U} := \uparrow \mathbf{I}$ .

(2) Note that for each  $a' \in \text{sc}(\uparrow \mathbf{I})$ , we have  $\text{sc}(\mathbf{I}) \cap \downarrow a' \neq \emptyset$  because  $a' \in \uparrow \mathbf{I}$ . Fixing one element in  $\text{sc}(\mathbf{I}) \cap \downarrow a'$  for each  $a' \in \text{sc}(\uparrow \mathbf{I})$  yields a map  $\mathbf{c}: \text{sc}(\uparrow \mathbf{I}) \rightarrow \text{sc}(\mathbf{I}) = \text{sc}(\uparrow \mathbf{I})$ . We call such  $\mathbf{c}$  a *choice map*.

The following is used in the computations below.

**Lemma 3.1.16.** Let  $M \in \text{mod } A$ ,  $x, y \in \mathbf{P}$ , and  $a \in M(x)$ . Then the composite of morphisms  $P_y \xrightarrow{P_{y,x}} P_x \xrightarrow{\rho_a^M} M$  is given by

$$\rho_a^M \cdot P_{y,x} = \rho_{M(p_{y,x})(a)}^M.$$

*Proof.* Let  $z \in \mathbf{P}$  and  $q \in P_y(z)$ . Then by Notation 2.5.21, we have

$$(\rho_a^M \cdot P_{y,x})(q) = \rho_a^M(q \cdot p_{y,x}) = M(q \cdot p_{y,x})(p) = (M(q)M(p_{y,x}))(p).$$

On the other hand,

$$\rho_{M(p_{y,x})(a)}^M(q) = M(q)(M(p_{y,x})(a)).$$

Therefore, the assertion holds.  $\square$

Now we are in a position to give a projective presentation of  $V_{\mathbf{I}}$  for any interval  $\mathbf{I}$  of  $\mathbf{P}$ .

**Proposition 3.1.17.** *Let  $\mathbf{I}$  be an interval of  $\mathbf{P}$ . Then  $V_{\mathbf{I}}$  has the following (not necessarily minimal) projective presentation:*

$$P_{\text{sc}_1(\mathbf{I})} \oplus P_{\text{sc}(\uparrow \mathbf{I})} \xrightarrow{\varepsilon_1} P_{\text{sc}(\mathbf{I})} \xrightarrow{\varepsilon_0} V_{\mathbf{I}} \rightarrow 0.$$

Here  $\varepsilon_0 = [\rho_{1_a}^{V_{\mathbf{I}}}]_{a \in \text{sc}(\mathbf{I})}$ , where we set  $1_u := 1_{\mathbb{k}} \in \mathbb{k} = V_{\mathbf{I}}(u)$  for all  $u \in \mathbf{I}$ ;  $\varepsilon_1 := \varepsilon_1(\mathbf{c}) := [\varepsilon_1^{\uparrow \mathbf{I}}, \varepsilon_{11}]$ , where  $\varepsilon_{11} := \varepsilon_{11}(\mathbf{c}) := [\delta_{a, \mathbf{c}(a')} P_{a', \mathbf{c}(a')}]_{(a, a') \in \text{sc}(\mathbf{I}) \times \text{sc}(\uparrow \mathbf{I})}$ , and  $\varepsilon_1^{\uparrow \mathbf{I}}$  is as in Notation 3.1.15.

*Proof.* By Lemma 3.1.6, we can write  $\mathbf{I} = \mathbf{U} \setminus \mathbf{U}'$  in terms of two up-sets  $\mathbf{U} = \uparrow \mathbf{I}$  and  $\mathbf{U}' = \uparrow \mathbf{I}'$ , and hence  $V_{\mathbf{I}} \cong V_{\mathbf{U}}/V_{\mathbf{U}'}$ . By Proposition 3.1.12 and Lemma 3.1.5,  $V_{\mathbf{U}}$  has the following (not necessarily minimal) projective presentation:

$$P_{\text{sc}_1(\mathbf{U})} \xrightarrow{\varepsilon_1^{\mathbf{U}}} P_{\text{sc}(\mathbf{U})} \xrightarrow{\varepsilon_0^{\mathbf{U}}} V_{\mathbf{U}} \rightarrow 0. \quad (3.1.11)$$

By Remark 2.2.13 (2),  $P_{\text{sc}(\mathbf{U})} = P_{\text{sc}(\mathbf{I})}$ , and  $P_{\text{sc}_1(\mathbf{U})} = P_{\text{sc}_1(\mathbf{I})}$ . Since the natural projection  $\pi: V_{\mathbf{U}} \rightarrow V_{\mathbf{I}}$  is just the restriction on  $\mathbf{I}$ , it follows by Lemma 3.1.14 that  $\varepsilon_0 = \pi \varepsilon_0^{\mathbf{U}} = \pi \circ [\rho_{1_a}^{V_{\mathbf{U}}}]_{a \in \text{sc}(\mathbf{U})} = [\rho_{1_a}^{V_{\mathbf{I}}}]_{a \in \text{sc}(\mathbf{I})}$ .

On the other hand, the up-set  $\mathbf{U}'$  is the disjoint union  $\mathbf{U}_1 \sqcup \cdots \sqcup \mathbf{U}_k$  of some connected up-sets  $\mathbf{U}_1, \dots, \mathbf{U}_k$  with  $k \in \mathbb{Z}_{\geq 1}$ , and hence  $V_{\mathbf{U}'} \cong V_{\mathbf{U}_1} \oplus \cdots \oplus V_{\mathbf{U}_k}$ . By Propositions 3.1.12 and 3.1.13, there is an epimorphism starting from a projective module:

$$P_{\text{sc}(\mathbf{U}')} \xrightarrow{\varepsilon_0^{\mathbf{U}'}} V_{\mathbf{U}'}, \quad (3.1.12)$$

where  $\varepsilon_0^{\mathbf{U}'} := [\rho_{1_a}^{V_{\mathbf{U}_1}}]_{a \in \text{sc}(\mathbf{U}_1)} \oplus \cdots \oplus [\rho_{1_a}^{V_{\mathbf{U}_k}}]_{a \in \text{sc}(\mathbf{U}_k)} = [\rho_{1_a}^{V_{\mathbf{U}'}}]_{a \in \text{sc}(\mathbf{U}'})$ . If the equality

$$\varepsilon_0^{\mathbf{U}} \varepsilon_{11} = \iota \varepsilon_0^{\mathbf{U}'} : P_{\text{sc}(\mathbf{U}')} \rightarrow V_{\mathbf{U}} \quad (3.1.13)$$

holds, where  $\iota: V_{\mathbf{U}'} \cong V_{\mathbf{U}_1} \oplus \cdots \oplus V_{\mathbf{U}_k} \rightarrow V_{\mathbf{U}}$  is the inclusion, then by combining (3.1.11), (3.1.12) and Lemma 3.1.14, we obtain a projective presentation of  $V_{\mathbf{I}}$  having the form

$$P_{\text{sc}_1(\mathbf{I})} \oplus P_{\text{sc}(\mathbf{U}')} \xrightarrow{[\varepsilon_1^{\mathbf{U}}, \varepsilon_{11}]} P_{\text{sc}(\mathbf{I})} \xrightarrow{\varepsilon_0^{\mathbf{I}}} V_{\mathbf{I}} \rightarrow 0 \quad (3.1.14)$$

as claimed. The equality (3.1.13) is verified as follows:

$$\begin{aligned}
\varepsilon_0^{\mathbf{U}} \varepsilon_{11} &= [\rho_{1_a}^{V_{\mathbf{U}}}]_{a \in \text{sc}(\mathbf{I})} \cdot [\delta_{a, \mathbf{c}(a')} \mathbf{P}_{a', \mathbf{c}(a')}]_{(a, a') \in \text{sc}(\mathbf{I}) \times \text{sc}(\mathbf{U}')} \\
&= \left[ \sum_{a \in \text{sc}(\mathbf{I})} \delta_{a, \mathbf{c}(a')} \rho_{1_a}^{V_{\mathbf{U}}} \cdot \mathbf{P}_{a', \mathbf{c}(a')} \right]_{a' \in \text{sc}(\mathbf{U}')} \\
&= [\rho_{1_{\mathbf{c}(a')}}^{V_{\mathbf{U}}} \cdot \mathbf{P}_{a', \mathbf{c}(a')}]_{a' \in \text{sc}(\mathbf{U}')} \\
&\stackrel{*}{=} [\rho_{V_{\mathbf{U}}(p_{a', \mathbf{c}(a')})(1_{\mathbf{c}(a')})}^{V_{\mathbf{U}}}]_{a' \in \text{sc}(\mathbf{U}')} \\
&= [\rho_{1_{a'}}^{V_{\mathbf{U}}}]_{a' \in \text{sc}(\mathbf{U}')} \\
&= \iota \circ [\rho_{1_{a'}}^{V_{\mathbf{U}'}}]_{a' \in \text{sc}(\mathbf{U}')} \\
&= \iota \varepsilon_0^{\mathbf{U}'}.
\end{aligned}$$

In the above, the equality  $(\stackrel{*}{=})$  follows by Lemma 3.1.16.  $\square$

### 3.1.1 The case where $V_{\mathbf{I}}$ is injective

Assume that  $V_{\mathbf{I}}$  is injective in  $\text{mod } A$ . Then  $b := \max \mathbf{I}$  exists, and we have  $\mathbf{I} = \downarrow b$ . Since  $V_{\mathbf{I}}$  is indecomposable injective,  $\text{soc } V_{\mathbf{I}} = V_{\{b\}}$  is a simple module at  $b$ , and  $\varepsilon_0^b = \rho_{1_b}: P_b \rightarrow V_{\{b\}}$  is the projective cover of  $V_{\{b\}}$ . Hence by Lemma 3.1.14 and Proposition 3.1.17, we have the following.

**Notation 3.1.18.** Let  $\mathbf{I}$  be an interval of  $\mathbf{P}$  with the maximum element  $b$ . Fixing one element  $a \in \text{sc}(\mathbf{I})$  induces another choice map  $\mathbf{c}': \{b\} \rightarrow \text{sc}(\mathbf{I})$  by  $\mathbf{c}'(b) := a$ .

**Theorem 3.1.19.** Let  $\mathbf{I}$  be an interval of  $\mathbf{P}$ . Assume that  $V_{\mathbf{I}}$  is injective, i.e.,  $\mathbf{I} = \downarrow b$  with  $b = \max \mathbf{I}$ . Then  $V_{\mathbf{I}}$  and  $V_{\mathbf{I}}/\text{soc } V_{\mathbf{I}}$  have projective presentations of the following forms.

$$\begin{aligned}
P_{\text{sc}_1(\mathbf{I})} \oplus P_{\text{sc}(\uparrow \mathbf{I})} &\xrightarrow{\varepsilon_1} P_{\text{sc}(\mathbf{I})} \xrightarrow{\varepsilon_0} V_{\mathbf{I}} \rightarrow 0, \\
(P_{\text{sc}_1(\mathbf{I})} \oplus P_{\text{sc}(\uparrow \mathbf{I})}) \oplus P_b &\xrightarrow{\varepsilon'_1} P_{\text{sc}(\mathbf{I})} \rightarrow V_{\mathbf{I}}/\text{soc } V_{\mathbf{I}} \rightarrow 0.
\end{aligned}$$

Here  $\varepsilon_1 := [\varepsilon_1^{\uparrow \mathbf{I}}, \varepsilon_{11}]$  and  $\varepsilon'_1 := [\varepsilon_1, \varepsilon''_1]$ , where  $\varepsilon_{11}, \varepsilon_1^{\uparrow \mathbf{I}}$  are given in Proposition 3.1.17, and  $\varepsilon''_1 := \varepsilon''_1(\mathbf{c}') := [\delta_{a, \mathbf{c}'(b)} \mathbf{P}_{b, \mathbf{c}'(b)}]_{(a, b) \in \text{sc}(\mathbf{I}) \times \{b\}}$ . Therefore, we have

$$d_M(V_{\mathbf{I}}) = \text{rank} \begin{bmatrix} M(\varepsilon_1) \\ M(\varepsilon''_1) \end{bmatrix} - \text{rank} [M(\varepsilon_1)]. \quad (3.1.15)$$

*Proof.* We only need to show the statement for  $V_{\mathbf{I}}/\text{soc } V_{\mathbf{I}}$ . By Lemma 3.1.14, it suffices to check  $\varepsilon_0 \varepsilon''_1 = \iota \varepsilon_0^{\{b\}}$ , in which  $\iota: V_{\{b\}} \rightarrow V_{\mathbf{I}}$  is the inclusion.

$$\begin{aligned}
\varepsilon_0 \varepsilon''_1 &= [\rho_{1_a}^{V_{\mathbf{I}}}]_{a \in \text{sc}(\mathbf{I})} \cdot [\delta_{a, \mathbf{c}'(b)} \mathbf{P}_{b, \mathbf{c}'(b)}]_{(a, b) \in \text{sc}(\mathbf{I}) \times \{b\}} = \sum_{a \in \text{sc}(\mathbf{I})} \delta_{a, \mathbf{c}'(b)} \rho_{1_a}^{V_{\mathbf{I}}} \cdot \mathbf{P}_{b, a} \\
&= \rho_{1_{\mathbf{c}'(b)}}^{V_{\mathbf{I}}} \cdot \mathbf{P}_{b, \mathbf{c}'(b)} \stackrel{*}{=} \rho_{V_{\mathbf{I}}(p_{b, \mathbf{c}'(b)})(1_{\mathbf{c}'(b)})}^{V_{\mathbf{I}}} = \rho_{1_b}^{V_{\mathbf{I}}} = \iota \rho_{1_b}^{V_{\{b\}}} = \iota \varepsilon_0^{\{b\}},
\end{aligned}$$

where we applied Lemma 3.1.16 to have the equality  $(\stackrel{*}{=})$ . Finally, (3.1.15) follows by applying Lemma 3.1.2 and Theorem 2.6.12.  $\square$

### 3.1.2 The case where $V_{\mathbf{I}}$ is non-injective

To apply Theorem 2.6.12 and Lemma 3.1.2, we need to compute projective presentations of  $\tau^{-1}V_{\mathbf{I}}$  and  $E$ . We first do it for  $\tau^{-1}V_{\mathbf{I}}$ .

Recall that  $\tau^{-1}M = \text{Tr } DM$  for all  $M \in \text{mod } A$ , where for each  $N \in \text{mod } A^{\text{op}}$ , the transpose  $\text{Tr } N$  of  $N$  is defined as the cokernel of some  $f^t$  with  $P_1 \xrightarrow{f} P_0 \rightarrow N \rightarrow 0$  a minimal projective presentation of  $N$ . By applying Proposition 3.1.17, we first obtain a projective presentation of  $DV_{\mathbf{I}}$  as follows. For this sake, we note that there exists an isomorphism  $DV_{\mathbf{I}} \cong V_{\mathbf{I}^{\text{op}}}$  in  $\text{mod } A^{\text{op}}$ , and by Lemma 3.1.6,  $\mathbf{I} = \mathbf{W} \setminus \mathbf{W}'$ , where  $\mathbf{W} := \downarrow \mathbf{I}$  and  $\mathbf{W}' := \downarrow \mathbf{I} \setminus \mathbf{I}$  are two down-sets of  $\mathbf{P}$ . By the duality,  $\mathbf{W}^{\text{op}} = (\downarrow_{\mathbf{P}} \mathbf{I})^{\text{op}} = \uparrow_{\mathbf{P}^{\text{op}}} \mathbf{I}^{\text{op}}$  and  $(\mathbf{W}')^{\text{op}} = \uparrow_{\mathbf{P}^{\text{op}}} \mathbf{I}^{\text{op}} \setminus \mathbf{I}^{\text{op}}$ . Hence  $DV_{\mathbf{I}} \cong V_{\mathbf{I}^{\text{op}}} \cong V_{\mathbf{W}^{\text{op}}} / V_{(\mathbf{W}')^{\text{op}}}$ , where

$$\begin{aligned} \text{sc}(\mathbf{W}^{\text{op}}) &= \text{sk}(\mathbf{W}) = \text{sk}(\mathbf{I}), \quad \text{sc}_1(\mathbf{W}^{\text{op}}) = \text{sk}_1(\mathbf{W}) = \text{sk}_1(\mathbf{I}), \text{ and} \\ \text{sc}((\mathbf{W}')^{\text{op}}) &= \text{sk}(\mathbf{W}') = \text{sk}(\downarrow \mathbf{I}). \end{aligned}$$

**Notation 3.1.20.** Let  $\mathbf{I}$  an interval of  $\mathbf{P}$ .

(1) Note that  $\text{sk}(\mathbf{I}) = \text{sk}(\downarrow \mathbf{I})$ , and hence also  $\text{sk}_1(\mathbf{I}) = \text{sk}_1(\downarrow \mathbf{I})$ .

(2) Note that for each  $b' \in \text{sk}(\downarrow \mathbf{I})$ , we have  $\text{sk}(\mathbf{I}) \cap \uparrow b' \neq \emptyset$  because  $b' \in \downarrow \mathbf{I}$ . Fixing one element  $b \in \text{sk}(\mathbf{I}) \cap \uparrow b'$  for each  $b' \in \text{sk}(\downarrow \mathbf{I})$  yields a choice map  $\mathbf{d}: \text{sk}(\downarrow \mathbf{I}) \rightarrow \text{sk}(\mathbf{I}) = \text{sk}(\downarrow \mathbf{I})$  that sends  $b'$  to  $b$ .

**Proposition 3.1.21.** *Let  $\mathbf{I}$  be an interval of  $\mathbf{P}$ . Then the interval module  $DV_{\mathbf{I}}$  has the following (not necessarily minimal) projective presentation<sup>1</sup> in  $\text{mod } A^{\text{op}}$ :*

$$P'_{\text{sk}(\downarrow \mathbf{I})} \oplus P'_{\text{sk}_1(\mathbf{I})} \xrightarrow{\psi_1} P'_{\text{sk}(\mathbf{I})} \xrightarrow{\psi_0} DV_{\mathbf{I}} \rightarrow 0. \quad (3.1.16)$$

Here  $\psi_0 := [\lambda_{1_b}^{V_{\mathbf{I}}}]_{b \in \text{sk}(\mathbf{I})}$  and  $\psi_1 := \psi_1(\mathbf{d}) := [\psi_{11}, \psi_1^{\downarrow \mathbf{I}}]$ , where  $\psi_{11} := \psi_{11}(\mathbf{d})$  is given by  $\psi_{11} := [\delta_{b, \mathbf{d}(b')} P'_{b', \mathbf{d}(b')}]_{(b, b') \in \text{sk}(\mathbf{I}) \times \text{sk}(\downarrow \mathbf{I})}$ , and

$$\psi_1^{\downarrow \mathbf{I}} := [\tilde{\mathbf{P}}_{b, \mathbf{b}_d}]_{(b, \mathbf{b}_d) \in \text{sk}(\mathbf{I}) \times \text{sk}_1(\mathbf{I})}, \quad (3.1.17)$$

where the entry is given by

$$\tilde{\mathbf{P}}_{b, \mathbf{b}_d} := \begin{cases} P'_{d, b}, & \text{if } b = \underline{\mathbf{b}}, \\ -P'_{d, b}, & \text{if } b = \overline{\mathbf{b}}, \\ \mathbf{0}, & \text{if } b \notin \mathbf{b}, \end{cases} \quad (3.1.18)$$

for all  $\mathbf{b}_d \in \text{sk}_1(\mathbf{I})$  and  $b \in \text{sk}(\mathbf{I})$ . □

We note here that  $\psi_0$  is a projective cover of  $DV_{\mathbf{I}}$  in (3.1.16) because it induces an isomorphism  $\text{top } P'_{\text{sk}(\mathbf{I})} \cong \text{top } DV_{\mathbf{I}}$ , but  $\psi_1: P'_{\text{sk}(\downarrow \mathbf{I})} \oplus P'_{\text{sk}_1(\mathbf{I})} \rightarrow \text{Im } \psi_1$  is not always a projective cover. By Corollary 2.6.6, we can set

$$P'_{\text{sk}(\downarrow \mathbf{I})} \oplus P'_{\text{sk}_1(\mathbf{I})} = P_1^t \oplus P_2^t \quad (3.1.19)$$

with  $\Psi: P_1^t \rightarrow \text{Im } \psi_1$  a projective cover, where  $\psi_1 = [\Psi, \mathbf{0}]$  is a matrix expression of  $\psi_1$  with respect to the decomposition  $P_1^t \oplus P_2^t$ .

---

<sup>1</sup>We changed the order of direct summands as in  $P'_{\text{sk}(\downarrow \mathbf{I})} \oplus P'_{\text{sk}_1(\mathbf{I})}$  because we wanted to put matrices dependent on choice maps closer to each other in the final formula.

**Proposition 3.1.22.** *In the setting above, we can give a projective presentation of  $\tau^{-1}V_{\mathbf{I}} \oplus P_2$  as follows:*

$$P_{\text{sk}(\mathbf{I})} \xrightarrow{\pi_1 = \begin{bmatrix} \Psi^t \\ \mathbf{0} \end{bmatrix}} P_1 \oplus P_2 \xrightarrow{(\text{coker } \Psi^t) \oplus 1_{P_2}} \tau^{-1}V_{\mathbf{I}} \oplus P_2 \rightarrow 0. \quad (3.1.20)$$

Here by changing the decomposition of the middle term to  $P_1 \oplus P_2 = P_{\text{sk}(\downarrow \mathbf{I})} \oplus P_{\text{sk}_1(\mathbf{I})}$ , we have

$$\pi_1 = \pi_1(\mathbf{d}) = \begin{bmatrix} \pi_{11} \\ \pi_1^{\downarrow \mathbf{I}} \end{bmatrix} := \begin{bmatrix} \psi_{11}^t \\ (\psi_1^{\downarrow \mathbf{I}})^t \end{bmatrix} : P_{\text{sk}(\mathbf{I})} \rightarrow P_{\text{sk}(\downarrow \mathbf{I})} \oplus P_{\text{sk}_1(\mathbf{I})}$$

that is induced from (3.1.16), where  $\pi_{11} := \pi_{11}(\mathbf{d})$  is given by

$$\pi_{11} := \psi_{11}^t = [\delta_{b, \mathbf{d}(b')} \mathbf{P}_{\mathbf{d}(b'), b'}]_{(b', b) \in \text{sk}(\downarrow \mathbf{I}) \times \text{sk}(\mathbf{I})},$$

and by (3.1.17) and (3.1.18), the precise form of  $\pi_1^{\downarrow \mathbf{I}} := (\psi_1^{\downarrow \mathbf{I}})^t$  is given as follows:

$$\begin{aligned} \pi_1^{\downarrow \mathbf{I}} &= [\tilde{\mathbf{P}}_{b, \mathbf{b}_d}]_{(b, \mathbf{b}_d) \in \text{sk}(\mathbf{I}) \times \text{sk}_1(\mathbf{I})}^t = [\tilde{\mathbf{P}}_{b, \mathbf{b}_d}]_{(\mathbf{b}_d, b) \in \text{sk}_1(\mathbf{I}) \times \text{sk}(\mathbf{I})}^t \\ &=: [\hat{\mathbf{P}}_{b, \mathbf{b}_d}]_{(\mathbf{b}_d, b) \in \text{sk}_1(\mathbf{I}) \times \text{sk}(\mathbf{I})}, \end{aligned}$$

where the entry is given by

$$\hat{\mathbf{P}}_{b, \mathbf{b}_d} := \begin{cases} \mathbf{P}_{b, d}, & \text{if } b = \underline{\mathbf{b}}, \\ -\mathbf{P}_{b, d}, & \text{if } b = \overline{\mathbf{b}}, \\ \mathbf{0}, & \text{if } b \notin \mathbf{b}, \end{cases} \quad (3.1.21)$$

for all  $b \in \text{sk}(\mathbf{I})$  and  $\mathbf{b}_d \in \text{sk}_1(\mathbf{I})$ .

*Proof.* By (3.1.16) and (3.1.19),  $DV_{\mathbf{I}}$  has a minimal projective presentation

$$P_1^t \xrightarrow{\Psi} P_{\text{sk}(\mathbf{I})}^t \xrightarrow{\psi_0} DV_{\mathbf{I}} \rightarrow 0. \quad (3.1.22)$$

Hence by applying  $(\star)^t := \text{Hom}_{A^{\text{op}}}(\star, A^{\text{op}})$  to  $\Psi$  in (3.1.22), we have a minimal projective presentation

$$P_{\text{sk}(\mathbf{I})} \xrightarrow{\Psi^t} P_1 \xrightarrow{\text{coker } \Psi^t} \tau^{-1}V_{\mathbf{I}} \rightarrow 0 \quad (3.1.23)$$

of  $\tau^{-1}V_{\mathbf{I}} = \text{Tr } DV_{\mathbf{I}}$  in  $\text{mod } A$ . Hence the assertion follows.  $\square$

Note that in the projective presentation (3.1.20) of  $\tau^{-1}V_{\mathbf{I}} \oplus P_2$ , both of the projective terms and the form of the morphism  $\psi_1^t$  between them is explicitly given, whereas those in the projective presentation (3.1.23), the forms of  $P_1$  and  $\Psi^t$  are not clear. Therefore, we will use the former presentation (3.1.20) in our computation. Fortunately, as seen in (3.1.26), the unnecessary  $P_2$  does not disturb it because we can give an explicit form of the projective presentation of  $E \oplus P_2$  as follows.

**Proposition 3.1.23.** *Let  $\mathbf{I}$  be an interval of  $\mathbf{P}$  with  $\text{sc}(\mathbf{I}) = \{a_1, \dots, a_n\}$  and  $\text{sk}(\mathbf{I}) = \{b_1, \dots, b_m\}$  ( $m \geq 2$ ). Choose any choice maps  $\mathbf{c}: \text{sc}(\uparrow \mathbf{I}) \rightarrow \text{sc}(\mathbf{I})$  and  $\mathbf{d}: \text{sk}(\downarrow \mathbf{I}) \rightarrow \text{sk}(\mathbf{I})$ , and set  $\varepsilon_1 := \varepsilon_1(\mathbf{c})$ ,  $\pi_1 := \pi_1(\mathbf{d})$ . Choose also any  $i \in [n]$ . Then there exists some  $j \in [m]$  such that  $b_j \geq a_i$ . Let  $\lambda := \lambda(b_j, a_i) := [\lambda_{b, a}]_{(b, a) \in \text{sk}(\mathbf{I}) \times \text{sc}(\mathbf{I})}$  to be the*

matrix with  $\lambda_{b,a} = P_{b_j, a_i}$  if  $(b, a) = (b_j, a_i)$  and  $\lambda_{b,a} = 0$  otherwise. Then the following is a projective presentation of  $E \oplus P_2$ :

$$(P_{\text{sc}(\mathbf{I})} \oplus P_{\text{sc}(\uparrow \mathbf{I})}) \oplus P_{\text{sk}(\mathbf{I})} \xrightarrow{\mu_E} P_{\text{sc}(\mathbf{I})} \oplus (P_{\text{sk}(\downarrow \mathbf{I})} \oplus P_{\text{sk}_1(\mathbf{I})}) \xrightarrow{\varepsilon_E} E \oplus P_2 \rightarrow 0.$$

Here  $\mu_E$  is given by

$$\mu_E := \left[ \begin{array}{c|c} \varepsilon_1 & \lambda \\ \hline \mathbf{0} & \pi_1 \end{array} \right],$$

where  $\varepsilon_1$  and  $\pi_1$  are given in Propositions 3.1.17 and 3.1.22, respectively.

*Proof.* Without loss of generality, we may assume that  $a_i = a_1$  and  $b_j = b_1$ . Then

$$\lambda = \begin{bmatrix} P_{b_1, a_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

By [38, Section 3.6], an almost split sequence (2.6.13) can be obtained as a pushout of the sequence (3.1.23) along a morphism  $\eta: P_{\text{sk}(\mathbf{I})} \rightarrow V_{\mathbf{I}}$  as follows:

$$\begin{array}{ccccccc} P_{\text{sk}(\mathbf{I})} & \xrightarrow{\Psi^t} & P_1 & \xrightarrow{\text{coker } \Psi^t} & \tau^{-1}V_{\mathbf{I}} & \longrightarrow & 0 \\ \downarrow \eta & & \downarrow \theta & & \parallel & & \\ V_{\mathbf{I}} & \xrightarrow{\mu} & E & \xrightarrow{\varepsilon} & \tau^{-1}V_{\mathbf{I}} & \longrightarrow & 0 \end{array} \quad (3.1.24)$$

Here,  $\eta$  is the composite of morphisms

$$P_{\text{sk}(\mathbf{I})} \xrightarrow{\text{can.}} \text{top } P_{\text{sk}(\mathbf{I})} \xrightarrow{\sim} \text{soc } \nu P_{\text{sk}(\mathbf{I})} \xrightarrow{\sim} \text{soc } V_{\mathbf{I}} \xrightarrow{\alpha} S \hookrightarrow \text{soc } V_{\mathbf{I}} \hookrightarrow V_{\mathbf{I}},$$

where  $\nu$  is the Nakayama functor  $\nu := D \circ \text{Hom}_A(\star, A)$ ,  $S$  is any simple  $A\text{-End}_A(V_{\mathbf{I}})$ -subbimodule of  $\text{soc } V_{\mathbf{I}}$ , and  $\alpha$  is a retraction.

Here we claim that any simple  $A$ -submodule  $S$  of  $\text{soc } V_{\mathbf{I}}$  is automatically a simple  $A\text{-End}_A(V_{\mathbf{I}})$ -subbimodule of  $\text{soc } V_{\mathbf{I}}$ . Indeed, this follows from the fact that  $\text{soc } V_{\mathbf{I}} = \bigoplus_{i \in [m]} V_{\{b_i\}}$ , where  $V_{\{b_i\}}$  are mutually non-isomorphic simple  $A$ -modules. More precisely, it is enough to show that  $f(S) \subseteq S$  for any  $f \in \text{End}_A(V_{\mathbf{I}})^{\text{op}}$  because if this is shown, then  $S$  turns out to be a right  $\text{End}_A(V_{\mathbf{I}})$ -submodule and a simple  $A\text{-End}_A(V_{\mathbf{I}})$ -subbimodule of  $\text{soc } V_{\mathbf{I}}$ . By the fact above,  $S \cong V_{\{b_i\}}$  for a unique  $i \in [m]$ , and hence  $\text{pr}_j(S) = 0$  for all  $j \in [m] \setminus \{i\}$ , where  $\text{pr}_j: \text{soc } V_{\mathbf{I}} \rightarrow V_{\{b_j\}}$  is the canonical projection. Thus  $S \subseteq V_{\{b_i\}}$ , which shows that  $S = V_{\{b_i\}}$  because the both hand sides are simple. Now if  $f = 0$ , then  $f(S) = 0 \subseteq S$ ; otherwise  $f(S) \cong S$ , and then we have  $f(S) = V_{\{b_i\}} = S$  by applying the argument above to the simple  $A$ -submodule  $f(S)$  of  $\text{soc } V_{\mathbf{I}}$ . This proves our claim.

Therefore, we may take  $S := V_{\{b_1\}}$ , and

$$\eta := \left[ \rho_{1_{b_1}}^{V_{\mathbf{I}}}, \mathbf{0}, \dots, \mathbf{0} \right] : P_{\text{sk}(\mathbf{I})} = P_{b_1} \oplus P_{b_2} \oplus \dots \oplus P_{b_m} \rightarrow V_{\mathbf{I}}.$$

By assumption,  $a_1 \leq b_1$  in  $\mathbf{I}$ . Hence we have a commutative diagram

$$\begin{array}{ccc} & P_{\text{sc}(\mathbf{I})} = P_{a_1} \oplus \dots \oplus P_{a_n} & \\ \eta' := \left[ \begin{array}{c|c} P_{b_1, a_1} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \end{array} \right] \nearrow & & \downarrow \varepsilon_0 = \left[ \rho_{1_{a_1}}^{V_{\mathbf{I}}}, \dots, \rho_{1_{a_n}}^{V_{\mathbf{I}}} \right] \\ P_{\text{sk}(\mathbf{I})} & \xrightarrow{\eta} & V_{\mathbf{I}} \end{array}$$

Indeed, by Lemma 3.1.16, we have

$$\varepsilon_0 \eta' = [\rho_{1_{a_1}}^{V_{\mathbf{I}}} P_{b_1, a_1}, \mathbf{0}, \dots, \mathbf{0}] = [\rho_{V_{\mathbf{I}}(p_{b_1, a_1})(1_{a_1})}^{V_{\mathbf{I}}}, \mathbf{0}, \dots, \mathbf{0}] = [\rho_{1_{b_1}}^{V_{\mathbf{I}}}, \mathbf{0}, \dots, \mathbf{0}] = \eta.$$

The pushout diagram (3.1.24) yields the exact sequence

$$P_{\text{sk}(\mathbf{I})} \xrightarrow{[\frac{\eta}{\Psi^t}]} V_{\mathbf{I}} \oplus P_1 \xrightarrow{[\mu, -\theta]} E \rightarrow 0.$$

Since  $P_1 \oplus P_2 = P_{\text{sk}(\downarrow \mathbf{I})} \oplus P_{\text{sk}_1(\mathbf{I})}$  and  $\pi_1 = [\frac{\Psi^t}{\mathbf{0}}]$ , this yields the exact sequence

$$P_{\text{sk}(\mathbf{I})} \xrightarrow{[\frac{\eta}{\pi_1}]} V_{\mathbf{I}} \oplus (P_{\text{sk}(\downarrow \mathbf{I})} \oplus P_{\text{sk}_1(\mathbf{I})}) \xrightarrow{\pi} E \oplus P_2 \rightarrow 0,$$

where  $\pi := [\mu, -\theta] \oplus \mathbf{0}$ . This is extended to the following commutative diagram with the bottom row exact:

$$\begin{array}{ccccccc} (P_{\text{sc}_1(\mathbf{I})} \oplus P_{\text{sc}(\uparrow \mathbf{I})}) \oplus P_{\text{sk}(\mathbf{I})} & \xrightarrow{\mu_E} & P_{\text{sc}(\mathbf{I})} \oplus (P_{\text{sk}(\downarrow \mathbf{I})} \oplus P_{\text{sk}_1(\mathbf{I})}) & \xrightarrow{\varepsilon_E} & E \oplus P_2 & \longrightarrow & 0 \\ \parallel & & \downarrow [\frac{\varepsilon_0}{\mathbf{0}} \frac{\mathbf{0}}{\mathbf{1}}] & & \parallel & & \\ (P_{\text{sc}_1(\mathbf{I})} \oplus P_{\text{sc}(\uparrow \mathbf{I})}) \oplus P_{\text{sk}(\mathbf{I})} & \xrightarrow{[\frac{\mathbf{0}}{\mathbf{0}} \frac{\eta}{\pi_1}]} & V_{\mathbf{I}} \oplus (P_{\text{sk}(\downarrow \mathbf{I})} \oplus P_{\text{sk}_1(\mathbf{I})}) & \xrightarrow{\pi} & E \oplus P_2 & \longrightarrow & 0 \end{array},$$

where we set  $\mu_E := [\frac{\varepsilon_1}{\mathbf{0}} \frac{\eta'}{\pi_1}]$  and  $\varepsilon_E := \pi \circ [\frac{\varepsilon_0}{\mathbf{0}} \frac{\mathbf{0}}{\mathbf{1}}]$ , which is an epimorphism as the composite of epimorphisms.

It remains to show that  $\varepsilon_E$  is a cokernel morphism of  $\mu_E$ . By the commutativity of the diagram and the exactness of the bottom row, we see that  $\varepsilon_E \mu_E = 0$ . Let  $[f, g] : P_{\text{sc}(\mathbf{I})} \oplus (P_{\text{sk}(\downarrow \mathbf{I})} \oplus P_{\text{sk}_1(\mathbf{I})}) \rightarrow X$  be a morphism with  $[f, g] \cdot \mu_E = 0$ . Then  $f \varepsilon_1 = 0$ . Since  $\varepsilon_0$  is a cokernel morphism of  $\varepsilon_1$ , there exists some  $f' : V_{\mathbf{I}} \rightarrow X$  such that  $f = f' \varepsilon_0$ . Then we have  $[f, g] = [f', g] \cdot [\frac{\varepsilon_0}{\mathbf{0}} \frac{\mathbf{0}}{\mathbf{1}}]$ . Now  $[f', g] \cdot [\frac{\mathbf{0}}{\mathbf{0}} \frac{\eta}{\pi_1}] = [f', g] [\frac{\varepsilon_0}{\mathbf{0}} \frac{\mathbf{0}}{\mathbf{1}}] \mu_E = [f, g] \cdot \mu_E = 0$ . Hence  $[f', g]$  factors through  $\pi$ , that is,  $[f', g] = h \pi$  for some  $h : E \oplus P_2 \rightarrow X$ . Therefore, we have  $[f, g] = h \pi [\frac{\varepsilon_0}{\mathbf{0}} \frac{\mathbf{0}}{\mathbf{1}}] = h \varepsilon_E$ . The uniqueness of  $h$  follows from the fact that  $\varepsilon_E$  is an epimorphism. As a consequence,  $\varepsilon_E$  is a cokernel morphism of  $\mu_E$ .  $\square$

We are now in a position to state the formula of  $d_M(V_{\mathbf{I}})$  in this case.

**Theorem 3.1.24.** *Let  $M \in \text{mod } A$  and  $\mathbf{I}$  an interval of  $\mathbf{P}$  with  $\text{sc}(\mathbf{I}) = \{a_1, \dots, a_n\}$  and  $\text{sk}(\mathbf{I}) = \{b_1, \dots, b_m\}$ . Choose any choice maps*

$$\mathbf{c} : \text{sc}(\uparrow \mathbf{I}) \rightarrow \text{sc}(\mathbf{I}) \text{ and } \mathbf{d} : \text{sk}(\downarrow \mathbf{I}) \rightarrow \text{sk}(\mathbf{I}),$$

*and set  $\varepsilon_1 := \varepsilon_1(\mathbf{c})$ ,  $\pi_1 := \pi_1(\mathbf{d})$  as in Propositions 3.1.17 and 3.1.22. Choose also any  $(j, i) \in [m] \times [n]$  such that  $b_j \geq a_i$ , and set  $\lambda := \lambda(b_j, a_i)$  as in Proposition 3.1.23. Assume that  $V_{\mathbf{I}}$  is non-injective (i.e.,  $m \geq 2$ ). Then*

$$d_M(V_{\mathbf{I}}) = \text{rank} \left[ \begin{array}{c|c} M(\varepsilon_1) & \mathbf{0} \\ \hline M(\lambda) & M(\pi_1) \end{array} \right] - \text{rank } M(\varepsilon_1) - \text{rank } M(\pi_1). \quad (3.1.25)$$

*Proof.* Because  $V_{\mathbf{I}}$  is not injective, the value of  $d_M(V_{\mathbf{I}})$  can be computed from the three terms of the almost split sequence (2.6.13) by using Theorem 2.6.12 as follows:

$$\begin{aligned} d_M(V_{\mathbf{I}}) &= \dim \operatorname{Hom}_A(V_{\mathbf{I}}, M) - \dim \operatorname{Hom}_A(E, M) \\ &\quad + \dim \operatorname{Hom}_A(\tau^{-1}V_{\mathbf{I}}, M) \\ &= \dim \operatorname{Hom}_A(V_{\mathbf{I}}, M) - \dim \operatorname{Hom}_A(E \oplus P_2, M) \\ &\quad + \dim \operatorname{Hom}_A(\tau^{-1}V_{\mathbf{I}} \oplus P_2, M), \end{aligned} \quad (3.1.26)$$

where  $P_2$  is a direct summand of  $P_{\operatorname{sk}(\downarrow \mathbf{I})} \oplus P_{\operatorname{sk}_1(\mathbf{I})}$  as in (3.1.19). Hence the assertion follows by Lemma 3.1.2, and Propositions 3.1.17, 3.1.22 and 3.1.23.  $\square$

**Remark 3.1.25.** We note here that the formula (3.1.25) covers all cases, regardless of whether  $V_{\mathbf{I}}$  is injective or not.

Summarizing Theorems 3.1.19 and 3.1.24, we obtain the following.

**Theorem 3.1.26.** *Let  $M \in \operatorname{mod} A$  and  $\mathbf{I}$  an interval of  $\mathbf{P}$  with  $\operatorname{sc}(\mathbf{I}) = \{a_1, \dots, a_n\}$  and  $\operatorname{sk}(\mathbf{I}) = \{b_1, \dots, b_m\}$ . Choose any choice maps*

$$\mathbf{c}: \operatorname{sc}(\uparrow \mathbf{I}) \rightarrow \operatorname{sc}(\mathbf{I}) \text{ and } \mathbf{d}: \operatorname{sk}(\downarrow \mathbf{I}) \rightarrow \operatorname{sk}(\mathbf{I}),$$

*and any  $(j, i) \in [m] \times [n]$  such that  $b_j \geq a_i$ . Set  $\lambda := \lambda(b_j, a_i)$  as in Proposition 3.1.23. Then*

$$d_M(V_{\mathbf{I}}) = \operatorname{rank} \left[ \begin{array}{c|c} M(\varepsilon_1) & \mathbf{0} \\ \hline M(\lambda) & M(\pi_1) \end{array} \right] - \operatorname{rank} M(\varepsilon_1) - \operatorname{rank} M(\pi_1). \quad (3.1.27)$$

*Here we collect definitions of  $\varepsilon_1 := \varepsilon_1(\mathbf{c})$  and  $\pi_1 := \pi_1(\mathbf{d})$  given in Propositions 3.1.17 and 3.1.22:  $\varepsilon_1 := [\varepsilon_1^{\uparrow \mathbf{I}}, \varepsilon_{11}]$ , where*

$$\begin{aligned} \varepsilon_{11} &:= [\delta_{a, \mathbf{c}(a')} \mathbf{P}_{a', \mathbf{c}(a')}]_{(a, a') \in \operatorname{sc}(\mathbf{I}) \times \operatorname{sc}(\uparrow \mathbf{I})}, \text{ and} \\ \varepsilon_1^{\uparrow \mathbf{I}} &:= [\tilde{\mathbf{P}}_{a, \mathbf{a}_c}]_{(a, \mathbf{a}_c) \in \operatorname{sc}(\mathbf{I}) \times \operatorname{sc}_1(\mathbf{I})} \end{aligned}$$

*with the entries given by*

$$\tilde{\mathbf{P}}_{a, \mathbf{a}_c} := \begin{cases} \mathbf{P}_{c, a}, & \text{if } a = \underline{\mathbf{a}}, \\ -\mathbf{P}_{c, a}, & \text{if } a = \bar{\mathbf{a}}, \\ \mathbf{0}, & \text{if } a \notin \mathbf{a}, \end{cases}$$

*for all  $\mathbf{a}_c \in \operatorname{sc}_1(\mathbf{I})$  and  $a \in \operatorname{sc}(\mathbf{I})$ ; and  $\pi_1 := \begin{bmatrix} \pi_{11} \\ \pi_1^{\downarrow \mathbf{I}} \end{bmatrix}$ , where*

$$\begin{aligned} \pi_{11} &:= [\delta_{b, \mathbf{d}(b')} \mathbf{P}_{\mathbf{d}(b'), b'}]_{(b', b) \in \operatorname{sk}(\downarrow \mathbf{I}) \times \operatorname{sk}(\mathbf{I})}, \text{ and} \\ \pi_1^{\downarrow \mathbf{I}} &:= [\hat{\mathbf{P}}_{b, \mathbf{b}_d}]_{(\mathbf{b}_d, b) \in \operatorname{sk}_1(\mathbf{I}) \times \operatorname{sk}(\mathbf{I})}, \end{aligned}$$

*with the entries given by*

$$\hat{\mathbf{P}}_{b, \mathbf{b}_d} := \begin{cases} \mathbf{P}_{b, d}, & \text{if } b = \underline{\mathbf{b}}, \\ -\mathbf{P}_{b, d}, & \text{if } b = \bar{\mathbf{b}}, \\ \mathbf{0}, & \text{if } b \notin \mathbf{b}, \end{cases}$$

*for all  $b \in \operatorname{sk}(\mathbf{I})$  and  $\mathbf{b}_d \in \operatorname{sk}_1(\mathbf{I})$ .*

### 3.2 The case of 2D-grids

We specialize the general formula (3.1.27) to the case where  $\mathbf{P} = \mathbf{G}_{m,n}$  for some  $m, n \geq 2$  to make the formula easier to see. Denote by the maximum element  $(m, n)$  (resp. minimum element  $(0, 0)$ ) of  $\mathbf{P}$  by  $\omega$  (resp.  $\hat{0}$ ).

Following [14], we introduce the subsequent definition. Recall the definitions of antichains provided in Definition 2.2.2 (2) and convex hulls provided in Definition 2.2.15 (4).

**Definition 3.2.1.** We denote by  $\text{Ac}(\mathbf{P})$  the set of all full subposets of  $\mathbf{P}$  that are antichains. For any  $\mathbf{K}, \mathbf{L} \in \text{Ac}(\mathbf{P})$ , we define  $\mathbf{K} \leq \mathbf{L}$  if for all  $x \in \mathbf{K}$ , there exists  $z_x \in \mathbf{L}$  such that  $x \leq z_x$ , and for all  $z \in \mathbf{L}$ , there exists  $x_z \in \mathbf{K}$  such that  $x_z \leq z$ . In this case, we define  $[\mathbf{K}, \mathbf{L}] := \{y \in \mathbf{P} \mid x \leq y \leq z \text{ for some } x \in \mathbf{K} \text{ and for some } z \in \mathbf{L}\}$  and set it to be the full subposet of  $\mathbf{P}$ .

**Lemma 3.2.2.**  $\{[\mathbf{K}, \mathbf{L}] \mid \mathbf{K}, \mathbf{L} \in \text{Ac}(\mathbf{P}), \mathbf{K} \leq \mathbf{L}\}$  forms the set of all convex subposets of  $\mathbf{P}$ .

*Proof.* Let  $\mathbf{K}, \mathbf{L} \in \text{Ac}(\mathbf{P})$  such that  $\mathbf{K} \leq \mathbf{L}$ . First, let us show that  $[\mathbf{K}, \mathbf{L}] = \text{conv}(\mathbf{K} \cup \mathbf{L})$ . Let  $y \in \text{conv}(\mathbf{K} \cup \mathbf{L})$ . By definition, there exist  $x_0, z_0 \in \mathbf{K} \cup \mathbf{L}$  such that  $x_0 \leq y \leq z_0$ . Now assume both  $x_0 \in \mathbf{K}$  and  $z_0 \in \mathbf{K}$ . In this case, since  $\mathbf{K} \in \text{Ac}(\mathbf{P})$  and  $x_0$  and  $z_0$  are comparable, then necessarily  $x_0 = z_0$  and so  $y = x_0 = z_0 \in \mathbf{K} \subseteq [\mathbf{K}, \mathbf{L}]$ . Similarly, if both  $x_0 \in \mathbf{L}$  and  $z_0 \in \mathbf{L}$ , we have  $y \in \mathbf{L} \subseteq [\mathbf{K}, \mathbf{L}]$ . So either  $x_0 \in \mathbf{K}, z_0 \in \mathbf{L}$  or  $x_0 \in \mathbf{L}, z_0 \in \mathbf{K}$ . If  $x_0 \in \mathbf{K}, z_0 \in \mathbf{L}$ , then by definition we have  $y \in [\mathbf{K}, \mathbf{L}]$ . Now assume we have  $x_0 \in \mathbf{L}, z_0 \in \mathbf{K}$ . Since  $\mathbf{K} \leq \mathbf{L}$ , there exists  $k \in \mathbf{K}$  such that  $k \leq x_0$ . So we have  $k \leq x_0 \leq y \leq z_0$  with both  $k, z_0 \in \mathbf{K}$ . Therefore  $k = z_0$  because  $\mathbf{K} \in \text{Ac}(\mathbf{P})$ , and so  $y = k = z_0 \in \mathbf{K} \subseteq [\mathbf{K}, \mathbf{L}]$ . This proves that  $[\mathbf{K}, \mathbf{L}] \supseteq \text{conv}(\mathbf{K} \cup \mathbf{L})$ , and so  $[\mathbf{K}, \mathbf{L}] = \text{conv}(\mathbf{K} \cup \mathbf{L})$ . In particular,  $[\mathbf{K}, \mathbf{L}]$  is a convex set. Now let  $\mathbf{S}$  be a convex subposet of  $\mathbf{P}$ . Because  $\mathbf{P}$  is finite, we can define  $\mathbf{K} := \text{sc}(\mathbf{S})$ ,  $\mathbf{L} := \text{sk}(\mathbf{S})$ , and we have  $\mathbf{K} \leq \mathbf{L}$ . Then it is clear that  $\mathbf{S} = \text{conv}(\mathbf{K} \cup \mathbf{L})$ , and so  $\mathbf{S} = [\mathbf{K}, \mathbf{L}]$ .  $\square$

Hence we have  $\mathbb{I} = \{[\mathbf{K}, \mathbf{L}] \mid \mathbf{K}, \mathbf{L} \in \text{Ac}(\mathbf{P}), \mathbf{K} \leq \mathbf{L}, [\mathbf{K}, \mathbf{L}] \text{ is connected}\}$ . In particular, we have  $\mathbf{I} = [\text{sc}(\mathbf{I}), \text{sk}(\mathbf{I})]$  for all  $\mathbf{I} \in \mathbb{I}$  and

$$\text{Seg}(\mathbf{P}) = \{\mathbf{I} \in \mathbb{I}(\mathbf{P}) \mid |\text{sc}(\mathbf{I})| = 1 = |\text{sk}(\mathbf{I})|\}.$$

In what follows, we sometimes write  $\mathbf{I} = [\text{sc}(\mathbf{I}), \text{sk}(\mathbf{I})]$  for  $\mathbf{I} \in \mathbb{I}(\mathbf{P})$ .

**Notation 3.2.3.** Set  $\text{sc}(\uparrow \mathbf{I}) = \text{sc}(\mathbf{I}) = \{a_1, \dots, a_k\}$ ,  $\text{sc}(\uparrow \uparrow \mathbf{I}) = \{a'_1, \dots, a'_{k'}\}$  and  $\text{sk}(\downarrow \mathbf{I}) = \text{sk}(\mathbf{I}) = \{b_1, \dots, b_l\}$ ,  $\text{sk}(\downarrow \downarrow \mathbf{I}) = \{b'_1, \dots, b'_{l'}\}$ . Throughout this section, we always assume that if  $(x_i, y_i)$  is the coordinate of  $a_i$  in  $\mathbf{G}_{m,n}$  for each  $i \in [k]$ , then  $i < j$  implies  $y_i < y_j$ , and the same for  $a'_i, b_i$ , and  $b'_i$ . Then we have

$$\begin{aligned} \uparrow \mathbf{I} &= [\{a_1, \dots, a_k\}, \omega], \quad \uparrow \uparrow \mathbf{I} = [\{a'_1, \dots, a'_{k'}\}, \omega], \quad \text{and} \\ \downarrow \mathbf{I} &= [0, \{b_1, \dots, b_l\}], \quad \downarrow \downarrow \mathbf{I} = [0, \{b'_1, \dots, b'_{l'}\}]. \end{aligned}$$

In this case, we set

$$\begin{aligned} \tilde{\text{sc}}_1(\mathbf{I}) &= \{a_{i,i+1} := a_i \vee a_{i+1} \mid i \in [k-1]\}, \quad \text{and} \\ \tilde{\text{sk}}_1(\mathbf{I}) &= \{b_{i,i+1} := b_i \wedge b_{i+1} \mid i \in [l-1]\}. \end{aligned}$$

We note that  $\tilde{\text{sc}}_1(\mathbf{I}) \subseteq \text{sc}_1(\mathbf{I})$  and  $\tilde{\text{sk}}_1(\mathbf{I}) \subseteq \text{sk}_1(\mathbf{I})$ .

With these notations, we have a specialization of Theorem 3.1.26. However, before stating this, we give the minimal projective presentations of  $V_{\mathbf{I}}$ ,  $\tau^{-1}V_{\mathbf{I}}$ , and the middle term  $E$  of the almost split sequence starting from  $V_{\mathbf{I}}$  without proofs (for the two latter modules,  $V_{\mathbf{I}}$  is assumed to be non-injective) because in the general case, we did not give them. We restate [5, Proposition 39] under these notations as follows.

**Lemma 3.2.4.** *Let  $\mathbf{U}$  be an up-set. Then the interval module  $V_{\mathbf{U}}$  has the following minimal projective presentation:*

$$P_{\tilde{\text{sc}}_1(\mathbf{U})} \xrightarrow{\varepsilon_1^{\mathbf{U}}} P_{\text{sc}(\mathbf{U})} \xrightarrow{\varepsilon_0^{\mathbf{U}}} V_{\mathbf{U}} \rightarrow 0,$$

where  $\varepsilon_0^{\mathbf{U}} = [\rho_{1_a}^{\mathbf{U}}]_{a \in \text{sc}(\mathbf{U})}$  and

$$\varepsilon_1^{\mathbf{U}} := \begin{bmatrix} P_{a_{12}, a_1} & & & & \\ -P_{a_{12}, a_2} & P_{a_{23}, a_2} & & & \\ & -P_{a_{23}, a_3} & \ddots & & \\ & & \ddots & P_{a_{k-1}, k, a_{k-1}} & \\ & & & -P_{a_{k-1}, k, a_k} & \end{bmatrix} \quad (\text{blank entries are zeros}) \quad (3.2.28)$$

**Remark 3.2.5.** We note that some columns are missing compared with (3.1.5) because  $\tilde{\text{sc}}_1(\mathbf{U}) \subseteq \text{sc}_1(\mathbf{U})$  and the above projective presentation is minimal in the 2D-grid setting and the missing columns are eliminated by a sequence of fundamental operations on the matrix. We refer the reader to Remarks 5.3.9 to 5.3.11 and Lemma 5.3.12 later in the text for similar reasoning.

This together with the presentation of the up-set  $\uparrow\mathbf{I}$  gives the following ([5, Proposition 41]).

**Proposition 3.2.6.** *For each  $a' \in \text{sc}(\uparrow\mathbf{I})$ , set  $\mathbf{c}(a') := a \in \text{sc}(\mathbf{I})$  if and only if  $\text{pr}_2(a) = \min\{\text{pr}_2(c) \mid c \leq a', c \in \text{sc}(\mathbf{I})\}$ . Then the interval module  $V_{\mathbf{I}}$  has the following minimal projective presentation:*

$$P_{\tilde{\text{sc}}_1(\mathbf{I})} \oplus P_{\text{sc}(\uparrow\mathbf{I})} \xrightarrow{\varepsilon_1 := [\varepsilon_1^{\uparrow\mathbf{I}}, \varepsilon_{11}]} P_{\text{sc}(\mathbf{I})} \xrightarrow{\varepsilon_0} V_{\mathbf{I}} \rightarrow 0,$$

where  $\varepsilon_0 = [\rho_{1_p}^{V_{\mathbf{I}}}]_{p \in \text{sc}(\mathbf{I})}$ ,  $\varepsilon_{11} := [\delta_{a, \mathbf{c}(a')} P_{a', \mathbf{c}(a')}]_{(a, a') \in \text{sc}(\mathbf{I}) \times \text{sc}(\uparrow\mathbf{I})}$ , and  $\varepsilon_1^{\uparrow\mathbf{I}}$  is given in (3.2.28) by letting  $\mathbf{U} = \uparrow\mathbf{I}$ .

### 3.2.1 The case where $V_{\mathbf{I}}$ is injective

Assume that  $V_{\mathbf{I}}$  is injective. Then  $V_{\mathbf{I}} \cong I_{b_1}$ , and hence  $\mathbf{I} = [\hat{0}, b_1]$ ,  $\mathbf{U} = \mathbf{P}$ , and  $\uparrow\mathbf{I} = [\{a'_1, a'_2\}, \omega]$ . Moreover,  $V_{\mathbf{I}}/\text{soc } V_{\mathbf{I}} \cong V_{\mathbf{J}}$  is also an interval module for the interval  $\mathbf{J} = [\hat{0}, \{b_1 - (1, 0), b_1 - (0, 1)\}]$ , where delete  $b_1 - (1, 0)$  or  $b_1 - (0, 1)$  if it is not in  $\mathbf{P}$ . Then the pair  $(\uparrow\mathbf{J}, \uparrow\mathbf{J})$  is a unique pair of up-sets in  $\mathbf{P}$  such that  $\mathbf{J} = \uparrow\mathbf{J} \setminus \uparrow\mathbf{J}$ . These are given by  $\uparrow\mathbf{J} = \mathbf{P}$ , and  $\uparrow\mathbf{J} = [\{a'_1, a'_2, b_1\}, \omega]$ . Hence by Theorem 2.6.12, Proposition 3.2.6 and Lemma 3.1.2, we have the following.

**Theorem 3.2.7.** *Assume that  $V_{\mathbf{I}}$  is injective. Then  $V_{\mathbf{I}}$  and  $V_{\mathbf{I}}/\text{soc } V_{\mathbf{I}}$  have minimal projective presentations of the following forms.*

$$\begin{aligned} P_{\text{sc}(\uparrow \mathbf{I})} &\xrightarrow{\varepsilon'_1} P_0 \rightarrow V_{\mathbf{I}} \rightarrow 0, \\ P_{\text{sc}(\uparrow \mathbf{I})} \oplus P_{b_1} &\xrightarrow{\varepsilon''_1} P_0 \rightarrow V_{\mathbf{I}}/\text{soc } V_{\mathbf{I}} \rightarrow 0, \end{aligned}$$

where  $\varepsilon'_1 = [P_{a'_1, \hat{0}}, P_{a'_2, \hat{0}}]$ , and  $\varepsilon''_1 = [P_{a'_1, \hat{0}}, P_{a'_2, \hat{0}}, P_{b_1, \hat{0}}]$ . Therefore, we have

$$d_M(V_{\mathbf{I}}) = \text{rank} \begin{bmatrix} M_{a'_1, \hat{0}} \\ M_{a'_2, \hat{0}} \\ M_{b_1, \hat{0}} \end{bmatrix} - \text{rank} \begin{bmatrix} M_{a'_1, \hat{0}} \\ M_{a'_2, \hat{0}} \end{bmatrix}.$$

### 3.2.2 The case where $V_{\mathbf{I}}$ is non-injective

Now, assume that  $V_{\mathbf{I}}$  is non-injective.

**Proposition 3.2.8.** *For each  $b' \in \text{sk}(\downarrow \mathbf{I})$ , set  $\mathbf{d}(b') := b \in \text{sk}(\mathbf{I})$  if and only if  $\text{pr}_2(b) = \min\{\text{pr}_2(d) \mid b' \leq d \in \text{sk}(\mathbf{I})\}$ . Then the interval module  $DV_{\mathbf{I}}$  has the following minimal projective presentation:*

$$P'_{\text{sk}(\downarrow \mathbf{I})} \oplus P'_{\tilde{\text{sk}}_1(\mathbf{I})} \xrightarrow{\psi_1 := [\psi_{11}, \psi_1^{\downarrow \mathbf{I}}]} P'_{\text{sk}(\mathbf{I})} \xrightarrow{\psi_0} DV_{\mathbf{I}} \rightarrow 0,$$

where  $\psi_0 = [\lambda_{1_b}^{\mathbf{I}}]_{b \in \text{sk}(\mathbf{I})}$ ,  $\psi_{11} := [\delta_{b, \mathbf{d}(b')} P'_{\mathbf{d}(b'), b'}]_{(b, b') \in \text{sk}(\mathbf{I}) \times \text{sk}(\downarrow \mathbf{I})}$  and

$$\psi_1^{\downarrow \mathbf{I}} := \begin{bmatrix} P'_{b_1, b_{12}} & & & & \\ -P'_{b_2, b_{12}} & P'_{b_2, b_{23}} & & & \\ & -P'_{b_3, b_{23}} & \ddots & & \\ & & \ddots & P'_{b_{l-1}, b_{l-1, l}} & \\ & & & -P'_{b_l, b_{l-1, l}} & \end{bmatrix}.$$

Therefore,  $\tau^{-1}V_{\mathbf{I}} = \text{Tr } DV_{\mathbf{I}}$  has a minimal projective presentation of the following form:

$$P_{\text{sk}(\mathbf{I})} \xrightarrow{\pi_1 := [\pi_{11}^{\downarrow \mathbf{I}}]} P_{\text{sk}(\downarrow \mathbf{I})} \oplus P_{\tilde{\text{sk}}_1(\mathbf{I})} \xrightarrow{\pi_0} \tau^{-1}V_{\mathbf{I}} \rightarrow 0,$$

where  $\pi_0$  is a projective cover of  $\tau^{-1}V_{\mathbf{I}}$ ,  $\pi_{11} := [\delta_{b', b} P_{\mathbf{d}(b'), b'}]_{(b', b) \in \text{sk}(\downarrow \mathbf{I}) \times \text{sk}(\mathbf{I})}$  and

$$\pi_1^{\downarrow \mathbf{I}} := \begin{bmatrix} P_{b_1, b_{12}} & -P_{b_2, b_{12}} & & & \\ & P_{b_2, b_{23}} & -P_{b_3, b_{23}} & & \\ & & \ddots & \ddots & \\ & & & P_{b_{l-1}, b_{l-1, l}} & -P_{b_l, b_{l-1, l}} \end{bmatrix}.$$

**Proposition 3.2.9.** *Assume that  $a_1 \leq b_1$  without loss of generality. Then the following is a projective presentation of  $E$ :*

$$(P_{\tilde{\text{sc}}_1(\mathbf{I})} \oplus P_{\text{sc}(\uparrow \mathbf{I})}) \oplus P_{\text{sk}(\mathbf{I})} \xrightarrow{\mu_E} P_{\text{sc}(\mathbf{I})} \oplus (P_{\text{sk}(\downarrow \mathbf{I})} \oplus P_{\tilde{\text{sk}}_1(\mathbf{I})}) \xrightarrow{\varepsilon_E} E \rightarrow 0.$$

Here  $\mu_E$  is given by

$$\mu_E := \left[ \begin{array}{c|cc} \varepsilon_1 & P_{b_1, a_1} & \mathbf{0} \\ \hline & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & & \pi_1 \end{array} \right],$$

where  $\varepsilon_1$  and  $\pi_1$  are given in Propositions 3.2.6 and 3.2.8, respectively.

**Theorem 3.2.10.** *Let  $M \in \text{mod } A$  and  $\mathbf{I}$  an interval of  $\mathbf{P}$ . Without loss of generality, we may assume that  $a_1 \leq b_1$  in  $\mathbf{P}$ . Then*

$$d_M(V_{\mathbf{I}}) = \text{rank} \left[ \begin{array}{c|cc} M(\varepsilon_1) & & \mathbf{0} \\ \hline M_{b_1, a_1} & \mathbf{0} & \\ \hline \mathbf{0} & \mathbf{0} & M(\pi_1) \end{array} \right] - \text{rank } M(\varepsilon_1) - \text{rank } M(\pi_1). \quad (3.2.29)$$

Here  $M(\varepsilon_1)$ ,  $M(\pi_1)$  have the forms

$$M(\varepsilon_1) = \begin{bmatrix} M(\varepsilon_1^{\uparrow \mathbf{I}}) \\ M(\varepsilon_{11}) \end{bmatrix} \quad \text{and} \quad M(\pi_1) = [M(\pi_{11}), M(\pi_1^{\downarrow \mathbf{I}})],$$

where  $M(\varepsilon_{11})$ ,  $M(\varepsilon_1^{\uparrow \mathbf{I}})$ ,  $M(\pi_{11})$  and  $M(\pi_1^{\downarrow \mathbf{I}})$  are given by

$$\begin{aligned} M(\varepsilon_{11}) &= [\delta_{a, \mathbf{c}(a')} M_{a', \mathbf{c}(a')}]_{(a', a) \in \text{sc}(\uparrow \mathbf{I}) \times \text{sc}(\mathbf{I})}, \\ M(\varepsilon_1^{\uparrow \mathbf{I}}) &= \begin{bmatrix} M_{a_{12}, a_1} & -M_{a_{12}, a_2} & & & \\ & M_{a_{23}, a_2} & -M_{a_{23}, a_3} & & \\ & & \ddots & \ddots & \\ & & & M_{a_{k-1}, a_{k-1}} & -M_{a_{k-1}, a_k} \end{bmatrix}, \\ M(\pi_{11}) &= [\delta_{b, \mathbf{d}(b')} M_{\mathbf{d}(b'), b'}]_{(b, b') \in \text{sk}(\mathbf{I}) \times \text{sk}(\downarrow \mathbf{I})}, \quad \text{and} \\ M(\pi_1^{\downarrow \mathbf{I}}) &= \begin{bmatrix} M_{b_1, b_{12}} & & & & \\ -M_{b_2, b_{12}} & M_{b_2, b_{23}} & & & \\ & -M_{b_3, b_{23}} & \ddots & & \\ & & \ddots & M_{b_{l-1}, b_{l-1, l}} & \\ & & & -M_{b_l, b_{l-1, l}} \end{bmatrix}. \end{aligned}$$

**Remark 3.2.11.** We express  $M(\varepsilon_1)$  and  $M(\pi_1)$  in block-matrix form as

$$[M(\varepsilon_1)_1, M(\varepsilon_1)_2] \quad \text{and} \quad \begin{bmatrix} M(\pi_1)_1 \\ M(\pi_1)_2 \end{bmatrix},$$

respectively, where  $M(\varepsilon_1)_1$  has  $\dim M(a_1)$  columns and  $M(\pi_1)_1$  has  $\dim M(b_1)$  rows. Then the matrix in the first term of (3.2.29), denoted by  $R(M, \mathbf{I})$ , has the following form:

$$R(M, \mathbf{I}) = \begin{bmatrix} M(\varepsilon_1)_1 & M(\varepsilon_1)_2 & \mathbf{0} \\ M_{b_1, a_1} & \mathbf{0} & M(\pi_1)_1 \\ \mathbf{0} & \mathbf{0} & M(\pi_1)_2 \end{bmatrix}.$$

We denote by  $E_r$  the identity matrix of rank  $r$ . By elementary column transformations within the second block column and elementary row transformations within the first block row, we can transform  $M(\varepsilon_1)_2$  to the normal form  $E_{r_1} \oplus \mathbf{0}$ ; and by elementary column transformations within the third block column and elementary

row transformations within the third block row, we can transform  $M(\pi_1)_2$  to the normal form  $E_{r_2} \oplus \mathbf{0}$ , where the obtained matrix  $R(M, \mathbf{I})_1$  is equivalent to  $R(M, \mathbf{I})$ , and has the form:

$$R(M, \mathbf{I})_1 = \left[ \begin{array}{c|cc|cc} M_{11} & E_{r_1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ M_{21} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline M_{b_1, a_1} & \mathbf{0} & \mathbf{0} & M_{34} & M_{35} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & E_{r_2} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array} \right] \sim \left[ \begin{array}{c|cc|cc} \mathbf{0} & E_{r_1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ M_{21} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline M_{b_1, a_1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & M_{35} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & E_{r_2} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array} \right].$$

By the same way, we can transform  $M_{21}$  and  $M_{35}$  to the normal forms:

$$R(M, \mathbf{I}) \sim \left[ \begin{array}{c|cc|cc} \mathbf{0} & E_{r_1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline E_{r'_1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline M_a & M_b & \mathbf{0} & \mathbf{0} & E_{r'_2} & \mathbf{0} \\ M_c & M_d & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & E_{r_2} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array} \right] \sim M_d \oplus E_{r_1} \oplus E_{r'_1} \oplus E_{r_2} \oplus E_{r'_2},$$

where the last equivalence is obtained by transforming  $M_a, M_b, M_c$  to  $\mathbf{0}$  by using  $E_{r'_1}$  and  $E_{r'_2}$ . Then we have

$$d_M(V_{\mathbf{I}}) = \text{rank } M_d$$

because  $\text{rank } R(M, \mathbf{I}) = \text{rank } M_d + r_1 + r'_1 + r_2 + r'_2$ ,  $\text{rank } M(\varepsilon_1) = r_1 + r'_1$ , and  $\text{rank } M(\pi_1) = r_2 + r'_2$ .

**Example 3.2.12.** In the following diagrams, let  $\mathbf{P} = \mathbf{G}_{4,2}$  be given by the Hasse quiver on the left, and  $M \in \text{mod } A$  be given by the diagram on the right:

$$\begin{array}{ccccccc} 1' & \longrightarrow & 2' & \longrightarrow & 3' & \longrightarrow & 4' \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ 1 & \longrightarrow & 2 & \longrightarrow & 3 & \longrightarrow & 4 \end{array} \quad \begin{array}{ccccccc} \mathbb{K}^2 & \xrightarrow{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} & \mathbb{K}^2 & \xrightarrow{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} & \mathbb{K}^2 & \xrightarrow{\mathbf{0}} & \mathbb{K} \\ \mathbf{0} \uparrow & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \uparrow & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \uparrow & & \mathbf{0} \uparrow & & \\ \mathbb{K} & \xrightarrow{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}} & \mathbb{K}^3 & \xrightarrow{\mathbf{1}} & \mathbb{K}^3 & \xrightarrow{\mathbf{1}} & \mathbb{K}^3 \end{array}$$

Let  $\mathbf{I} := [\{2, 1'\}, \{4, 3'\}]$  be an interval of  $\mathbf{P}$ . Then  $\dim V_{\mathbf{I}} := \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$ , and  $a_1 = 2$ ,  $a_2 = 1'$ ,  $a_{12} = 2'$ ,  $b_1 = 4$ ,  $b_2 = 3'$ ,  $b_{12} = 3$ ,  $a'_1 = 4'$ ,  $b'_1 = 1$ . Therefore,

$$\begin{aligned} R(M, \mathbf{I}) &= \begin{bmatrix} M_{a_{12}, a_1} & -M_{a_{12}, a_2} & \mathbf{0} & \mathbf{0} \\ M_{a'_1, a_1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ M_{b_1, a_1} & \mathbf{0} & M_{b_1, b'_1} & M_{b_1, b_{12}} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -M_{b_2, b_{12}} \end{bmatrix} \\ &= \begin{bmatrix} M_{2', 2} & -M_{2', 1'} & \mathbf{0} & \mathbf{0} \\ M_{4', 2} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ M_{4, 2} & \mathbf{0} & M_{4, 1} & M_{4, 3} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -M_{3', 3} \end{bmatrix} \\ &= \left[ \begin{array}{c|cc|cc} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} & \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \mathbf{0} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \end{array} \right] \sim \left[ \begin{array}{c|cc|cc} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \mathbf{0} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{array} \right]. \end{aligned}$$

Hence  $d_M(V_{\mathbf{I}}) = 2$ . Noting that  $M \cong V_{\mathbf{I}}^2 \oplus V_{[1,4]} \oplus V_{[4',4']}$ , we see that this gives a correct answer.

Define another  $N \in \text{mod } A$  based on  $M$ , by replacing the linear maps  $M_{4',3'}$  and  $M_{4',4}$  with  $\begin{bmatrix} 1 & 0 \end{bmatrix}$  and  $\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$  in  $N$ , respectively, while keeping the other structural linear maps of  $N$  the same as those of  $M$ . Then

$$R(N, \mathbf{I}) = \left[ \begin{array}{c|c|c|c} \begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{smallmatrix} & \begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix} & \mathbf{0} & \mathbf{0} \\ \hline \begin{smallmatrix} 1 & 0 & 0 \end{smallmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix} & \mathbf{0} & \begin{smallmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{smallmatrix} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{smallmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \end{smallmatrix} \end{array} \right] \sim \left[ \begin{array}{c|c|c|c} \begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix} & \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} & \mathbf{0} & \mathbf{0} \\ \hline \begin{smallmatrix} 1 & 0 & 0 \end{smallmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \begin{smallmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{smallmatrix} & \mathbf{0} & \begin{smallmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{smallmatrix} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{smallmatrix} \end{array} \right].$$

Hence  $d_N(V_{\mathbf{I}}) = 1$ , which coincides with the answer obtained from the decomposition  $N \cong V_{\mathbf{I}} \oplus V_{[1,4]} \oplus V_{[\{2,1'\},4']}$ . These decompositions can be easily seen by drawing the structure quivers of  $M, N$ :

$$\begin{array}{ccccccc} 1'_x & \longrightarrow & 2'_x & \longrightarrow & 3'_x & \dashrightarrow & 4' \\ & & \uparrow & & \uparrow & & \uparrow \\ 1'_y & \longrightarrow & 2'_y & \longrightarrow & 3'_y & & \vdots \\ & & \uparrow & & \uparrow & & \vdots \\ & & 2_x & \longrightarrow & 3_x & \longrightarrow & 4_x \\ & & & & \uparrow & & \vdots \\ & & & & 2_y & \longrightarrow & 3_y \longrightarrow 4_y \\ & & & & & & \vdots \\ 1 & \longrightarrow & 2_z & \longrightarrow & 3_z & \longrightarrow & 4_z \end{array},$$

where  $M$  is given by solid arrows, and  $N$  is given by both solid and broken arrows, bases of  $M(i)$  are denoted by  $i$  or  $i_a$  ( $a \in \{x, y, z\}$ ) for all  $i \in \mathbf{P}$ .

# Chapter 4

## Essential-cover technique

In Chapter 3, Theorem 3.1.26 provides a general and explicit formula for computing the interval multiplicity in theory, taking the persistence module as the input. Nevertheless, the persistence module is usually latent in practical analysis and hard to obtain in most situations. Thus, how to compute the interval multiplicity directly from the level of filtration without computing the persistent homology in advance becomes a critical problem to be solved from the TDA perspective. This is also the key step to bringing our theory to the ground of applications. For this reason, we will introduce a potential technique in this chapter to achieve the purpose.

### 4.1 Definition of essential covers

As a generalization of persistence modules defined over incidence categories, we refer to any covariant linear functor  $B \rightarrow \text{mod } \mathbb{k}$  as a  $B$ -module where  $B$  denotes a linear category.

We first introduce the following notion to consider matrices with entries morphisms in a linear category in a natural way.

**Definition 4.1.1.** (1) For each linear category  $B$ , a linear category  $\bigoplus B$ , called the *formal additive hull* of  $B$ , is defined as follows: The set of objects is given by

$$(\bigoplus B)_0 := \{(x_i)_{i \in [m]} = (x_1, \dots, x_m) \mid x_1, \dots, x_m \in B_0, m \geq 1\} \sqcup \{0\}.$$

Let  $x = (x_i)_{i \in [m]}$  and  $y = (y_j)_{j \in [n]}$  with  $m, n \geq 1$ . Then we set

$$\begin{aligned} (\bigoplus B)(x, y) &:= \{ [\alpha_{ji}]_{(j,i) \in [n] \times [m]} \mid \alpha_{ji} \in B(x_i, y_j) \text{ for all } (j, i) \in [n] \times [m] \}, \\ (\bigoplus B)(x, 0) &:= \{J_{0,m}\}, (\bigoplus B)(0, y) := \{J_{n,0}\}, (\bigoplus B)(0, 0) := \{J_{0,0}\}, \end{aligned}$$

where  $J_{0,m}, J_{n,0}, J_{0,0}$  are the empty matrices of size  $(0, m), (n, 0), (0, 0)$ , respectively. The composition of nonempty matrices is given by the matrix multiplication. If morphisms  $\alpha, \beta$  have size  $(k, p), (q, l)$  with  $k, l, p, q \geq 0$ , respectively, then the composite  $\alpha\beta$  is defined if and only if the domain of  $\alpha$  coincides with the codomain of  $\beta$ , only if  $p = q$ , and in that case, the size of  $\alpha\beta$  is  $(k, l)$  as in the usual matrix multiplication. The composite of nonempty matrix and an empty matrix is an empty matrix, and that of empty matrix and empty matrix is empty matrix except for the case where  $J_{n,0}J_{0,m} = 0_{n,m}$ , the zero matrix of size  $(n, m)$  with  $m, n \geq 1$ . As easily seen,  $\bigoplus B$  is a pre-additive category, and we have

$$\begin{aligned} (x_i)_{i \in [m]} &\cong (x_1) \oplus \dots \oplus (x_m), \\ (x_i)_{i \in [m]} \oplus (y_j)_{j \in [n]} &\cong (x_1, \dots, x_n, y_1, \dots, y_n), \text{ and} \\ (x_1) \oplus \dots \oplus (x_m) &\cong (x_1 \oplus \dots \oplus x_m) \text{ if } x_1 \oplus \dots \oplus x_m \text{ exists in } B \end{aligned}$$

for all  $x_1, \dots, x_m, y_1, \dots, y_n \in B_0$ . Thus  $\bigoplus B$  turns out to be an additive category.

We regard  $B$  as a full subcategory of  $\bigoplus B$  by the embedding  $(f: x \rightarrow y) \mapsto ([f]: (x) \rightarrow (y))$  for all morphisms  $f$  in  $B$ . In the sequel, we will frequently consider the case where  $B = \mathbb{k}[\mathbf{S}]$  for a finite poset  $\mathbf{S}$ .

Note that if  $B$  is additive, then we have an equivalence  $\eta_B: \bigoplus B \rightarrow B$  that sends 0 to 0,  $(x_i)_{i \in [m]}$  to  $\bigoplus_{i \in [m]} x_i$ , empty matrices to zero morphisms, and each morphism

$$[\alpha_{ji}]_{(j,i) \in [n] \times [m]}: (x_i)_{i \in [m]} \rightarrow (y_j)_{j \in [n]}$$

in  $\bigoplus B$  to  $[\alpha_{ji}]_{(j,i) \in [n] \times [m]}: \bigoplus_{i \in [m]} x_i \rightarrow \bigoplus_{j \in [n]} y_j$  in  $B$ .

(2) Let  $F: B \rightarrow C$  be a linear functor between linear categories. Then a functor  $\bigoplus F: \bigoplus B \rightarrow \bigoplus C$  is defined as follows: We set  $(\bigoplus F)(0) := 0$  and  $(\bigoplus F)((x_i)_{i \in [m]}) := (F(x_i))_{i \in [m]}$  for each nonzero object  $(x_i)_{i \in [m]} \in (\bigoplus B)_0$ . For each morphism

$$\alpha := [\alpha_{ji}]_{(j,i) \in [n] \times [m]}: (x_i)_{i \in [m]} \rightarrow (y_j)_{j \in [n]},$$

we set

$$(\bigoplus F)(\alpha) := [F(\alpha_{ji})]_{(j,i) \in [n] \times [m]}: (F(x_i))_{i \in [m]} \rightarrow (F(y_j))_{j \in [n]}.$$

Finally  $F(\mathbf{J}) := \mathbf{J}$  for all  $\mathbf{J} \in \{\mathbf{J}_{n,0}, \mathbf{J}_{0,m} \mid m, n \geq 0\}$ . For example,  $\mathbf{J}_{0,m}: (x_i)_{i \in [m]} \rightarrow 0$  is sent to  $\mathbf{J}_{0,m}: (F(x_i))_{i \in [m]} \rightarrow 0$ . If there is no confusion, we denote  $\bigoplus F$  simply by  $F$ .

**Example 4.1.2.** Let  $\zeta: \mathbf{Z} \rightarrow \mathbf{P}$  be an order-preserving map between posets. Then by (2.2.1) and (2.2.2) we have a linear functor  $\mathbb{k}[\zeta]: \mathbb{k}[\mathbf{Z}] \rightarrow \mathbb{k}[\mathbf{P}]$ , which yields a linear functor  $\bigoplus \mathbb{k}[\zeta]: \bigoplus \mathbb{k}[\mathbf{Z}] \rightarrow \bigoplus \mathbb{k}[\mathbf{P}]$ . If  $\alpha := [\alpha_{ji}]_{(j,i) \in [n] \times [m]}$  is a morphism in  $\bigoplus \mathbb{k}[\mathbf{Z}]$ , we denote  $(\bigoplus \mathbb{k}[\zeta])(\alpha)$  simply by  $\zeta(\alpha) = [\zeta(\alpha_{ji})]_{(j,i) \in [n] \times [m]}$ .

**Proposition 4.1.3.** *Let  $B$  be a linear category and  $\mathcal{C}$  an additive linear category. Then each linear functor  $F: B \rightarrow \mathcal{C}$  uniquely extends to a linear functor  $\hat{F}: \bigoplus B \rightarrow \mathcal{C}$ , which we denote by the same letter  $F$  if there seems to be no confusion.*

*Proof.* Define a linear functor  $\hat{F}: \bigoplus B \rightarrow \mathcal{C}$  as the composite  $\hat{F} := \eta_{\mathcal{C}} \circ (\bigoplus F)$ . Namely, for each morphism  $\alpha = [\alpha_{ji}]_{(j,i) \in [n] \times [m]}: (x_i)_{i \in [m]} \rightarrow (y_j)_{j \in [n]}$  in  $\bigoplus B$ , we set

$$\hat{F}(\alpha) := [F(\alpha_{ij})]_{j,i}: \bigoplus_{i \in [m]} F(x_i) \rightarrow \bigoplus_{j \in [n]} F(y_j).$$

It is easy to see that this is the unique extension of  $F$ . □

Since each finitely generated projective module over  $\mathbb{k}[\mathbf{P}]$  is isomorphic to a finite direct sum of representable functors  $P_x := \mathbb{k}[\mathbf{P}](x, -)$ ,  $(x \in \mathbf{P})$ , we have the following by applying the proposition above to the case where  $B = \mathbb{k}[\mathbf{P}] = A$ .

**Corollary 4.1.4.** *The Yoneda embedding  $Y^A: A^{\text{op}} \rightarrow \text{prj } A$ ,  $x \mapsto P_x := A(x, -)$  extends to an equivalence  $Y^A: (\bigoplus A)^{\text{op}} \rightarrow \text{prj } A$ ,  $(x_i)_{i \in [m]} \mapsto \bigoplus_{i \in [m]} P_{x_i}$ . Note that  $Y^A$  maps each morphism  $p_{y,x}: x \rightarrow y$  in  $\mathbf{P}$  to  $\mathbf{P}_{y,x}: P_y \rightarrow P_x$ . Therefore, it maps each morphism*

$$[p_{y_j, x_i}]_{(j,i) \in [n] \times [m]}: (x_i)_{i \in [m]} \rightarrow (y_j)_{j \in [n]}$$

in  $\bigoplus A$  to the morphism

$$[\mathbf{P}_{y_j, x_i}]_{(i,j) \in [m] \times [n]} = {}^t[\mathbf{P}_{y_j, x_i}]_{(j,i) \in [n] \times [m]} : \bigoplus_{j \in [n]} P_{y_j} \rightarrow \bigoplus_{i \in [m]} P_{x_i}$$

in  $\text{prj } A$ .

Similarly, the Yoneda embedding  $Y_A: A \rightarrow \text{prj}(A^{\text{op}})$ ,  $x \mapsto P'_x := A(-, x)$  extends to an equivalence  $Y_A: \bigoplus A \rightarrow \text{prj } A^{\text{op}}$ ,  $(x_i)_{i \in [m]} \mapsto \bigoplus_{i \in [m]} P'_{x_i}$ .

**Definition 4.1.5.** Let  $\mathbf{I}$  be an interval of  $\mathbf{P}$ . Choose any choice maps  $\mathbf{c}: \text{sc}(\uparrow \mathbf{I}) \rightarrow \text{sc}(\mathbf{I})$  and  $\mathbf{d}: \text{sk}(\downarrow \mathbf{I}) \rightarrow \text{sk}(\mathbf{I})$ , and set  $\varepsilon_1 := \varepsilon_1(\mathbf{c})$ ,  $\pi_1 := \pi_1(\mathbf{d})$  as in Propositions 3.1.17 and 3.1.22. Choose also any  $(b, a) \in \text{sk}(\mathbf{I}) \times \text{sc}(\mathbf{I})$  such that  $b \geq a$ , and set  $\lambda := \lambda(b, a)$  as in Proposition 3.1.23. Then by Corollary 4.1.4, there exists a unique morphism  $\mathbf{g}_1$  (resp.  $\mathbf{g}_2$ ,  $\mathbf{g}_3$ ) in  $\bigoplus \mathbb{k}[\mathbf{P}]$  such that  $Y^A(\mathbf{g}_1) = {}^t\varepsilon_1$  (resp.  $Y^A(\mathbf{g}_2) = {}^t\pi_1$ ,  $Y^A(\mathbf{g}_3) = {}^t\lambda$ ). We set

$$\mathbf{g}(\mathbf{c}, \mathbf{d}, (b, a)) := \left[ \begin{array}{c|c} \mathbf{g}_1 & \mathbf{0} \\ \hline \mathbf{g}_3 & \mathbf{g}_2 \end{array} \right].$$

The following are the explicit forms of  $\mathbf{g}_1$ ,  $\mathbf{g}_2$ ,  $\mathbf{g}_3$ :

$$\mathbf{g}_1 := \left[ \begin{array}{c} [\tilde{p}_{a, \mathbf{a}_c}]_{(\mathbf{a}_c, a) \in \text{sc}_1(\mathbf{I}) \times \text{sc}(\mathbf{I})} \\ [\delta_{a, \mathbf{c}(a')} p_{a', \mathbf{c}(a')}]_{(a', a) \in \text{sc}(\uparrow \mathbf{I}) \times \text{sc}(\mathbf{I})} \end{array} \right] \text{ with the entries given by}$$

$$\tilde{p}_{a, \mathbf{a}_c} := \begin{cases} p_{c, a}, & \text{if } a = \underline{\mathbf{a}}, \\ -p_{c, a}, & \text{if } a = \overline{\mathbf{a}}, \\ \mathbf{0}, & \text{if } a \notin \mathbf{a}, \end{cases}$$

for all  $\mathbf{a}_c \in \text{sc}_1(\mathbf{I})$  and  $a \in \text{sc}(\mathbf{I})$ ; and

$$\mathbf{g}_2 := \left[ [\delta_{b, \mathbf{d}(b')} p_{\mathbf{d}(b'), b'}]_{(b, b') \in \text{sk}(\mathbf{I}) \times \text{sk}(\downarrow \mathbf{I})}, [\hat{p}_{b, \mathbf{b}_d}]_{(\mathbf{b}_d, b) \in \text{sk}(\mathbf{I}) \times \text{sk}_1(\mathbf{I})} \right],$$

with the entries given by

$$\hat{p}_{b, \mathbf{b}_d} := \begin{cases} p_{b, d}, & \text{if } b = \underline{\mathbf{b}}, \\ -p_{b, d}, & \text{if } b = \overline{\mathbf{b}}, \\ \mathbf{0}, & \text{if } b \notin \mathbf{b}, \end{cases}$$

for all  $b \in \text{sk}(\mathbf{I})$  and  $\mathbf{b}_d \in \text{sk}_1(\mathbf{I})$ ; and  $\mathbf{g}_3$  is the block matrix with the size  $\text{sk}(\mathbf{I}) \times \text{sc}(\mathbf{I})$ , the  $(b, a)$ -entry of  $\mathbf{g}_3$ , given by  $p_{b, a}$ , is the only nonzero entry.

**Notation 4.1.6.** Let  $B$  be a linear category,  $W$  a  $B$ -module, and  $m, n$  positive integers, and consider a morphism  $\mathbf{g} = [g_{ji}]_{(j,i) \in [n] \times [m]} : (x_i)_{i \in [m]} \rightarrow (y_j)_{j \in [n]}$  in  $\bigoplus B$ . Then by applying the convention in Proposition 4.1.3 in the case where  $\mathcal{C} = \text{mod } \mathbb{k}$ , we write

$$W(\mathbf{g}) := \hat{W}(\mathbf{g}) = [W(g_{ij})]_{j,i} : \bigoplus_{i \in [m]} W(x_i) \rightarrow \bigoplus_{j \in [n]} W(y_j).$$

By Definition 4.1.5 and Notation 4.1.6, Theorem 3.1.26 can be restated as follows.

**Theorem 4.1.7.** Let  $M \in \text{mod } A$  and  $\mathbf{I}$  an interval of  $\mathbf{P}$ . Choose any choice maps  $\mathbf{c}: \text{sc}(\uparrow \mathbf{I}) \rightarrow \text{sc}(\mathbf{I})$  and  $\mathbf{d}: \text{sk}(\downarrow \mathbf{I}) \rightarrow \text{sk}(\mathbf{I})$ , and any  $(b, a) \in \text{sk}(\mathbf{I}) \times \text{sc}(\mathbf{I})$  with  $b \geq a$ . Set  $\mathbf{g} := \mathbf{g}(\mathbf{c}, \mathbf{d}, (b, a))$  as in Definition 4.1.5. Then

$$d_M(V_{\mathbf{I}}) = \text{rank } M(\mathbf{g}) - \text{rank } M(\mathbf{g}_1) - \text{rank } M(\mathbf{g}_2). \quad (4.1.1)$$

Let  $\zeta: \mathbf{Z} \rightarrow \mathbf{P}$  be an order-preserving map with  $\mathbf{Z}$  a (possibly infinite) poset. Recall the  $\mathbb{k}$ -linearization  $\mathbb{k}[\zeta]$  of  $\zeta$  given in (2.2.1) and (2.2.2), and the restriction functor  $R_\zeta$  induced by  $\zeta$  in Definition 2.5.5. For brevity we write  $R$  instead of  $R_\zeta$  if there seems to be no confusion.

**Definition 4.1.8.** Let  $B$  be a linear category, and  $1 \leq m, n \in \mathbb{Z}$ . For each  $(j, i) \in [n] \times [m]$ , let  $\alpha_{ji}: x_{ji} \rightarrow y_{ji}$  be a morphism in  $B$ . Then the family  $(\alpha_{ji})_{(j,i) \in [n] \times [m]}$  is said to satisfy the *matrix condition* if  $[\alpha_{ji}]_{(j,i) \in [n] \times [m]}$  is a morphism in the category  $\bigoplus B$ , namely, if for any  $i, p \in [m]$  and  $j, q \in [n]$ , we have  $x_{ji} = x_{qp}$  if  $i = p$  and  $y_{ji} = y_{qp}$  if  $j = q$ . If this is the case, then by setting  $x_i := x_{1,i}$  and  $y_j := y_{j,1}$ , we have  $\alpha_{ji} \in B(x_i, y_j)$  for all  $(j, i) \in [n] \times [m]$ , and  $\alpha := [\alpha_{ji}]_{(j,i) \in [n] \times [m]}: (x_i)_{i \in [m]} \rightarrow (y_j)_{j \in [n]}$  becomes a morphism in  $\bigoplus B$ .

**Lemma 4.1.9.** Let  $\zeta: \mathbf{Z} \rightarrow \mathbf{P}$  be an order-preserving map with  $\mathbf{Z}$  a poset, and take a morphism  $\alpha := [\alpha_{ji}]_{(j,i) \in [n] \times [m]}: (x_i)_{i \in [m]} \rightarrow (y_j)_{j \in [n]}$  in  $\bigoplus \mathbb{k}[\mathbf{P}]$ . Then the following are equivalent:

- (1) There exists a morphism  $\alpha': x' \rightarrow y'$  in  $\bigoplus \mathbb{k}[\mathbf{Z}]$  such that  $\zeta(\alpha') = \alpha$  (see Example 4.1.2 for  $\zeta(\alpha')$ ).
- (2) There exist maps  $\zeta': \{x_i \mid i \in [m]\} \rightarrow \mathbf{Z}$  and  $\zeta'': \{y_j \mid j \in [n]\} \rightarrow \mathbf{Z}$  with  $\zeta\zeta' = \mathbb{1}$  and  $\zeta\zeta'' = \mathbb{1}$  (i.e., these are sections of  $\zeta$ ) such that for any nonzero entry  $\alpha_{ji}: x_i \rightarrow y_j$  of  $\alpha$ , there exists some  $\alpha'_{ji}: \zeta'(x_i) \rightarrow \zeta''(y_j)$  with  $\zeta(\alpha'_{ji}) = \alpha_{ji}$ .

*Proof.* (1)  $\Rightarrow$  (2). Assume (1). If the morphism  $\alpha'$  has the form  $\alpha': (x'_i)_{i \in [m]} \rightarrow (y'_j)_{j \in [n]}$ , then the map  $x_i \mapsto x'_i$  (resp.  $y_j \mapsto y'_j$ ) defines the desired section  $\zeta'$  of  $\zeta$  (resp.  $\zeta''$  of  $\zeta$ ).

(2)  $\Rightarrow$  (1). Assume (2). Then for any  $(j, i) \in [n] \times [m]$ , there exists a morphism  $\alpha'_{ji}: \zeta'(x_i) \rightarrow \zeta''(y_j)$  such that  $\zeta(\alpha'_{ji}) = \alpha_{ji}$ . Then the family  $(\alpha'_{ji})_{(j,i) \in [n] \times [m]}$  satisfies the matrix condition, and hence  $\alpha' := [\alpha'_{ji}]_{(j,i) \in [n] \times [m]}: (\zeta'(x_i))_{i \in [m]} \rightarrow (\zeta''(y_j))_{j \in [n]}$  is a morphism in  $\bigoplus \mathbb{k}[\mathbf{Z}]$ , and satisfies  $\zeta(\alpha') = \alpha$ .  $\square$

**Definition 4.1.10.** Let  $\zeta: \mathbf{Z} \rightarrow \mathbf{P}$  be an order-preserving map with  $\mathbf{Z}$  a poset, and  $\alpha: x \rightarrow y$  a morphism in  $\bigoplus \mathbb{k}[\mathbf{P}]$ . We say that  $\zeta$  *covers*  $\alpha$  if one of the conditions in Lemma 4.1.9 is satisfied. Note that if this is the case, then  $\zeta$  covers all submatrices of  $\alpha$ .

**Definition 4.1.11.** Let  $\zeta: \mathbf{Z} \rightarrow \mathbf{P}$  be an order-preserving map with  $\mathbf{Z}$  a poset, and  $\mathbf{I}$  an interval of  $\mathbf{P}$ . Then we say that  $\zeta$  *essentially covers*  $\mathbf{I}$  if there exists some morphism  $\mathbf{g} := \begin{bmatrix} \mathbf{g}_1 & \mathbf{0} \\ \mathbf{g}_3 & \mathbf{g}_2 \end{bmatrix}$  in  $\bigoplus \mathbb{k}[\mathbf{P}]$  such that for any  $M \in \text{mod } A$  we have a formula

$$d_M(V_{\mathbf{I}}) = \text{rank} \begin{bmatrix} M(\mathbf{g}_1) & \mathbf{0} \\ M(\mathbf{g}_3) & M(\mathbf{g}_2) \end{bmatrix} - \text{rank} \begin{bmatrix} M(\mathbf{g}_1) & \mathbf{0} \\ \mathbf{0} & M(\mathbf{g}_2) \end{bmatrix}, \quad (4.1.2)$$

and  $\zeta$  covers  $\mathbf{g}$ .

**Remark 4.1.12.** (1) We remark here that Theorem 4.1.7 guarantees that such a morphism  $\mathbf{g}$  in  $\bigoplus \mathbb{k}[\mathbf{P}]$  can be always taken such that the first condition (4.1.2) is always satisfied. We also caution the reader that  $\mathbf{g}$  is not unique in general, for example, due to the choice maps  $\mathbf{c}: \text{sc}(\uparrow \mathbf{I}) \rightarrow \text{sc}(\mathbf{I})$ ,  $\mathbf{d}: \text{sk}(\downarrow \mathbf{I}) \rightarrow \text{sk}(\mathbf{I})$ , and a choice of pair  $(b, a) \in \text{sk}(\mathbf{I}) \times \text{sc}(\mathbf{I})$  with  $b \geq a$ .

- (2) In Definition 4.1.11, we allow the cases where any matrices among  $\mathbf{g}_1, \mathbf{g}_2$  are  $\mathbf{g}_3$  empty matrices. We also note the reader that the formula (4.1.2) is a natural generalization of the well-known rank formula appearing in the one-parameter persistence case. In more detail, suppose  $\mathbf{P} := \mathbb{A}_n$  and the interval  $\mathbf{I} := [s, t]$  ( $s, t \in [n]$ ). If we set  $\mathbf{g}_1 := [p_{t+1, s}]$ ,  $\mathbf{g}_2 := [p_{t, s-1}]$ , and  $\mathbf{g}_3 := [p_{t, s}]$ , then for any  $M \in \text{mod } \mathbb{k}[\mathbf{P}]$  the right-hand side of (4.1.2) yields

$$\begin{aligned}
 & \text{rank} \left[ \begin{array}{c|c} M_{t+1, s} & \mathbf{0} \\ \hline M_{t, s} & M_{t, s-1} \end{array} \right] - \text{rank } M_{t+1, s} - \text{rank } M_{t, s-1} \\
 \stackrel{(a)}{=} & \text{rank} \left[ \begin{array}{c|c} \mathbf{0} & -M_{t+1, s-1} \\ \hline M_{t, s} & M_{t, s-1} \end{array} \right] - \text{rank } M_{t+1, s} - \text{rank } M_{t, s-1} \\
 \stackrel{(b)}{=} & \text{rank} \left[ \begin{array}{c|c} \mathbf{0} & -M_{t+1, s-1} \\ \hline M_{t, s} & \mathbf{0} \end{array} \right] - \text{rank } M_{t+1, s} - \text{rank } M_{t, s-1} \\
 = & \text{rank } M_{t, s} + \text{rank } M_{t+1, s-1} - \text{rank } M_{t+1, s} - \text{rank } M_{t, s-1} \\
 = & d_M(V_{\mathbf{I}}), \tag{4.1.3}
 \end{aligned}$$

where the equality  $\stackrel{(a)}{=}$  follows by the elementary operations of adding the second row block multiplied by  $-M_{t+1, t}$  from the left to the first row block, and the equality  $\stackrel{(b)}{=}$  follows by the elementary transformation of adding the first column block multiplied by  $-M_{s, s-1}$  to the second column. The last equality is known as the formula of the persistent Betti numbers and the multiplicity in persistent homology (see [35, Chapter VII]).

**Lemma 4.1.13.** *Let  $B$  be a linear category,  $W$  a  $B$ -module, and  $\mathbf{g}$  a morphism in  $\bigoplus B$ . Assume that we have a direct sum decomposition  $W \cong W_1 \oplus W_2$  of  $B$ -modules. Then we have an equivalence  $W(\mathbf{g}) \cong W_1(\mathbf{g}) \oplus W_2(\mathbf{g})$  of linear maps. In particular, the equality*

$$\text{rank } W(\mathbf{g}) = \text{rank } W_1(\mathbf{g}) + \text{rank } W_2(\mathbf{g})$$

*holds.*

*Proof.* Let  $f: W \rightarrow W_1 \oplus W_2$  be an isomorphism of  $B$ -modules. Then for any  $i \in [m], j \in [n]$ , we have a commutative diagram

$$\begin{array}{ccc}
 W(x_i) & \xrightarrow{W(g_{ji})} & W(y_j) \\
 f_{x_i} \downarrow & & \downarrow f_{y_j} \\
 W_1(x_i) \oplus W_2(x_i) & \xrightarrow{W_1(g_{ji}) \oplus W_2(g_{ji})} & W_1(y_j) \oplus W_2(y_j)
 \end{array}$$

with both  $f_{x_i}$  and  $f_{y_j}$  isomorphisms. This yields the commutative diagram

$$\begin{array}{ccc}
\bigoplus_{i \in [m]} W(x_i) & \xrightarrow[\quad = W(\mathbf{g}) \quad]{[W(g_{ji})]_{j,i}} & \bigoplus_{j \in [n]} W(y_j) \\
\bigoplus_{i \in [m]} f_{x_i} \downarrow & & \downarrow \bigoplus_{j \in [n]} f_{y_j} \\
\bigoplus_{i \in [m]} (W_1(x_i) \oplus W_2(x_i)) & \xrightarrow{[W_1(g_{ji}) \oplus W_2(g_{ji})]_{j,i}} & \bigoplus_{j \in [n]} (W_1(y_j) \oplus W_2(y_j)) \\
\sigma_x \downarrow & & \downarrow \sigma_y \\
(\bigoplus_{i \in [m]} W_1(x_i)) \oplus (\bigoplus_{i \in [m]} W_2(x_i)) & \xrightarrow[\quad = W_1(\mathbf{g}) \oplus W_2(\mathbf{g}) \quad]{[W_1(g_{ji})]_{j,i} \oplus [W_2(g_{ji})]_{j,i}} & (\bigoplus_{j \in [n]} W_1(y_j)) \oplus (\bigoplus_{j \in [n]} W_2(y_j))
\end{array},$$

where  $\sigma_x, \sigma_y$  are given by the permutation matrices corresponding to the permutation  $\sigma_k$  (for  $k = m, n$ , respectively) of the set  $[2k]$  defined by

$$\text{for all } i \in [2k], \quad \sigma_k(i) := \begin{cases} \ell, & \text{if } i = 2\ell - 1 \text{ for some } \ell \in [k], \\ k + \ell, & \text{if } i = 2\ell \text{ for some } \ell \in [k]. \end{cases}$$

The nonzero entries of which are the identity maps. Then since all vertical maps above are isomorphisms, the assertion holds.  $\square$

Before giving the main theorem, we need the following notation.

**Notation 4.1.14.** Let  $M \in \text{mod } \mathbb{k}[\mathbf{P}]$ . If  $M \cong L^n \oplus N$  with  $n \geq 0$  such that  $N$  has no direct summand isomorphic to  $L$ , then we set  $\bar{d}_M(L) := n$ . In particular, if  $L$  is indecomposable, then  $\bar{d}_M(L)$  coincides with  $d_M(L)$ . Moreover, by the Krull–Schmidt theorem, we easily see that if  $L = \bigoplus_{i \in [m]} L_i$  for some  $m \geq 1$  with each  $L_i$  indecomposable, then  $\bar{d}_M(L) = \min_{i \in [m]} d_M(L_i)$ .

**Theorem 4.1.15.** Let  $\zeta: \mathbf{Z} \rightarrow \mathbf{P}$  be an order-preserving map with  $\mathbf{Z}$  a poset, and  $R: \text{mod } \mathbb{k}[\mathbf{P}] \rightarrow \text{mod } \mathbb{k}[\mathbf{Z}]$  the restriction functor induced by  $\zeta$ . Take an interval  $\mathbf{I}$  in  $\mathbf{P}$ . If  $\zeta$  essentially covers  $\mathbf{I}$ , then for any  $M \in \text{mod } A$ , we have

$$d_M(V_{\mathbf{I}}) = \bar{d}_{R(M)}(R(V_{\mathbf{I}})).$$

*Proof.* Assume that  $\zeta$  essentially covers  $\mathbf{I}$ . Then there exists  $\mathbf{g} := \begin{bmatrix} \mathbf{g}_1 & \mathbf{0} \\ \mathbf{g}_3 & \mathbf{g}_2 \end{bmatrix}$  in  $\bigoplus \mathbb{k}[\mathbf{P}]$  such that for any  $M \in \text{mod } A$  we have a formula

$$d_M(V_{\mathbf{I}}) = \text{rank} \begin{bmatrix} M(\mathbf{g}_1) & \mathbf{0} \\ M(\mathbf{g}_3) & M(\mathbf{g}_2) \end{bmatrix} - \text{rank} \begin{bmatrix} M(\mathbf{g}_1) & \mathbf{0} \\ \mathbf{0} & M(\mathbf{g}_2) \end{bmatrix},$$

and  $\zeta$  covers  $\mathbf{g}$ , say  $\zeta(\mathbf{g}') = \mathbf{g}$  for some  $\mathbf{g}'$  in  $\bigoplus \mathbb{k}[\mathbf{Z}]$ . Let  $g_{vu}$  (resp.  $g'_{vu}$ ) be the  $(v, u)$ -entry of  $\mathbf{g}$  (resp.  $\mathbf{g}'$ ). Then

$$M(g_{vu}) = M(\zeta(g'_{vu})) = R(M)(g'_{vu}).$$

Thus we have  $M(\mathbf{g}_i) = R(M)(\mathbf{g}'_i)$  for all  $i = 1, 2, 3$ . Hence

$$d_M(V_{\mathbf{I}}) = \text{rank} \begin{bmatrix} R(M)(\mathbf{g}'_1) & \mathbf{0} \\ R(M)(\mathbf{g}'_3) & R(M)(\mathbf{g}'_2) \end{bmatrix} - \text{rank} \begin{bmatrix} R(M)(\mathbf{g}'_1) & \mathbf{0} \\ \mathbf{0} & R(M)(\mathbf{g}'_2) \end{bmatrix}. \quad (4.1.4)$$

Set here  $r := d_M(V_{\mathbf{I}})$ ,  $s := \bar{d}_{R(M)}(R(V_{\mathbf{I}}))$ . Then it is enough to show that  $r = s$ . By the former, we have  $M \cong V_{\mathbf{I}}^r \oplus N$  for some module  $N$  in  $\text{mod } \mathbb{k}[\mathbf{P}]$ , which shows that  $R(M) \cong R(V_{\mathbf{I}})^r \oplus R(N)$ . Hence we have  $r \leq s$ . On the other hand, by the latter we have an isomorphism  $R(M) \cong R(V_{\mathbf{I}})^s \oplus L$  for some module  $L$  in  $\text{mod } \mathbb{k}[\mathbf{Z}]$ . Then by Lemma 4.1.13, we have the following equalities:

$$\begin{aligned} \text{rank} \begin{bmatrix} R(M)(\mathbf{g}'_1) & \mathbf{0} \\ R(M)(\mathbf{g}'_3) & R(M)(\mathbf{g}'_2) \end{bmatrix} &= s \text{rank} \begin{bmatrix} R(V_{\mathbf{I}})(\mathbf{g}'_1) & \mathbf{0} \\ R(V_{\mathbf{I}})(\mathbf{g}'_3) & R(V_{\mathbf{I}})(\mathbf{g}'_2) \end{bmatrix} \\ &+ \text{rank} \begin{bmatrix} L(\mathbf{g}'_1) & \mathbf{0} \\ L(\mathbf{g}'_3) & L(\mathbf{g}'_2) \end{bmatrix}, \end{aligned} \quad (4.1.5)$$

and

$$\begin{aligned} \text{rank} \begin{bmatrix} R(M)(\mathbf{g}'_1) & \mathbf{0} \\ \mathbf{0} & R(M)(\mathbf{g}'_2) \end{bmatrix} &= s \text{rank} \begin{bmatrix} R(V_{\mathbf{I}})(\mathbf{g}'_1) & \mathbf{0} \\ \mathbf{0} & R(V_{\mathbf{I}})(\mathbf{g}'_2) \end{bmatrix} \\ &+ \text{rank} \begin{bmatrix} L(\mathbf{g}'_1) & \mathbf{0} \\ \mathbf{0} & L(\mathbf{g}'_2) \end{bmatrix}. \end{aligned} \quad (4.1.6)$$

Note that the formula (4.1.4) holds also for  $M = V_{\mathbf{I}}$ . Thus we have

$$d_{V_{\mathbf{I}}}(V_{\mathbf{I}}) = \text{rank} \begin{bmatrix} R(V_{\mathbf{I}})(\mathbf{g}'_1) & \mathbf{0} \\ R(V_{\mathbf{I}})(\mathbf{g}'_3) & R(V_{\mathbf{I}})(\mathbf{g}'_2) \end{bmatrix} - \text{rank} \begin{bmatrix} R(V_{\mathbf{I}})(\mathbf{g}'_1) & \mathbf{0} \\ \mathbf{0} & R(V_{\mathbf{I}})(\mathbf{g}'_2) \end{bmatrix}. \quad (4.1.7)$$

By the equalities (4.1.4), (4.1.5), (4.1.6), and (4.1.7), we see that

$$\begin{aligned} r = d_M(V_{\mathbf{I}}) &= s \cdot d_{V_{\mathbf{I}}}(V_{\mathbf{I}}) + \text{rank} \begin{bmatrix} L(\mathbf{g}'_1) & \mathbf{0} \\ L(\mathbf{g}'_3) & L(\mathbf{g}'_2) \end{bmatrix} - \text{rank} \begin{bmatrix} L(\mathbf{g}'_1) & \mathbf{0} \\ \mathbf{0} & L(\mathbf{g}'_2) \end{bmatrix} \\ &= s + \text{rank} \begin{bmatrix} L(\mathbf{g}'_1) & \mathbf{0} \\ L(\mathbf{g}'_3) & L(\mathbf{g}'_2) \end{bmatrix} - \text{rank} \begin{bmatrix} L(\mathbf{g}'_1) & \mathbf{0} \\ \mathbf{0} & L(\mathbf{g}'_2) \end{bmatrix} \geq s. \end{aligned}$$

Hence we have  $r = s$ , and the proof is completed.  $\square$

We provide an example to explain how to use the essential-cover technique to compute interval multiplicities from the filtration level.

**Example 4.1.16.** Let  $\mathbf{P} = \mathbf{G}_{5,2}$  (Example 2.2.10 (1)) and let  $\mathcal{F}$  be a  $\mathbf{P}$ -filtration shown in Figure 4.1(a). Set  $M := H_1(-; \mathbb{Z}/2\mathbb{Z}) \circ \mathcal{F}$ . We compute the interval multiplicity of  $V_{\mathbf{I}}$  in  $M$  where

$$\mathbf{I} := \begin{array}{ccccccc} (1, 2) & \longrightarrow & (2, 2) & \longrightarrow & (3, 2) & & \\ & & \uparrow & & \uparrow & & \\ & & (2, 1) & \longrightarrow & (3, 1) & \longrightarrow & (4, 1) \longrightarrow (5, 1) \end{array}.$$

By Definition 4.1.11 the inclusion map  $\iota: \mathbf{Z} \hookrightarrow \mathbf{P}$  essentially covers  $\mathbf{I}$ , where the subposet  $\mathbf{Z}$  of  $\mathbf{P}$  is given by

$$\mathbf{Z} := \begin{array}{ccccccc} & & & \curvearrowright & & & \\ (1, 2) & \longrightarrow & (2, 2) & & (3, 2) & \longrightarrow & (4, 2) \\ & & \uparrow & \nearrow & \uparrow & & \\ (1, 1) & & (2, 1) & & (3, 1) & \longrightarrow & (5, 1) \\ & & & \curvearrowleft & & & \end{array}.$$

By Theorem 4.1.15 it suffices to consider the  $\mathbf{Z}$ -filtration  $\mathcal{F}' := \mathcal{F} \circ \iota$  shown in Figure 4.1(b), in which arrows represent inclusions. Noticing that  $\mathbf{Z}$  is a finite zigzag poset, the restriction  $R(M) = H_1(-; \mathbb{Z}/2\mathbb{Z}) \circ \mathcal{F}'$  is the zigzag persistent homology. Now, one only needs to decompose this zigzag persistent homology and count the multiplicity of the following interval

$$\mathbf{I}' := \begin{array}{ccccccc} & (1, 2) & \longrightarrow & (2, 2) & & (3, 2) & \\ & & & \uparrow & \nearrow & \uparrow & \\ & (2, 1) & & & & (3, 1) & \longrightarrow (5, 1) \end{array}$$

(the dimension vector of  $V_{\mathbf{I}'} \in \text{mod } \mathbb{k}[\mathbf{Z}]$  is  $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$ ) in the barcodes of  $R(M)$ . This kind of problem is studied in the filtration level (for instance, see [23, 29, 43]).

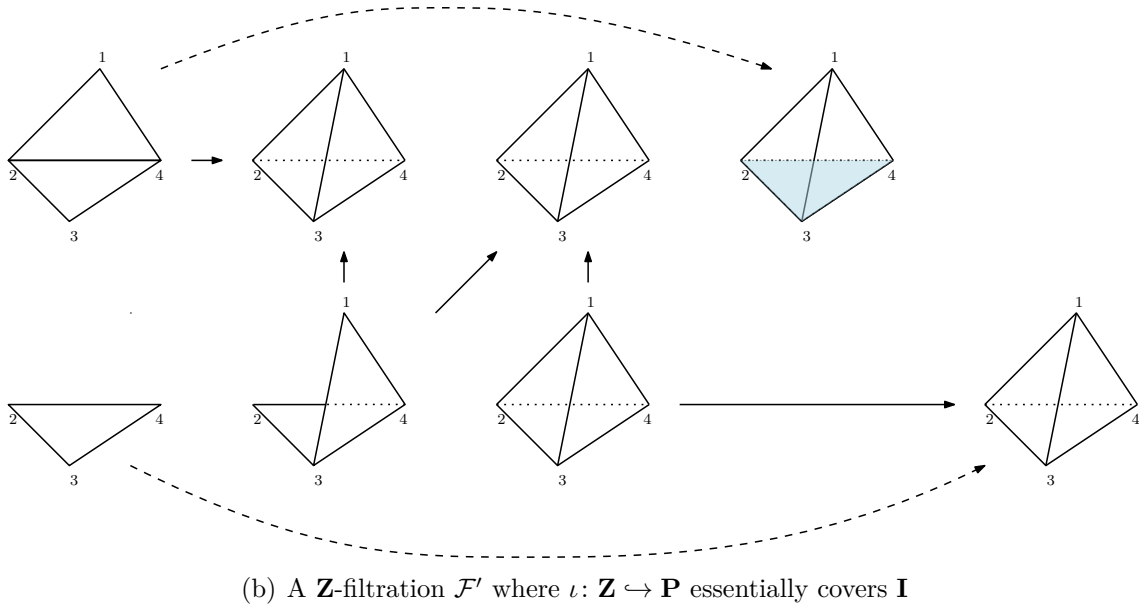
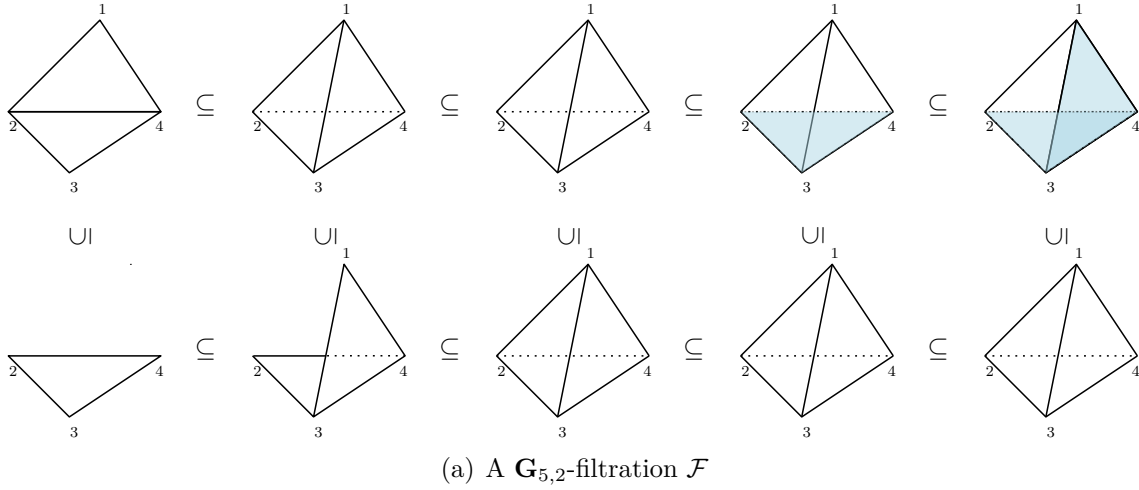


Figure 4.1: The illustration of computing interval multiplicities from the filtration level, as demonstrated in Example 4.1.16

**Remark 4.1.17.** In Example 4.1.16, we see that the multiplicity of  $\mathbf{I}'$  in the barcodes of  $R(M)$  is 0. Indeed, the decomposition of  $R(M)$  is given by

$$R(M) = \begin{bmatrix} 1101 \\ 0000 \end{bmatrix} \oplus \begin{bmatrix} 0010 \\ 0011 \end{bmatrix} \oplus \begin{bmatrix} 0001 \\ 0000 \end{bmatrix} \oplus \begin{bmatrix} 0110 \\ 0111 \end{bmatrix} \oplus \begin{bmatrix} 1110 \\ 1111 \end{bmatrix},$$

where all direct summands are interval modules presented by their dimension vectors.

The following example shows that in some case, the essential cover of intervals starts from a directed tree formed by connecting several zigzag posets.

**Example 4.1.18.** Let  $\mathbf{P} = \mathbf{G}_{6,2}$  and consider the following interval of  $\mathbf{P}$ :

$$\mathbf{I} := \begin{array}{ccccccc} (2, 2) & \longrightarrow & (3, 2) & \longrightarrow & (4, 2) & & \\ & & \uparrow & & \uparrow & & \\ & & (3, 1) & \longrightarrow & (4, 1) & \longrightarrow & (5, 1) \end{array}.$$

We compute the interval multiplicity of  $V_{\mathbf{I}}$ . For brevity we set  $a_1 := (3, 1)$ ,  $a_2 := (2, 2)$ ,  $b_1 := (5, 1)$ ,  $b_2 := (4, 2)$  by adopting Notation 3.2.3. Then  $a_{12} = a_1 \vee a_2 = (3, 2)$ ,  $b_{12} = b_1 \wedge b_2 = (4, 1)$ ,  $\text{sc}(\uparrow \mathbf{I}) = \{a'_1, a'_2\} = \{(6, 1), (5, 2)\}$ , and  $\text{sk}(\downarrow \mathbf{I}) = \{b'_1, b'_2\} = \{(2, 1), (1, 2)\}$ .

By Theorem 4.1.7, there exists a morphism  $\mathbf{g} = \begin{bmatrix} \mathbf{g}_1 & \mathbf{0} \\ \mathbf{g}_3 & \mathbf{g}_2 \end{bmatrix}$  in  $\oplus \mathbb{k}[\mathbf{P}]$  such that (4.1.2) holds. Here  $\mathbf{g}$  has the form:

$$\mathbf{g} := \left[ \begin{array}{c|c} \mathbf{g}_1 & \mathbf{0} \\ \hline \mathbf{g}_3 & \mathbf{g}_2 \end{array} \right] = \left[ \begin{array}{cc|ccc} p_{a_{12}, a_1} & -p_{a_{12}, a_2} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ p_{a'_1, a_1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & p_{a'_2, a_2} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & p_{b_1, b'_1} & \mathbf{0} & p_{b_1, b_{12}} \\ p_{b_2, a_1} & \mathbf{0} & \mathbf{0} & p_{b_2, b'_2} & -p_{b_2, b_{12}} \end{array} \right].$$

Hence if we take the (not full) subposet  $\mathbf{Z}$  of  $\mathbf{P}$  given by the right-hand side of Figure 4.2, then the inclusion map  $\iota: \mathbf{Z} \hookrightarrow \mathbf{P}$  essentially covers  $\mathbf{I}$ . We remark that in this example,  $\mathbf{Z}$  itself is not the zigzag poset, but the Hasse quiver of  $\mathbf{Z}$  is a directed tree formed by connecting two zigzag posets (shown in green and blue colors in Figure 4.2).

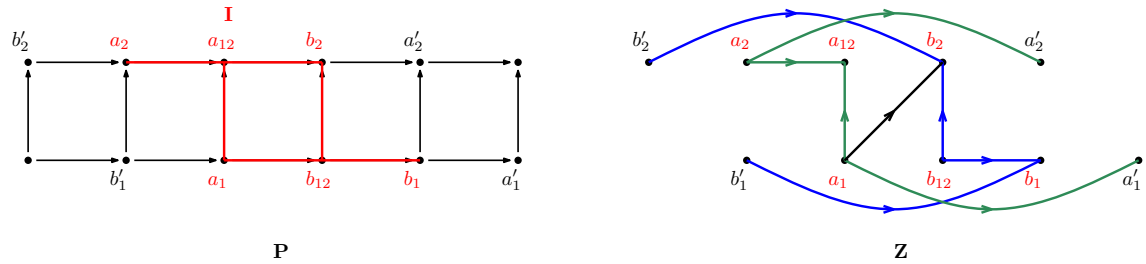


Figure 4.2: An illustration of the essential cover of  $\mathbf{I}$

## 4.2 Bipath persistence as zigzag persistence

Recently, Aoki et al. [3] provided a complete classification of posets whose module category only consists of interval-decomposable modules. They showed that every persistence module in  $\text{mod } \mathbb{k}[\mathbf{P}]$  is interval-decomposable if and only if  $\mathbf{P}$  is either a poset of Dynkin type A or a bipath poset. See Figure 4.3 for an illustration. The

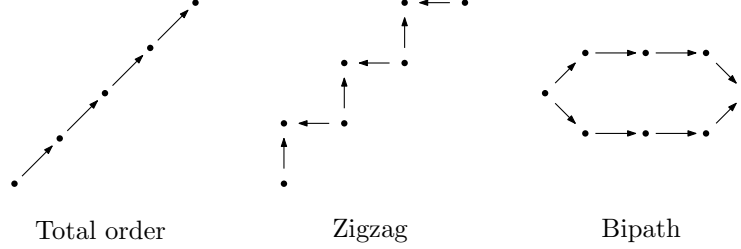
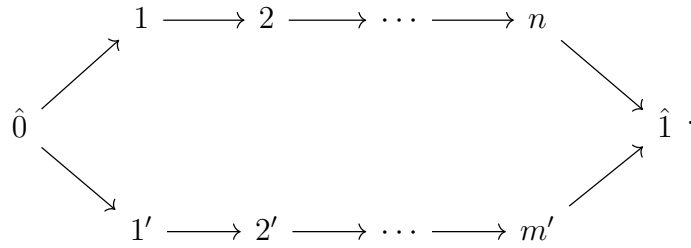


Figure 4.3: Posets that possess interval decomposability

former poset (involving totally ordered and zigzag posets) is well studied and applied in the one-parameter or zigzag persistent homology, while the latter is not commonly considered. They subsequently investigated the so-called bipath persistent homology in [2]. To obtain the visualization of the bipath persistence diagram, decomposing the bipath persistent homology in each dimension becomes the central task. In this section, we will apply our formula to compute the multiplicities of interval modules in the bipath poset setting, and inspired by the obtained formulas, we propose an alternate way of computing the bipath persistence diagram in the practical TDA pipeline, without obtaining the bipath persistent homology.

### 4.2.1 Fast decomposition of bipath persistence

To begin with, we review the definition of bipath poset. Let  $n, m \in \mathbb{Z}_{\geq 1}$ . Then the *bipath* poset  $\mathbf{B}_{n,m}$  is defined to be a poset having the following Hasse quiver:



We set  $[m]' := \{1', \dots, m'\}$ . The full subposet  $\mathbf{U} := \{\hat{0}, \hat{1}\} \sqcup [n]$  (resp.  $\mathbf{D} := \{\hat{0}, \hat{1}\} \sqcup [m]'$ ) is called the *upper* (resp. *lower*) *path* of  $\mathbf{B}_{n,m}$ . Throughout this subsection, we will let the ambient poset  $\mathbf{P}$  be  $\mathbf{B}_{n,m}$ . It is easy to check the interval  $\mathbf{I}$  in  $\mathbf{B}_{n,m}$  belongs to the following five types:

- (i)  $\mathbf{I} = \mathbf{B}_{n,m}$ .
- (ii)  $\mathbf{I} := [s, t] := \{x \in \mathbf{B}_{n,m} \mid s \leq x \leq t\}$  for some  $s, t \in [n]$ . We write  $\mathbb{I}_u$  to denote the set of all intervals having this type.

- (iii)  $\mathbf{I} := [s', t'] := \{x' \in \mathbf{B}_{n,m} | s' \leq x' \leq t'\}$  for some  $s', t' \in [m]'$ . We write  $\mathbb{I}_d$  to denote the set of all intervals having this type.
- (iv)  $\mathbf{I} := [\hat{0}, t] \cup [\hat{0}, t']$  for some  $t \in \mathbf{U} \setminus \{\hat{1}\}$  and  $t' \in \mathbf{D} \setminus \{\hat{1}\}$ . We write  $\mathbb{I}_l$  to denote the set of all intervals having this type.
- (v)  $\mathbf{I} := [s, \hat{1}] \cup [s', \hat{1}]$  for some  $s \in \mathbf{U} \setminus \{\hat{0}\}$  and  $s' \in \mathbf{D} \setminus \{\hat{0}\}$ . We write  $\mathbb{I}_r$  to denote the set of all intervals having this type.

$\mathbf{B}_{n,m}$ -persistence modules will be called the *bipath modules*. From now on, we provide the formula of  $d_M(V_{\mathbf{I}})$  case by case. Before doing so, we set the following conventions, which are used below:

$$\begin{aligned}
\hat{0} + 1 &:= 1, \quad n + 1 := \hat{1} \\
\hat{0} + 1' &:= 1', \quad t' + 1' := (t + 1)' \quad (t' \in [m - 1]'), \quad m' + 1' := \hat{1}, \\
1 - 1 &:= \hat{0}, \quad \hat{1} - 1 := n, \\
1' - 1' &:= \hat{0}, \quad s' - 1' := (s - 1)' \quad (s' \in [2', m']), \quad \hat{1} - 1' := m',
\end{aligned} \tag{4.2.8}$$

(i) Let  $\mathbf{I} = \mathbf{B}_{n,m}$ . Since  $V_{\mathbf{I}}$  is injective in  $\text{mod } \mathbb{k}[\mathbf{P}]$ , we apply (3.1.15) here and obtain

$$d_M(V_{\mathbf{I}}) = \text{rank } M_{\hat{1}, \hat{0}} \tag{4.2.9}$$

by noticing both  $\text{sc}_1(\mathbf{I})$  and  $\text{sc}(\uparrow \mathbf{I})$  are empty.

(ii) Let  $\mathbf{I} := [s, t] := \{x \in \mathbf{B}_{n,m} | s \leq x \leq t\}$  for some  $s, t \in [n]$ . Since  $V_{\mathbf{I}}$  is non-injective in  $\text{mod } \mathbb{k}[\mathbf{P}]$ , we apply (3.1.27) here. In this case,  $\text{sc}(\mathbf{I}) = \{s\}$ ,  $\text{sc}_1(\mathbf{I}) = \emptyset$ ,  $\text{sc}(\uparrow \mathbf{I}) = \{t + 1\}$ . On the other hand,  $\text{sk}(\mathbf{I}) = \{t\}$ ,  $\text{sk}_1(\mathbf{I}) = \emptyset$ ,  $\text{sk}(\downarrow \mathbf{I}) = \{s - 1\}$ . Then we obtain

$$d_M(V_{\mathbf{I}}) = \text{rank} \left[ \begin{array}{c|c} M_{t+1,s} & \mathbf{0} \\ \hline M_{t,s} & M_{t,s-1} \end{array} \right] - \text{rank } M_{t+1,s} - \text{rank } M_{t,s-1}. \tag{4.2.10}$$

(iii) Let  $\mathbf{I} := [s', t'] := \{x' \in \mathbf{B}_{n,m} | s' \leq x' \leq t'\}$  for some  $s', t' \in [m]'$ . This case is similar to the above case and thus we obtain

$$d_M(V_{\mathbf{I}}) = \text{rank} \left[ \begin{array}{c|c} M_{t'+1',s'} & \mathbf{0} \\ \hline M_{t',s'} & M_{t',s'-1'} \end{array} \right] - \text{rank } M_{t'+1',s'} - \text{rank } M_{t',s'-1'}. \tag{4.2.11}$$

(iv) Let  $\mathbf{I} := [\hat{0}, t] \cup [\hat{0}, t']$  for some  $t \in \mathbf{U} \setminus \{\hat{1}\}$  and  $t' \in \mathbf{D} \setminus \{\hat{1}\}$ .

**Case 1.** Suppose  $t = \hat{0}$  or  $t' = \hat{0}$ . Then  $V_{\mathbf{I}}$  is an injective module in  $\text{mod } \mathbb{k}[\mathbf{P}]$  with  $\max(\mathbf{I}) = \max\{t, t'\}$ . Hence we apply (3.1.15) here. In this case,  $\text{sc}(\mathbf{I}) = \{\hat{0}\}$ ,  $\text{sc}_1(\mathbf{I}) = \emptyset$ ,  $\text{sc}(\uparrow \mathbf{I}) = \{t + 1, t' + 1'\}$ . Then we obtain

$$d_M(V_{\mathbf{I}}) = \text{rank} \left[ \begin{array}{c|c} M_{t+1,\hat{0}} & \\ \hline M_{t'+1',\hat{0}} & \\ \hline M_{\max\{t,t'\},\hat{0}} & \end{array} \right] - \text{rank} \left[ \begin{array}{c} M_{t+1,\hat{0}} \\ M_{t'+1',\hat{0}} \end{array} \right]. \tag{4.2.12}$$

**Case 2.** Suppose  $t = n$  and  $t' = m$ . Then  $\text{sc}(\mathbf{I}) = \{\hat{0}\}$ ,  $\text{sc}_1(\mathbf{I}) = \emptyset$ ,  $\text{sc}(\uparrow \mathbf{I}) = \{\hat{1}\}$ . On the other hand,  $\text{sk}(\mathbf{I}) = \{t, t'\}$ ,  $\text{sk}_1(\mathbf{I}) = \{\hat{0}\}$ ,  $\text{sk}(\downarrow \mathbf{I}) = \emptyset$ . Then we obtain

$$d_M(V_{\mathbf{I}}) = \text{rank} \left[ \begin{array}{c|c} M_{\hat{1},\hat{0}} & \mathbf{0} \\ \hline M_{t,\hat{0}} & M_{t,\hat{0}} \\ \hline \mathbf{0} & -M_{t',\hat{0}} \end{array} \right] - \text{rank } M_{\hat{1},\hat{0}} - \text{rank} \left[ \begin{array}{c} M_{t,\hat{0}} \\ -M_{t',\hat{0}} \end{array} \right] \tag{4.2.13}$$

**Case 3.** Suppose  $t, t'$  are not in the above two cases. Then  $\text{sc}(\mathbf{I}) = \{\hat{0}\}$ ,  $\text{sc}_1(\mathbf{I}) = \emptyset$ ,  $\text{sc}(\uparrow\mathbf{I}) = \{t+1, t'+1\}$ . On the other hand,  $\text{sk}(\mathbf{I}) = \{t, t'\}$ ,  $\text{sk}_1(\mathbf{I}) = \{\hat{0}\}$ ,  $\text{sk}(\downarrow\mathbf{I}) = \emptyset$ . Then we obtain

$$d_M(V_{\mathbf{I}}) = \text{rank} \left[ \begin{array}{c|c} M_{t+1, \hat{0}} & \mathbf{0} \\ M_{t'+1', \hat{0}} & \mathbf{0} \\ \hline M_{t, \hat{0}} & M_{t, \hat{0}} \\ \mathbf{0} & -M_{t', \hat{0}} \end{array} \right] - \text{rank} \begin{bmatrix} M_{t+1, \hat{0}} \\ M_{t'+1', \hat{0}} \end{bmatrix} - \text{rank} \begin{bmatrix} M_{t, \hat{0}} \\ -M_{t', \hat{0}} \end{bmatrix}. \quad (4.2.14)$$

Notice that if we let  $t = n$  and  $t' = m$  in (4.2.14), then the result coincides with (4.2.13). Therefore, we can unify **Case 2** and **Case 3** and summarize the final result as follows:

**Case 1\*.** If  $t = \hat{0}$  or  $t' = \hat{0}$ , then we have

$$d_M(V_{\mathbf{I}}) = \text{rank} \left[ \begin{array}{c} M_{t+1, \hat{0}} \\ M_{t'+1', \hat{0}} \\ \hline M_{\max\{t, t'\}, \hat{0}} \end{array} \right] - \text{rank} \begin{bmatrix} M_{t+1, \hat{0}} \\ M_{t'+1', \hat{0}} \end{bmatrix}. \quad (4.2.15)$$

**Case 2\*.** If  $t \neq \hat{0}$  and  $t' \neq \hat{0}$ , then

$$d_M(V_{\mathbf{I}}) = \text{rank} \left[ \begin{array}{c|c} M_{t+1, \hat{0}} & \mathbf{0} \\ M_{t'+1', \hat{0}} & \mathbf{0} \\ \hline M_{t, \hat{0}} & M_{t, \hat{0}} \\ \mathbf{0} & -M_{t', \hat{0}} \end{array} \right] - \text{rank} \begin{bmatrix} M_{t+1, \hat{0}} \\ M_{t'+1', \hat{0}} \end{bmatrix} - \text{rank} \begin{bmatrix} M_{t, \hat{0}} \\ -M_{t', \hat{0}} \end{bmatrix}. \quad (4.2.16)$$

(v) Let  $\mathbf{I} := [s, \hat{1}] \cup [s', \hat{1}]$  for some  $s \in \mathbf{U} \setminus \{\hat{0}\}$  and  $s' \in \mathbf{D} \setminus \{\hat{0}\}$ . This case is just the dual of case (iv) above, and we analogously obtain the following.

**Case 1'.** If  $s = \hat{1}$  or  $s' = \hat{1}$ , then we have

$$d_M(V_{\mathbf{I}}) = \text{rank} [M_{\hat{1}, \min\{s, s'\}} \mid M_{\hat{1}, s-1} \quad M_{\hat{1}, s'-1'}] - \text{rank} [M_{\hat{1}, s-1} \quad M_{\hat{1}, s'-1'}]. \quad (4.2.17)$$

**Case 2'.** If  $s \neq \hat{1}$  and  $s' \neq \hat{1}$ , then we have

$$d_M(V_{\mathbf{I}}) = \text{rank} \left[ \begin{array}{cc|cc} M_{\hat{1}, s} & M_{\hat{1}, s'} & \mathbf{0} & \mathbf{0} \\ M_{\hat{1}, s} & \mathbf{0} & M_{\hat{1}, s-1} & M_{\hat{1}, s'-1'} \end{array} \right] - \text{rank} [M_{\hat{1}, s} \quad M_{\hat{1}, s'}] \\ - \text{rank} [M_{\hat{1}, s-1} \quad M_{\hat{1}, s'-1'}]. \quad (4.2.18)$$

The obtained formulas (4.2.9), (4.2.10), (4.2.11), (4.2.15), (4.2.16), (4.2.17), (4.2.18) suggest us to consider the essential covering of the bipath poset. As a rough description, it suffices to consider two special subposets of  $\mathbf{B}_{n,m}$  that are of Dynkin type  $\mathbb{A}$ , and decompose the restricted module in each respective module category. This strategy of decomposing the bipath persistence module can utilize the fast computation of zigzag persistence. Another remarkable advantage is that our strategy does not consider the basis changes at the global minimum and maximum elements, which is the key difference compared with the original decomposition method proposed by Aoki et al. [2].

Let  $\mathbf{Z}$  be a poset having the Hasse quiver

$$\begin{array}{ccccccc}
 \tilde{0} & \longrightarrow & 1 & \longrightarrow & 2 & \longrightarrow & \cdots & \longrightarrow & n \\
 & & & & & & & & \searrow \\
 \hat{0} & & & & & & & & \hat{1} \\
 & \searrow & & & & & & & \nearrow \\
 & & 1' & \longrightarrow & 2' & \longrightarrow & \cdots & \longrightarrow & m'
 \end{array} \quad (4.2.19)$$

and define the order-preserving map  $\zeta: \mathbf{Z} \rightarrow \mathbf{P}$  by  $\zeta(x) := x$  if  $x \in \mathbf{Z} \setminus \{\tilde{0}\}$ , and  $\zeta(\tilde{0}) := \hat{0}$ . Then we have the following.

**Proposition 4.2.1.** *Let  $\zeta$  be an order-preserving map defined above, and let  $R$  be the restriction functor induced by  $\zeta$ . Then for every interval  $\mathbf{I} \in \mathbb{I}_d \sqcup \mathbb{I}_u \sqcup \mathbb{I}_r \sqcup \{\mathbf{B}_{n,m}\}$  and every  $M \in \text{mod } \mathbb{k}[\mathbf{P}]$ , we have*

$$d_M(V_{\mathbf{I}}) = \bar{d}_{R(M)}(R(V_{\mathbf{I}})) = d_{R(M)}(R(V_{\mathbf{I}})). \quad (4.2.20)$$

*Proof.* We recall the notation  $\bar{d}$  as given in Notation 4.1.14. The second equality of (4.2.20) holds since  $R(V_{\mathbf{I}})$  is the indecomposable module in  $\text{mod } \mathbb{k}[\mathbf{Z}]$  for every interval  $\mathbf{I} \in \mathbb{I}_d \sqcup \mathbb{I}_u \sqcup \mathbb{I}_r \sqcup \{\mathbf{B}_{n,m}\}$ . It suffices to show that  $\zeta$  essentially covers every  $\mathbf{I} \in \mathbb{I}_d \sqcup \mathbb{I}_u \sqcup \mathbb{I}_r \sqcup \{\mathbf{B}_{n,m}\}$  by Theorem 4.1.15.

(i) Let  $\mathbf{I} = \mathbf{B}_{n,m}$ . Then it is obvious that there exists a morphism  $\mathbf{g}' := [p_{\hat{1},\hat{0}}]$  in  $\bigoplus \mathbb{k}[\mathbf{Z}]$  such that  $\zeta(\mathbf{g}') = [p_{\hat{1},\hat{0}}]$  in  $\bigoplus \mathbb{k}[\mathbf{P}]$ , and hence  $\zeta$  essentially covers  $\mathbf{B}_{n,m}$  by (4.2.9).

(ii) Let  $\mathbf{I} \in \mathbb{I}_d$ . This case is trivial by observing (4.2.11) and the definition of  $\zeta$ .

(iii) Let  $\mathbf{I} \in \mathbb{I}_u$ . Write  $\mathbf{I} := [s, t]$  for some  $s, t \in [n]$ . All cases are trivial except  $s = 1$ . If  $\mathbf{I} = [1, t]$ , then by (4.2.10) the morphism in  $\bigoplus \mathbb{k}[\mathbf{P}]$  can be taken as

$$\mathbf{g} := \left[ \begin{array}{c|c} p_{t+1,1} & \mathbf{0} \\ \hline p_{t,1} & p_{t,\hat{0}} \end{array} \right].$$

Let

$$\mathbf{g}' := \left[ \begin{array}{c|c} p_{t+1,1} & \mathbf{0} \\ \hline p_{t,1} & p_{t,\hat{0}} \end{array} \right].$$

Then  $\mathbf{g}'$  is the morphism in  $\bigoplus \mathbb{k}[\mathbf{Z}]$  satisfying  $\zeta(\mathbf{g}') = \mathbf{g}$ .

(iv) Let  $\mathbf{I} \in \mathbb{I}_r$ . This case is also trivial by observing (4.2.17), (4.2.18), and the definition of  $\zeta$ .

Therefore, the assertion follows.  $\square$

**Remark 4.2.2.** From this proposition, one can easily see that to compute four subdiagrams  $\mathcal{D}(\mathbb{I}_d)$ ,  $\mathcal{D}(\mathbb{I}_u)$ ,  $\mathcal{D}(\mathbb{I}_r)$  and  $\mathcal{D}(\mathbf{B}_{n,m})$  of the bipath persistence diagram (see the precise definition in [2]), it suffices to compute the persistence diagram of zigzag poset  $\mathbf{Z}$  and retrieve the corresponding intervals.

**Remark 4.2.3.** The above  $\zeta$  does not essentially cover the interval  $\mathbf{I} \in \mathbb{I}_l$ . To interpret this, let us take  $\mathbf{I} = [\hat{0}, 1]$  as an example. By (4.2.15) we may take

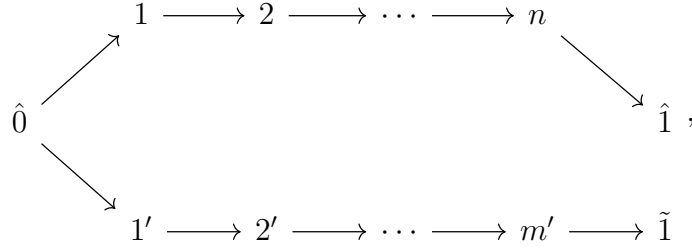
$$\mathbf{g} := \left[ \begin{array}{c} p_{2,\hat{0}} \\ \hline p_{1',\hat{0}} \\ \hline p_{1,\hat{0}} \end{array} \right],$$

and by the definition of  $\zeta$ ,  $\zeta(p_{2,\hat{0}}) = p_{2,\hat{0}}$ ,  $\zeta(p_{1',\hat{0}}) = p_{1',\hat{0}}$ ,  $\zeta(p_{1,\hat{0}}) = p_{1,\hat{0}}$ . However, we notice that the family

$$\begin{pmatrix} p_{2,\hat{0}} \\ \frac{p_{1',\hat{0}}}{p_{1,\hat{0}}} \end{pmatrix}$$

does not satisfy the matrix condition, and cannot be a morphism in  $\bigoplus \mathbb{k}[\mathbf{Z}]$ , and hence  $\zeta$  does not essentially cover  $\mathbf{I} = [\hat{0}, 1]$ .

For those intervals in  $\mathbb{I}_l$ , we shall consider another  $\mathbf{Z}'$  and  $\zeta'$ . Let  $\mathbf{Z}'$  be a poset having the Hasse quiver



and define the order-preserving map  $\zeta': \mathbf{Z}' \rightarrow \mathbf{P}$  by  $\zeta'(x) := x$  if  $x \in \mathbf{Z} \setminus \{\tilde{1}\}$ , and  $\zeta'(\tilde{1}) := \hat{1}$ . Then we have the following.

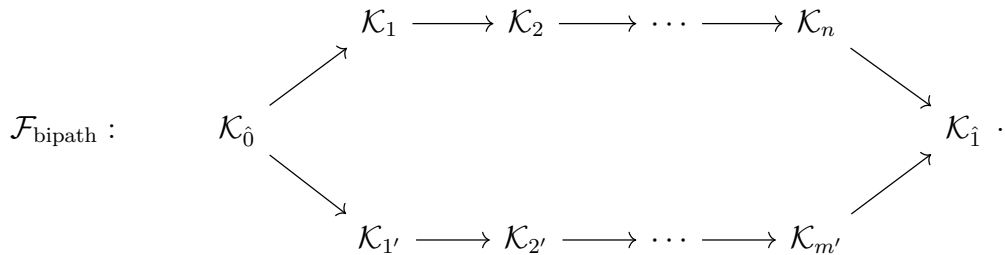
**Proposition 4.2.4.** *Let  $\zeta'$  be an order-preserving map defined above, and let  $R'$  be the restriction functor induced by  $\zeta'$ . Then for every interval  $\mathbf{I} \in \mathbb{I}_d \sqcup \mathbb{I}_u \sqcup \mathbb{I}_l \sqcup \{\mathbf{B}_{n,m}\}$  and every  $M \in \text{mod } \mathbb{k}[\mathbf{P}]$ , we have*

$$d_M(V_{\mathbf{I}}) = \bar{d}_{R'(M)}(R'(V_{\mathbf{I}})) = d_{R'(M)}(R'(V_{\mathbf{I}})). \quad (4.2.21)$$

*Proof.* It is similar to the proof of Proposition 4.2.1 and thus we leave this proof to the reader.  $\square$

**Remark 4.2.5.** From this proposition, one can easily see that to get the remaining sub-diagram  $\mathcal{D}(\mathbb{I}_l)$  of the bipath persistence diagram, it suffices to compute the persistence diagram of zigzag poset  $\mathbf{Z}'$  and retrieve the corresponding intervals.

**Summary of the procedure** To summarize, we propose our decomposition strategy in the following practical data analysis setting. Let  $\mathcal{F}_{\text{bipath}}: \mathbf{B}_{n,m} \rightarrow \mathbf{FinSimp}$  denote a bipath filtration, defined as follows.



We would like to obtain the  $q$ th bipath persistence diagram  $\mathcal{D}_q(\mathcal{F}_{\text{bipath}})$  of  $\mathcal{F}_{\text{bipath}}$ .

**Step 1.** Take the information of spaces and maps from  $\mathcal{F}_{\text{bipath}}$  and build the following zigzag filtration  $\mathcal{F}_{\triangleright}$ :

$$\mathcal{F}_{\triangleright} := \begin{array}{c} \mathcal{K}_0 \longrightarrow \mathcal{K}_1 \longrightarrow \mathcal{K}_2 \longrightarrow \cdots \longrightarrow \mathcal{K}_n \\ \searrow \\ \mathcal{K}_1 \\ \nearrow \\ \mathcal{K}_0 \longrightarrow \mathcal{K}_{1'} \longrightarrow \mathcal{K}_{2'} \longrightarrow \cdots \longrightarrow \mathcal{K}_{m'} \end{array} .$$

We can compute the zigzag persistence diagram  $\mathcal{D}_q(\mathcal{F}_{\triangleright})$  by utilizing the fast computation algorithm provided in [29], and obtain the sub-diagrams  $\mathcal{D}_q(\mathbb{I}_d)$ ,  $\mathcal{D}_q(\mathbb{I}_u)$ ,  $\mathcal{D}_q(\mathbb{I}_r)$ , and  $\mathcal{D}_q(\mathbf{B}_{n,m})$  based on Proposition 4.2.1.

**Step 2.** Take the information of spaces and maps from  $\mathcal{F}_{\text{bipath}}$  and build the following zigzag filtration  $\mathcal{F}_{\triangleleft}$ :

$$\mathcal{F}_{\triangleleft} := \begin{array}{c} \mathcal{K}_1 \longrightarrow \mathcal{K}_2 \longrightarrow \cdots \longrightarrow \mathcal{K}_n \longrightarrow \mathcal{K}_1 \\ \nearrow \\ \mathcal{K}_0 \\ \searrow \\ \mathcal{K}_{1'} \longrightarrow \mathcal{K}_{2'} \longrightarrow \cdots \longrightarrow \mathcal{K}_{m'} \longrightarrow \mathcal{K}_1 \end{array} .$$

Again, we compute the zigzag persistence diagram  $\mathcal{D}_q(\mathcal{F}_{\triangleleft})$  by utilizing the fast computation provided in [29], and obtain the remaining sub-diagram  $\mathcal{D}_q(\mathbb{I}_l)$  based on Proposition 4.2.4.

To obtain the stability result of the bipath persistence in the next subsection, we are motivated to find a single poset that essentially covers the bipath poset. Without loss of generality, we now assume the number of vertices in the lower path (i.e.,  $m$ ) is not greater than that in the upper path (i.e.,  $n$ ) of  $\mathbf{B}_{n,m}$ , in terms of the usual total order of  $\mathbb{Z}$ .

Let  $\mathbf{S}$  be a poset having the Hasse quiver:

$$\begin{array}{c} 1'' \longrightarrow 2'' \longrightarrow \cdots \longrightarrow m'' \\ \nearrow \\ \tilde{0} \\ \searrow \\ 1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n \\ \searrow \\ \hat{0} \\ \nearrow \\ 1' \longrightarrow 2' \longrightarrow \cdots \longrightarrow m' \end{array} , \quad (4.2.22)$$

where we set  $[m]'' := \{1'', \dots, m''\}$ . Now, we define the order-preserving map  $\zeta: \mathbf{S} \rightarrow \mathbf{P}$  by setting

$$\zeta(x) := \begin{cases} x, & \text{if } x \in [n] \sqcup [m]' \sqcup \{\hat{0}, \hat{1}\}, \\ \hat{0}, & \text{if } x = \tilde{0}, \\ i', & \text{if } x := i'' \in [m]''. \end{cases} \quad (4.2.23)$$

Then we have the following.

**Proposition 4.2.6.** *Let  $\zeta: \mathbf{S} \rightarrow \mathbf{P}$  be an order-preserving map defined above, and let  $R$  be the restriction functor induced by  $\zeta$ . Then for every interval  $\mathbf{I} \in \mathbb{I}(\mathbf{P})$  and every  $M \in \text{mod } \mathbb{k}[\mathbf{P}]$ , we have*

$$d_M(V_{\mathbf{I}}) = \bar{d}_{R(M)}(R(V_{\mathbf{I}})). \quad (4.2.24)$$

*Proof.* This is the direct consequence of Theorem 4.1.15. The verification is similar to the proof of Proposition 4.2.1 and thus we leave this proof to the reader.  $\square$

We remark here that in this case,  $\bar{d}$  appearing in (4.2.24) can not be changed to the usual symbol  $d$  for the multiplicity since  $R(V_{\mathbf{I}})$  may not be indecomposable in  $\text{mod } \mathbb{k}[\mathbf{S}]$ . For instance, considering an interval  $\mathbf{I} = [s', t']$  of  $\mathbf{P}$ .

## 4.2.2 Algebraic stability of bipath persistence

In this subsection, we will define an order-preserving map from an infinite zigzag to the bipath, which we call the *covering map* and is inspired by the universal covering of the circle. See Figure 4.4 for an illustration.

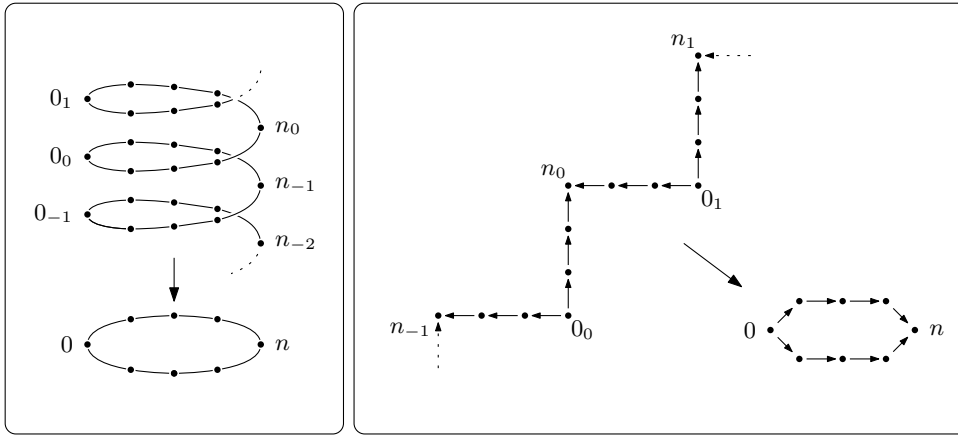
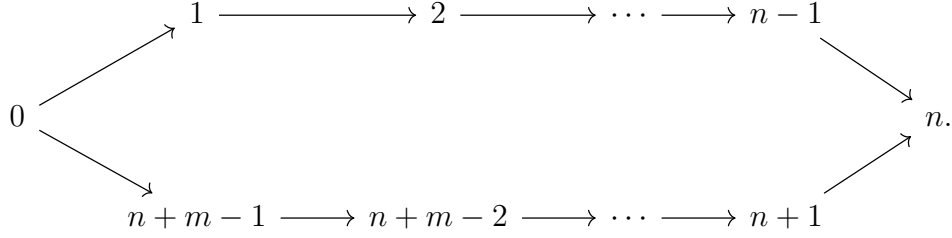


Figure 4.4: Illustration of the covering map

We translate elements of the theory of zigzag persistence to bipath persistence using this covering map. We use the restriction functor induced by the covering map to obtain an infinite zigzag persistence module  $R(M)$  from a bipath module  $M$ . Such a zigzag module  $R(M)$  has an infinite but periodic barcode, and the barcodes of  $R(M)$  and  $M$  determine each other, in the sense of Thanks to this translation between the bipath and zigzag, we are able to establish algebraic stability for bipath persistence. This is a consequence of the algebraic stability of zigzag modules,

which is, in turn, a consequence of the algebraic stability of block-decomposable modules [10, 19]. Specifically, we define the interleaving and bottleneck distance between bipath modules  $M$  and  $N$  to be the interleaving and bottleneck distance between  $R(M)$  and  $R(N)$ , as zigzag modules—much like one does for zigzag modules and block-decomposable modules [10, 19].

In what follows, we work on a fixed bipath poset  $\mathbf{B}$ , and for the sake of describing the covering map conveniently we fix  $n, m \in \mathbb{N}_{\geq 1}$  and set  $\mathbf{B} := \{0, \dots, n+m-1\}$  with the order given by two maximal chains  $0 < 1 < \dots < n$  and  $0 < n+m-1 < n+m-2 < \dots < n$ :



Under this labeling, we reformulate intervals of  $\mathbf{B}$  as follows (see Figure 4.5 for an illustration).

- (1) The *full interval*, which is the poset  $\mathbf{B}$  itself,
- (2) The *left interval* for some  $i \in \{0, 1, \dots, n-1\}$  and  $j \in \{0, n+m-1, \dots, n+1\}$ , such that

$$[i, j]_{\triangleleft} := \{z \in \mathbf{B} \mid 0 \leq z \leq i\} \cup \{z \in \mathbf{B} \mid 0 \leq z \leq j\}.$$

- (3) The *right interval* for some  $i \in \{1, \dots, n-1, n\}$  and  $j \in \{n+m-1, \dots, n+1, n\}$ , such that

$$[i, j]_{\triangleright} := \{z \in \mathbf{B} \mid i \leq z \leq n\} \cup \{z \in \mathbf{B} \mid j \leq z \leq n\}.$$

- (4) The *upper interval* for some  $i, j \in \{1, \dots, n-1\}$  such that

$$[i, j]_{\top} := \{z \in \mathbf{B} \mid i \leq z \leq j\}.$$

- (5) The *lower interval* for some  $i, j \in \{n+m-1, \dots, n+1\}$  such that

$$[i, j]_{\perp} := \{z \in \mathbf{B} \mid j \leq z \leq i\}.$$

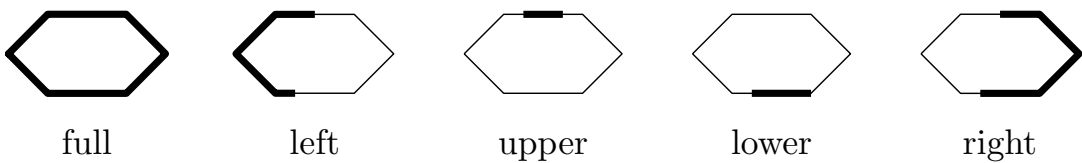


Figure 4.5: Types of intervals of the bipath poset

Since every bipath module is interval-decomposable, and by the Krull-Schmidt theorem, there is a uniquely defined multiset of intervals of  $\mathbf{B}$  for every bipath module  $M$ , denoted by  $\mathcal{B}(M)$ , such that

$$M \cong \bigoplus_{\mathbf{I} \in \mathcal{B}(M)} V_{\mathbf{I}}.$$

In what follows, instead of the *barcode*, we would call  $\mathcal{B}(M)$  the *arc code* of the bipath module  $M$ . See Figure 4.6 for an illustration.

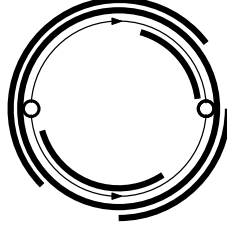


Figure 4.6: Illustration of an arc code

Considering the previously mentioned types of intervals, the arc code  $\mathcal{B}(M)$  can be written as the union

$$\mathcal{B}(M) = \mathcal{B}_{\text{full}}(M) \cup \mathcal{B}_{\triangleleft}(M) \cup \mathcal{B}_{\triangleright}(M) \cup \mathcal{B}_{\perp}(M) \cup \mathcal{B}_{\top}(M),$$

where  $\mathcal{B}_{\text{full}}(M)$  denotes the multiset of  $\mathcal{B}(M)$  consisting of full intervals,  $\mathcal{B}_{\triangleleft}(M)$  denotes the multiset consisting of left intervals, and analogously for the rest.

Now we are in the position to define the desired covering map. Instead of utilizing the finite zigzag poset  $\mathbf{S}$  provided at the end of Section 4.2.1, we will consider the standard infinite zigzag poset  $\mathbb{Z}\mathbb{Z}$  given in Example 2.2.11 (1). Recall that  $\mathbb{Z}\mathbb{Z}$  is the full subposet of  $\mathbb{Z}^{\text{op}} \times \mathbb{Z}$  given by  $\{(a, b) \in \mathbb{Z}^{\text{op}} \times \mathbb{Z} \mid a = b \text{ or } a = b + 1\}$ .

**Definition 4.2.7.** We define an order-preserving map  $\zeta: \mathbb{Z}\mathbb{Z} \rightarrow \mathbf{B}$  that we call the *covering map* as follows, where we set  $\mathcal{N} := n + m - 1$ ,

$$\begin{aligned} \zeta(k\mathcal{N} + 1, k\mathcal{N}) &= 0, \\ \zeta(k\mathcal{N} + i, k\mathcal{N} + i) &= \zeta(k\mathcal{N} + i + 1, k\mathcal{N} + i) = i, & \text{for } i \in [n - 1], \\ \zeta(k\mathcal{N} + n, k\mathcal{N} + n) &= n, \\ \zeta(k\mathcal{N} - i + 1, k\mathcal{N} - i) &= \zeta(k\mathcal{N} - i + 1, k\mathcal{N} - i + 1) = n + m - i, & \text{for } i \in [m - 1], \end{aligned}$$

for every  $k \in \mathbb{Z}$ . See Figure 4.7 for an illustration.

The restriction functor induced by the covering map  $\zeta: \mathbb{Z}\mathbb{Z} \rightarrow \mathbf{B}$  is simply denoted by  $R: \text{mod } \mathbb{k}[\mathbf{B}] \rightarrow \text{mod } \mathbb{k}[\mathbb{Z}\mathbb{Z}]$ . It is evident that the restriction functor  $R$  preserves finite direct sums.

Shortly, we describe the arc code of the bipath module  $M$  in terms of the barcode of  $R(M)$ . Indeed, persistence modules over  $\mathbb{Z}\mathbb{Z}$  have a barcode as mentioned in Chapter 2, that is, for every module  $N \in \text{mod } \mathbb{k}[\mathbb{Z}\mathbb{Z}]$  there is a unique multiset  $\mathcal{B}(N)$  of intervals such that  $N \cong \bigoplus_{\mathbf{I} \in \mathcal{B}(N)} V_{\mathbf{I}}$ , by a result of Botnan [17].

We describe the image under the restriction functor  $R$  of the different types of intervals of a bipath  $\mathbf{B}$  in the following lemma, illustrated in Figure 4.8.

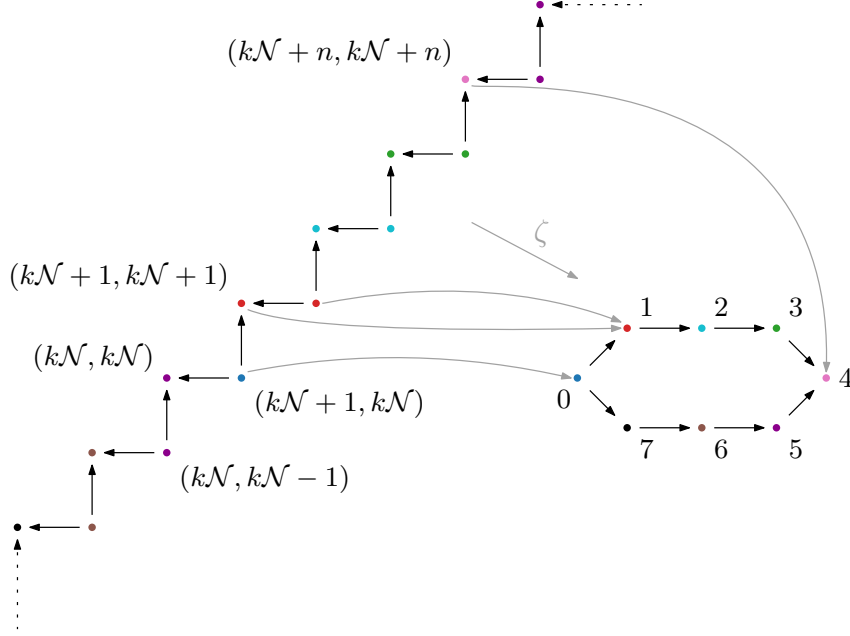


Figure 4.7: Illustration of the covering map  $\zeta: \mathbb{Z}\mathbb{Z} \rightarrow \mathbf{B}$ . In this case,  $n = m = 4$

**Lemma 4.2.8.** *For each type of interval module over the bipath poset  $\mathbf{B}$ , the following results hold.*

- For the full interval  $\mathbf{B}$ ,  $R(V_{\mathbf{B}}) = V_{\mathbb{Z}\mathbb{Z}}$ .
- For a left interval  $[i, j]_{\triangleleft}$ :
  - if  $j = 0$ , then  $R(V_{[i, 0]_{\triangleleft}}) = \bigoplus_{k \in \mathbb{Z}} V_{(k\mathcal{N}, k\mathcal{N}+i+1)_{\mathbb{Z}\mathbb{Z}}}$ .
  - if  $j \neq 0$ , then  $R(V_{[i, j]_{\triangleleft}}) = \bigoplus_{k \in \mathbb{Z}} V_{(k\mathcal{N}-n-m+j, k\mathcal{N}+i+1)_{\mathbb{Z}\mathbb{Z}}}$ .
- For a right interval  $[i, j]_{\triangleright}$ ,  $R(V_{[i, j]_{\triangleright}}) = \bigoplus_{k \in \mathbb{Z}} V_{[k\mathcal{N}+i, k\mathcal{N}+j]_{\mathbb{Z}\mathbb{Z}}}$ .
- For an upper interval  $[i, j]_{\top}$ ,  $R(V_{[i, j]_{\top}}) = \bigoplus_{k \in \mathbb{Z}} V_{[k\mathcal{N}+i, k\mathcal{N}+j+1)_{\mathbb{Z}\mathbb{Z}}}$ .
- For a lower interval  $[i, j]_{\perp}$ ,  $R(V_{[i, j]_{\perp}}) = \bigoplus_{k \in \mathbb{Z}} V_{(k\mathcal{N}+i-1, k\mathcal{N}+j]_{\mathbb{Z}\mathbb{Z}}}$ .

Here  $i, j \in \{0, 1, \dots, \mathcal{N}\}$  and  $\mathcal{N} = n + m - 1$ .

*Proof.* By the definition of  $R$ , the required lemma follows immediately.  $\square$

Note that each  $R(V_{\mathbf{I}})$  for an interval  $\mathbf{I}$  of  $\mathbf{B}$ , except for the full interval, is an infinitely *periodic* direct sum of interval modules over  $\mathbb{Z}\mathbb{Z}$  with disjoint support. Due to this periodicity and since  $R$  preserves finite direct sums, for a bipath module  $M$ , we have a free action of  $\mathbb{Z}$  on the barcode  $\mathcal{B}(R(M))$ , given by adding  $z\mathcal{N}$  to each endpoint of an interval  $\mathbf{J}$  for  $z \in \mathbb{Z}$  and  $\mathbf{J} \in \mathcal{B}(R(M))$ . We denote  $z \uparrow \mathbf{J} \in \mathcal{B}(R(M))$  as the interval resulting from the action of  $z$  on  $\mathbf{J}$ , and we set  $[\mathbf{J}] := \{z \uparrow \mathbf{J} \mid z \in \mathbb{Z}\}$ . In conclusion, we obtain the following theorem.

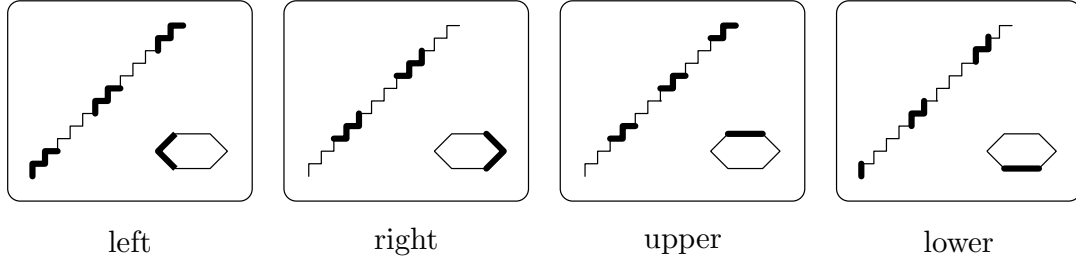


Figure 4.8: Illustration of interval modules over the bipath poset, and their image under the restriction functor  $R$ , as in Lemma 4.2.8

**Theorem 4.2.9.** *Let  $M$  be a bipath module and  $\mathcal{B}(M) := (\mathbb{I}(\mathbf{B}), d_M)$  the arc code of  $M$ . Then the multiset  $\mathcal{B}(R(M)) := (\mathbb{I}(\mathbb{Z}\mathbb{Z}), d_{R(M)})$  is the unique barcode of the zigzag module  $R(M)$  up to reordering and isomorphism. Moreover, there is a bijection  $\phi: \mathbb{I}(\mathbf{B}) \cong \mathbb{I}(\mathbb{Z}\mathbb{Z})/\mathbb{Z}$ . Here  $\mathbb{I}(\mathbb{Z}\mathbb{Z})/\mathbb{Z}$  denotes the orbit space of  $\mathbb{I}(\mathbb{Z}\mathbb{Z})$  induced by the action  $\uparrow$ .*

*Proof.* The first assertion follows by Theorem 2.5.15. The second assertion follows by Lemma 4.2.8, where the bijection  $\phi$  is given by Table 4.1.  $\square$

| $\mathbb{I}(\mathbf{B})$            | $\mathbb{I}(\mathbb{Z}\mathbb{Z})/\mathbb{Z}$  |
|-------------------------------------|------------------------------------------------|
| $\mathbf{B}$                        | $[\mathbb{Z}\mathbb{Z}]$                       |
| $[i, 0]_{\triangleleft}$            | $[(0, i + 1)_{\mathbb{Z}\mathbb{Z}}]$          |
| $[i, j]_{\triangleleft} (j \neq 0)$ | $[(-n - m + j, i + 1)_{\mathbb{Z}\mathbb{Z}}]$ |
| $[i, j]_{\triangleright}$           | $[i, j]_{\mathbb{Z}\mathbb{Z}}$                |
| $[i, j]_{\top}$                     | $[i, j + 1)_{\mathbb{Z}\mathbb{Z}}$            |
| $[i, j]_{\perp}$                    | $[i - 1, j]_{\mathbb{Z}\mathbb{Z}}$            |

Table 4.1: Correspondence between the intervals of  $\mathbf{B}$  and the orbits of  $\mathbb{I}(\mathbb{Z}\mathbb{Z})/\mathbb{Z}$

It is straightforward that for any  $\mathbf{I} \in \mathbb{I}(\mathbf{B})$  and any  $\mathbf{J} \in \mathbb{I}(\mathbb{Z}\mathbb{Z})$  with  $\phi(\mathbf{I}) = [\mathbf{J}]$ , we have  $d_M(V_{\mathbf{I}}) = d_{R(M)}(V_{\mathbf{J}}) = d_{R(M)}(V_{z \uparrow \mathbf{J}})$  for all  $z \in \mathbb{Z}$ . Therefore, the arc code  $\mathcal{B}(M) = (\mathbb{I}(\mathbf{B}), d_M)$  of  $M$  together with the covering map also gives a multiset  $(\mathbb{I}(\mathbb{Z}\mathbb{Z})/\mathbb{Z}, \mathbf{m}_M)$  where  $\mathbf{m}_M([\mathbf{J}]) := d_M(V_{\phi^{-1}([\mathbf{J}]}) = d_{R(M)}(V_{\mathbf{J}})$ .

The covering map  $\zeta: \mathbb{Z}\mathbb{Z} \rightarrow \mathbf{B}$  allows us to translate the arc code of the bipath module to the barcode of the infinite zigzag module, and vice versa. Furthermore, and as we do shortly, the covering map can be used to study the algebraic stability of bipath modules, via the algebraic stability of zigzag modules, which is well-established following Botnan and Lesnick [19] and Bakke Bjerkevik [10]. These stability results depend on the concept of block-decomposable modules, which we review first after recalling the definitions of interleavings and bottleneck distance.

We briefly review interleavings over  $\mathbb{R}^{\text{op}} \times \mathbb{R}$  and the bottleneck distance. Recall that for an  $\epsilon \in [0, \infty]$ , we define the *shift functor*  $(\cdot)(\epsilon): \text{mod } \mathbb{k}[\mathbb{R}^{\text{op}} \times \mathbb{R}] \rightarrow \text{mod } \mathbb{k}[\mathbb{R}^{\text{op}} \times \mathbb{R}]$  by sending  $M$  to the persistence module given by  $M(\epsilon)_{(x,y)} :=$

$M_{(x-\epsilon, y+\epsilon)}$  and

$$M(\epsilon) \left( (x, y) \xrightarrow{p(w, z), (x, y)} (w, z) \right) := M \left( p_{(w-\epsilon, z+\epsilon), (x-\epsilon, y+\epsilon)} \right).$$

On morphisms  $f: M \rightarrow N$ , we set  $f(\epsilon)_{(x, y)} := f_{(x-\epsilon, y+\epsilon)}$ . For every module  $M$ , we define a morphism  $\phi_\epsilon^M: M \rightarrow M(\epsilon)$  by the structure linear map  $M(p_{(x-\epsilon, y+\epsilon), (x, y)})$ . We say that two persistence modules  $M, N \in \text{mod } \mathbb{k}[\mathbb{R}^{\text{op}} \times \mathbb{R}]$  are  $\epsilon$ -interleaved if there exist morphisms  $f: M \rightarrow N(\epsilon)$  and  $g: N \rightarrow M(\epsilon)$  such that  $g(\epsilon) \circ f = \phi_{2\epsilon}^M$  and  $f(\epsilon) \circ g = \phi_{2\epsilon}^N$ .

As in [10], we describe matchings in terms of interleavings between interval modules. Let  $\mathcal{A}$  and  $\mathcal{B}$  be two multisets of intervals of  $\mathbb{R}^{\text{op}} \times \mathbb{R}$ . An  $\epsilon$ -matching between  $\mathcal{A}$  and  $\mathcal{B}$  is given by a bijection  $\sigma: \mathcal{A}' \rightarrow \mathcal{B}'$  for some subsets  $\mathcal{A}' \subset \mathcal{A}$  and  $\mathcal{B}' \subset \mathcal{B}$  such that

- every interval  $\mathbf{I}$  in  $\mathcal{A} \setminus \mathcal{A}'$  and  $\mathcal{B} \setminus \mathcal{B}'$  satisfies that  $V_{\mathbf{I}}$  is  $\epsilon$ -interleaved with the zero module, and
- for every  $\mathbf{I} \in \mathcal{A}'$ , the modules  $V_{\mathbf{I}}$  and  $V_{\sigma(\mathbf{I})}$  are  $\epsilon$ -interleaved.

Then, the *interleaving distance*  $d_I(M, N)$  between two persistence modules  $M, N$  over  $\mathbb{R}^{\text{op}} \times \mathbb{R}$  and the *bottleneck distance*  $d_B(\mathcal{A}, \mathcal{B})$  between two multisets of intervals  $\mathcal{A}$  and  $\mathcal{B}$  are given by

$$\begin{aligned} d_I(M, N) &:= \inf \{ \epsilon \mid M \text{ and } N \text{ are } \epsilon\text{-interleaved} \}, \\ d_B(\mathcal{A}, \mathcal{B}) &:= \inf \{ \epsilon \mid \text{there is an } \epsilon\text{-matching between } \mathcal{A} \text{ and } \mathcal{B} \}. \end{aligned}$$

In the sequel, for two persistence modules  $M$  and  $N$ ,  $d_B(M, N)$  denotes the bottleneck distance between the barcodes  $\mathcal{B}(M)$  and  $\mathcal{B}(N)$ . Namely,  $d_B(M, N) := d_B(\mathcal{B}(M), \mathcal{B}(N))$ .

Let  $\mathbb{U}$  be the full subposet of  $\mathbb{R}^{\text{op}} \times \mathbb{R}$  given by  $\{(a, b) \in \mathbb{R}^{\text{op}} \times \mathbb{R} \mid a \leq b\}$ . As in [10, 19], we are interested in the following four types of intervals of  $\mathbb{U}$ .

**Definition 4.2.10.** The following intervals of  $\mathbb{U}$  are called *blocks*, where  $a, b \in \mathbb{R} \cup \{\pm\infty\}$  and  $a \leq b$ ,

$$\begin{aligned} [a, b]_{\text{BL}} &:= \{ (c, d) \in \mathbb{U} \mid c \leq b, d \geq a \}, & [a, b]_{\text{BL}} &:= \{ (c, d) \in \mathbb{U} \mid a \leq d < b \}, \\ (a, b]_{\text{BL}} &:= \{ (c, d) \in \mathbb{U} \mid a < c \leq b \}, & (a, b]_{\text{BL}} &:= \{ (c, d) \in \mathbb{U} \mid c > a, d < b \}. \end{aligned}$$

A *block module* is an interval module  $V_{\mathbf{I}}$ , where  $\mathbf{I}$  is a block as defined above. A persistence module over  $\mathbb{U}$  is called *block-decomposable* if it decomposes into a direct sum of block modules.

The following establishes the algebraic stability of block-decomposable modules and is due to Bakke Bjerkevik [10, Theorem 4.18].

**Theorem 4.2.11** (Block isometry theorem). *For two block-decomposable modules  $M$  and  $N$ ,  $d_I(M, N) = d_B(M, N)$ .*

Consider the inclusion  $\iota: \mathbb{ZZ} \hookrightarrow \mathbb{R}^{\text{op}} \times \mathbb{R}$ . By composing first the left Kan extension along  $\iota$  (i.e., the left adjoint of the restriction functor induced by  $\iota$ ) and then the restriction functor of the inclusion  $\mathbb{U} \hookrightarrow \mathbb{R}^{\text{op}} \times \mathbb{R}$ , one obtains the *block*

*extension functor*  $\text{Ext}: \text{mod } \mathbb{k}[\mathbb{ZZ}] \rightarrow \text{mod } \mathbb{k}[\mathbb{U}]$ ; we refer to [10, 19] for more context on this functor and the definitions involved. The block extension functor preserves direct sums and sends barcodes of zigzag modules to barcodes of block-decomposable modules, by which we mean the following. See Figure 4.9 for an illustration.

**Lemma 4.2.12.** *Let  $M$  be a zigzag module. Then  $\text{Ext}(M)$  is block-decomposable, and*

$$\begin{aligned} \text{Ext}(V_{[a,b]\mathbb{ZZ}}) &= V_{[a,b]\text{BL}}, & \text{Ext}(V_{[a,b)\mathbb{ZZ}}) &= V_{[a,b)\mathbb{ZZ}}, \\ \text{Ext}(V_{(a,b]\mathbb{ZZ}}) &= V_{(a,b]\text{BL}}, & \text{Ext}(V_{(a,b)\mathbb{ZZ}}) &= V_{(a,b)\text{BL}}. \end{aligned}$$

Thus, we may write  $\text{Ext}(\mathcal{B}(M)) = \mathcal{B}(\text{Ext}(M))$  by identifying intervals with interval modules.

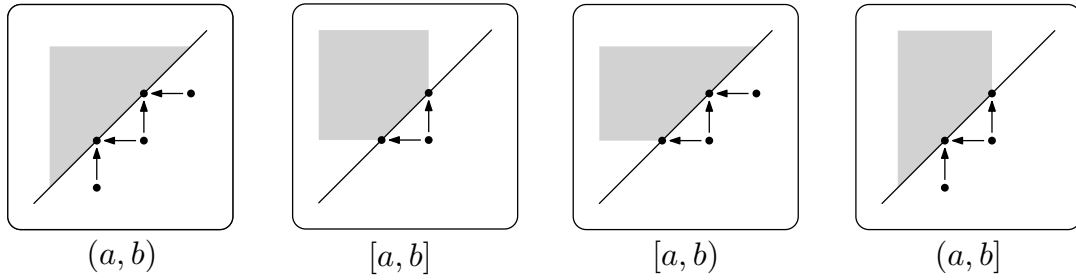


Figure 4.9: Illustration of the correspondence between the interval of  $\mathbb{ZZ}$  and the block of  $\mathbb{U}$ , as in Lemma 4.2.12

As in [10, 19], this lemma motivates the definitions of interleaving distance and bottleneck distance of two zigzag modules  $M$  and  $N$  by setting

$$d_I(M, N) := d_I(\text{Ext}(M), \text{Ext}(N)) \text{ and } d_B(M, N) := d_B(\text{Ext}(M), \text{Ext}(N)).$$

As such, Theorem 4.2.11 translates directly to zigzag modules.

Recall the restriction functor  $R: \text{mod } \mathbb{k}[\mathbf{B}] \rightarrow \text{mod } \mathbb{k}[\mathbb{ZZ}]$  induced by the covering map defined in Definition 4.2.7. Also recall that by Theorem 4.2.9,  $R$  preserves arc codes up to the action of  $\mathbb{Z}$ . In turn, and in parallel to the definition of interleaving and bottleneck distances in zigzag modules, for two bipath modules  $M$  and  $N$ , we define

$$d_I(M, N) := d_I(R(M), R(N)) \text{ and } d_B(M, N) := d_B(R(M), R(N)).$$

Combining this with the block isometry theorem (Theorem 4.2.11), we immediately obtain the isometry theorem for bipath persistence.

**Theorem 4.2.13** (Bipath isometry theorem). *Let  $M$  and  $N$  be bipath modules. Then  $d_I(M, N) = d_B(M, N)$ .*

# Chapter 5

## Interval rank invariants

As before, we always assume the poset  $\mathbf{P}$  is finite, and set  $A := \mathbb{k}[\mathbf{P}]$ . In this chapter, we introduce an invariant of persistence modules over finite posets, which extends the conventional rank invariant proposed by Carlsson and Zomorodian [22], and generalizes the generalized rank invariant proposed by Kim and Mémoli [40].

Recall that every  $\mathbf{P}$ -persistence module in  $\text{mod } A$  has the unique indecomposable decomposition (Theorem 2.5.15). Set  $\mathcal{L}$  to be the complete set of isoclasses of indecomposable modules in  $\text{mod } A$ . For any subset  $\mathcal{Q}$  of  $\mathcal{L}$ , the *additive closure* of  $\mathcal{Q}$ , denoted by  $\text{add } \mathcal{Q}$ , is defined as the smallest full additive subcategory which contains  $\mathcal{Q}$  and is closed under taking direct summands. In particular, the complete set of isoclasses of interval modules in  $\text{mod } A$  will be denoted by  $\mathcal{I}$ .

### 5.1 Algebraic invariants

**Definition 5.1.1.** Let  $\mathbf{P}$  be a poset and let  $A = \mathbb{k}[\mathbf{P}]$ . The *split Grothendieck group* of  $A$ , denoted by  $K^\oplus(A)$ , is defined to be the abelian group  $G/G'$ , where  $G$  is the free abelian group with basis  $\{[M] \in \mathcal{L} \mid M \in \text{mod } A\}$ , and  $G'$  is the subgroup of  $G$  generated by the set

$$\{[M] - [L] - [N] \in G \mid 0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0 \text{ is a split exact sequence}\}.$$

For each  $M \in \text{mod } A$ , we set  $\llbracket M \rrbracket$  to be the element of  $K^\oplus(A)$  corresponding to  $M$ . Similarly, we can define the split Grothendieck group of the additive closure  $\text{add } \mathcal{Q}$  where  $\mathcal{Q} \subseteq \mathcal{L}$ , and by  $K^\oplus(\mathcal{Q})$  we denote the split Grothendieck group of  $\text{add } \mathcal{Q}$ .

**Definition 5.1.2.** An *(algebraic) invariant* of the incidence category  $A$  is a map  $\mathbf{q}: (\text{mod } A)_0 \rightarrow G$  for some set  $G$ , satisfying  $\mathbf{q}(M) = \mathbf{q}(N)$  whenever  $M \cong N$  in  $\text{mod } A$ . Here and subsequently,  $(\text{mod } A)_0$  denotes the set of all  $\mathbf{P}$ -persistence modules.

**Remark 5.1.3.** It is equivalent to saying that an invariant is a map  $\mathbf{q}: (\text{mod } A)_0 \rightarrow G$  for some set  $G$  such that the following diagram commutes for some  $\mathbf{q}'$ :

$$\begin{array}{ccc} (\text{mod } A)_0 & \xrightarrow{\pi} & (\text{mod } A)_0 / \cong \\ \mathbf{q} \downarrow & \searrow \mathbf{q}' & \\ G & & \end{array}$$

Here  $(\text{mod } A)_0 / \cong$  is the set of isoclasses of  $\mathbf{P}$ -persistence modules, and  $\pi$  is the canonical projection.

If one further desires the additivity on the invariant, then the following concept of additive invariant is needed.

**Definition 5.1.4.** An *additive invariant* of the incidence category  $A$  is an algebraic invariant  $\mathbf{q}$  taking values in an abelian group  $G$ , satisfying  $\mathbf{q}(M \oplus N) = \mathbf{q}(M) + \mathbf{q}(N)$ .

**Remark 5.1.5.** We note that the concept of additive invariants defined in [15, Definition 2.15] is slightly different from the one we proposed here. Recalling that an additive invariant of  $\text{mod } A$  defined in [15] is a group homomorphism  $\mathbf{p}: K^\oplus(A) \rightarrow G$  for some abelian group  $G$ . From the following commutative diagram, one can easily see that the concept of additive invariant in this thesis is more general than that in [15], because every additive invariant  $\mathbf{q}$  factors through the canonical map that sends each module  $M$  with an indecomposable decomposition  $M = \bigoplus_{i \in [n]} M_i$  to  $\sum_{i \in [n]} \llbracket M_i \rrbracket$ .

$$\begin{array}{ccccc}
 & & (\text{mod } A)_0 & & \\
 & \swarrow \text{can} & \downarrow \mathbf{q} & \searrow \pi & \\
 K^\oplus(A) & & & & (\text{mod } A)_0 / \cong \\
 & \searrow \mathbf{p} & \downarrow & \swarrow & \\
 & & G & & 
 \end{array}$$

**Example 5.1.6.** Let  $\mathbf{P}$  be a poset and let  $A = \mathbb{k}[\mathbf{P}]$ .

- (1) The *dimension vector*, denoted by **dim**, is an additive invariant of  $A$  given by

$$\mathbf{dim}: (\text{mod } A)_0 \rightarrow \mathbb{Z}^{\mathbf{P}}, \quad M \mapsto \mathbf{dim} M,$$

where we recall that  $\mathbf{dim} M$  is the dimension vector of  $M$  defined in Definition 2.5.3. To extend the definition starting from  $K^\oplus(A)$ , for each  $\llbracket M \rrbracket \in K^\oplus(A)$  with  $\llbracket M \rrbracket = \llbracket M_+ \rrbracket - \llbracket M_- \rrbracket$  and  $M_+, M_- \in \text{mod } A$ , we set

$$\mathbf{dim} \llbracket M \rrbracket := \mathbf{dim} M_+ - \mathbf{dim} M_-. \quad (5.1.1)$$

- (2) The *rank invariant*, denoted by **rank**, is an additive invariant of  $A$  given by

$$\mathbf{rank}: (\text{mod } A)_0 \rightarrow \mathbb{Z}^{\text{Seg}(\mathbf{P})}, \quad M \mapsto ([x, y] \mapsto \text{rank } M(p_{y,x})),$$

where we denote the above function  $([x, y] \mapsto \text{rank } M(p_{y,x}))$  by writing

$$\mathbf{rank}_{[x,y]} M := \text{rank } M(p_{y,x}) = \text{rank } M_{y,x}.$$

It is evident that the rank invariant is a generalization of the dimension vector because  $[x, x]$  is trivially a segment of  $\mathbf{P}$  and  $\text{rank } M(p_{x,x}) = \text{dim } M(x)$ .

- (3) The *generalized rank invariant* [40], denoted by **GRI**, is an additive invariant of  $A$  given by

$$\mathbf{GRI}: (\text{mod } A)_0 \rightarrow \mathbb{Z}^{\mathbb{I}(\mathbf{P})}, \quad M \mapsto \left( \mathbf{I} \mapsto \text{rank} \left( \varprojlim M|_{\mathbf{I}} \xrightarrow{\mu_M} \varinjlim M|_{\mathbf{I}} \right) \right),$$

where  $M|_{\mathbf{I}}$  is regarded as a functor  $\mathbf{I} \rightarrow \text{mod } \mathbb{k}$ ,  $\varprojlim M|_{\mathbf{I}}$  (resp.  $\varinjlim M|_{\mathbf{I}}$ ) denotes the limit (resp. colimit) of  $M|_{\mathbf{I}}$  in  $\text{mod } \mathbb{k}$ , and  $\mu_M$  is the unique linear map determined by the universality of limit and colimit. We refer to Definition 5.3.20 for a thorough treatment. It is evident that the generalized rank invariant is a generalization of the rank invariant because  $\text{Seg}(\mathbf{P}) \subseteq \mathbb{I}(\mathbf{P})$  and  $\mathbf{GRI}(M)|_{\text{Seg}(\mathbf{P})} = \mathbf{rank} M$  holds for all  $M \in \text{mod } A$ .

- (4) The *multiplicity invariant*, denoted by  $\mathbf{mult}_{\mathcal{L}}$ , is an additive invariant of  $A$  given by

$$\mathbf{mult}_{\mathcal{L}}: (\text{mod } A)_0 \rightarrow \mathbb{Z}^{\mathcal{L}}, \quad M \mapsto d_M,$$

where we recall that  $d_M$  denotes the multiplicity function (Theorem 2.5.15). In particular, restricting to only interval modules provides another invariant called the *interval multiplicity invariant* of  $A$  given by

$$\mathbf{mult}_{\mathcal{I}}: (\text{mod } A)_0 \rightarrow \mathbb{Z}^{\mathcal{I}}, \quad M \mapsto d_M|_{\mathcal{I}}.$$

Note that we can compute  $\mathbf{mult}_{\mathcal{I}}$  by utilizing formulas provided in Chapter 3.

We say an additive invariant  $\mathbf{q}$  is *complete* if  $\mathbf{q}(M) = \mathbf{q}(N)$  implies  $M \cong N$ , and *incomplete* otherwise. It is clear that  $\mathbf{mult}_{\mathcal{L}}$  is complete by Theorem 2.5.15.

## 5.2 Compression systems and multiplicities

From now on, we simply write  $\mathbb{I}$  instead of  $\mathbb{I}(\mathbf{P})$  for brevity.

**Definition 5.2.1.** A compression system for  $\mathbf{P}$  is a family  $\xi = (\xi_{\mathbf{I}})_{\mathbf{I} \in \mathbb{I}}$  of order-preserving maps  $\xi_{\mathbf{I}}: \mathbf{I}^{\xi} \rightarrow \mathbf{P}$  from a connected finite poset  $\mathbf{I}^{\xi}$  satisfying the following conditions for each  $\mathbf{I} \in \mathbb{I}$ .

- (1)  $\xi_{\mathbf{I}}$  factors through the poset inclusion  $\iota_{\mathbf{I}}: \mathbf{I} \hookrightarrow \mathbf{P}$ ;
- (2) The image  $\xi_{\mathbf{I}}(\mathbf{I}^{\xi})$  contains  $\text{sc}(\mathbf{I}) \cup \text{sk}(\mathbf{I})$ .
- (3) If  $\mathbf{I} = [x, y] \in \text{Seg}(\mathbf{P})$ , then there exists a pair  $(x', y') \in [\leq]_{\mathbf{I}^{\xi}}$  such that  $(\xi_{\mathbf{I}}(x'), \xi_{\mathbf{I}}(y')) = (x, y)$ .

**Example 5.2.2** (tot). Let  $\mathbf{I} \in \mathbb{I}$ . Obviously, the poset inclusion  $\iota_{\mathbf{I}}: \mathbf{I} \hookrightarrow \mathbf{P}$  satisfies the above three conditions. In this case, we denote by  $\text{tot}_{\mathbf{I}}$  the poset inclusion  $\mathbf{I}^{\text{tot}} \hookrightarrow \mathbf{P}$  where  $\mathbf{I}^{\text{tot}} := \mathbf{I}$ . The family  $\text{tot} := (\text{tot}_{\mathbf{I}})_{\mathbf{I} \in \mathbb{I}}$  is called the *total* compression system for  $\mathbf{P}$ .

**Example 5.2.3** (ss). Let  $\mathbf{I} \in \mathbb{I}$ , and let  $\mathbf{I}^{\text{ss}} := \text{sk}(\mathbf{I}) \cup \text{sc}(\mathbf{I})$  be the full subposet of  $\mathbf{I}$ . Set  $\text{ss}_{\mathbf{I}}: \mathbf{I}^{\text{ss}} \hookrightarrow \mathbf{P}$  to be the poset inclusion. This defines a compression system  $\text{ss} := (\text{ss}_{\mathbf{I}})_{\mathbf{I} \in \mathbb{I}}$  for  $A$ , which is called the *source-sink* compression system for  $\mathbf{P}$ .

**Example 5.2.4.** Let  $\mathbf{P} := \mathbf{G}_{5,2}$  as in Example 2.2.10 (1), and  $\mathbf{I}$  be the interval with  $\text{sc}(\mathbf{I}) := \{(1, 2), (2, 1)\}$  and  $\text{sk}(\mathbf{I}) := \{(3, 2)\}$ . Take  $\mathbf{I}^{\xi} := \{(1, 2), (2, 2), (3, 2), (2, 1)\} \subseteq \mathbf{I}$ , and take  $\xi_{\mathbf{I}}: \mathbf{I}^{\xi} \rightarrow \mathbf{P}$  to be the inclusion, then this  $\xi_{\mathbf{I}}$  can be taken as a component of a compression system for  $\mathbf{P}$ , which satisfies  $\text{ss}_{\mathbf{I}} \neq \xi_{\mathbf{I}} \neq \text{tot}_{\mathbf{I}}$ .

We have the following observation for the poset  $\mathbf{I}^{\xi}$ .

**Proposition 5.2.5.** *If  $\mathbf{I}^{\xi}$  has a unique source and a unique sink, i.e.,  $|\text{sc}(\mathbf{I}^{\xi})| = |\text{sk}(\mathbf{I}^{\xi})| = 1$ , then  $\mathbf{I}$  also has a unique source and a unique sink.*

*Proof.* We can set  $\text{sc}(\mathbf{I}^\xi) = \{a\}$  and  $\text{sk}(\mathbf{I}^\xi) = \{b\}$  for some  $a, b \in \mathbf{I}^\xi$ . Assume

$$\text{sc}(\mathbf{I}) := \{x_1, \dots, x_n\} \text{ and } \text{sk}(\mathbf{I}) := \{y_1, \dots, y_m\}. \quad (5.2.2)$$

Since the image  $\xi_{\mathbf{I}}(\mathbf{I}^\xi)$  contains  $\text{sc}(\mathbf{I}) \cup \text{sk}(\mathbf{I})$ , there exist subsets  $\{a_1, \dots, a_n\}$  and  $\{b_1, \dots, b_m\}$  of  $\mathbf{I}^\xi$  such that  $\xi_{\mathbf{I}}(a_i) = x_i$ ,  $\xi_{\mathbf{I}}(b_j) = y_j$  for all  $i \in [n]$ ,  $j \in [m]$ . Then for any  $i \in [n]$ ,  $j \in [m]$ , the relations  $a \leq a_i$ ,  $b_j \leq b$  in  $\mathbf{I}^\xi$  show that  $\xi_{\mathbf{I}}(a) \leq x_i$ ,  $y_j \leq \xi_{\mathbf{I}}(b)$ . Hence  $\xi_{\mathbf{I}}(a) = x_i$ ,  $y_j = \xi_{\mathbf{I}}(b)$  by (5.2.2) for all  $i \in [n]$ ,  $j \in [m]$ . Thus  $n = 1 = m$ .  $\square$

**Remark 5.2.6.** The converse of the proposition above does not hold in general. For example, let  $\mathbf{P} = G_{2,2} = \mathbf{I}$ , and  $\mathbf{I}^\xi := \{(1,1), (1,2), (2,2)\}$  with the order relations defined by the Hasse quiver  $(1,1) \rightarrow (2,2) \leftarrow (1,2)$ , not a full subposet of  $\mathbf{P}$ . Define  $\xi_{\mathbf{I}}: \mathbf{I}^\xi \rightarrow \mathbf{I}$  as the inclusion. For other intervals  $\mathbf{J} \in \mathbb{I}$ , let  $\xi_{\mathbf{J}}: \mathbf{J}^\xi \rightarrow \mathbf{J}$  be the identity of  $\mathbf{J}$ . Then this  $\xi$  satisfies all three conditions in Definition 5.2.1 and thus it is a compression system for  $\mathbf{P}$ . In this case,  $\mathbf{I}$  has a unique source and a unique sink, but  $\mathbf{I}^\xi$  does not.

**Notation 5.2.7.** Let  $\mathbf{I} \in \mathbb{I}$ . Set  $F = \mathbb{k}[\iota_{\mathbf{I}}]: \mathbb{k}[\mathbf{I}] \rightarrow A$  to be the linearization of the inclusion map  $\iota_{\mathbf{I}}: \mathbf{I} \hookrightarrow \mathbf{P}$ , and  $R = R_{\mathbf{I}}: \text{mod } A \rightarrow \text{mod } \mathbb{k}[\mathbf{I}]$  the restriction functor induced by  $\iota_{\mathbf{I}}$ . Then the order-preserving map  $\xi_{\mathbf{I}}: \mathbf{I}^\xi \rightarrow \mathbf{P}$  factors through  $\iota_{\mathbf{I}}$  by the condition (1) in Definition 5.2.1, and the functor  $F_{\mathbf{I}} = \mathbb{k}[\xi_{\mathbf{I}}]: \mathbb{k}[\mathbf{I}^\xi] \rightarrow A$  factors through  $F$  as in the following diagrams:

$$\begin{array}{ccc} \mathbf{I}^\xi & \xrightarrow{\xi_{\mathbf{I}}} & \mathbf{P} \\ & \searrow \xi' & \nearrow \iota_{\mathbf{I}} \\ & \mathbf{I} & \end{array}, \quad \begin{array}{ccc} \mathbb{k}[\mathbf{I}^\xi] & \xrightarrow{F_{\mathbf{I}}} & A \\ & \searrow F' & \nearrow F \\ & \mathbb{k}[\mathbf{I}] & \end{array}.$$

Hence we have a factorization of the restriction functor  $R_{\mathbf{I}}$  by  $R$  as in the diagram

$$\begin{array}{ccc} \text{mod } \mathbb{k}[\mathbf{I}^\xi] & \xleftarrow{R_{\mathbf{I}}} & \text{mod } A \\ & \nwarrow R' \quad \nearrow R & \\ & \text{mod } \mathbb{k}[\mathbf{I}] & \end{array}.$$

The restriction functor  $R_{\mathbf{I}}$  sends all interval modules in  $\text{mod } A$  to indecomposable modules in  $\text{mod } \mathbb{k}[\mathbf{I}^\xi]$  by Proposition 2.5.14. Subsequently, we have the following definition.

**Definition 5.2.8.** Let  $M \in \text{mod } A$ , and  $\mathbf{I} \in \mathbb{I}$ . Then the number

$$c_M^\xi(\mathbf{I}) := d_{R_{\mathbf{I}}(M)}(R_{\mathbf{I}}(V_{\mathbf{I}}))$$

is called the *compression multiplicity* of  $\mathbf{I}$  in  $M$  under  $\xi$ .

Clearly,  $c_M^\xi$  can be viewed as a function  $c_M^\xi: \mathbb{I} \rightarrow \mathbb{Z}$ . We provide several properties of the compression multiplicity.

**Lemma 5.2.9.** Let  $\xi = (\xi_{\mathbf{I}})_{\mathbf{I} \in \mathbb{I}}$  be a compression system for  $\mathbf{P}$ , and let  $\mathbf{I}, \mathbf{J} \in \mathbb{I}$  with  $\mathbf{I} \leq \mathbf{J}$ . Then  $R_{\mathbf{I}}(V_{\mathbf{J}}) = R_{\mathbf{I}}(V_{\mathbf{I}})$  in  $\text{mod } \mathbb{k}[\mathbf{I}^\xi]$ .

*Proof.* For each  $x' \in \mathbf{I}^\xi$ , we have

$$R_{\mathbf{I}}(V_{\mathbf{J}})(x') = V_{\mathbf{J}}(F_{\mathbf{I}}(x')) = V_{\mathbf{J}}(\xi_{\mathbf{I}}(x')) = V_{\mathbf{J}}(\iota_{\mathbf{I}}(\xi'(x'))) = V_{\mathbf{J}}(\xi'(x')) = V_{\mathbf{I}}(\xi'(x')).$$

The last equality holds because  $\xi'(x') \in \mathbf{I} \subseteq \mathbf{J}$ . For every morphism  $p_{y',x'} \in \mathbb{k}[\mathbf{I}^\xi]_1$  where  $(x', y') \in [\leq]_{\mathbf{I}^\xi}$ , we have

$$R_{\mathbf{I}}(V_{\mathbf{J}})(p_{y',x'}) = V_{\mathbf{J}}(F_{\mathbf{I}}(p_{y',x'})) = V_{\mathbf{J}}(\mathbb{k}[\xi_{\mathbf{I}}](p_{y',x'})) = V_{\mathbf{J}}(p_{\xi_{\mathbf{I}}(y'), \xi_{\mathbf{I}}(x')}) = V_{\mathbf{I}}(p_{\xi_{\mathbf{I}}(y'), \xi_{\mathbf{I}}(x')}).$$

The last equality holds because  $p_{\xi_{\mathbf{I}}(y'), \xi_{\mathbf{I}}(x')}$  is the morphism in  $\mathbb{k}[\mathbf{P}](\xi'(x'), \xi'(y'))$  by noticing  $\xi_{\mathbf{I}}(x') = \xi'(x')$  and  $\xi_{\mathbf{I}}(y') = \xi'(y')$ .  $\square$

**Lemma 5.2.10.** *Let  $\mathbf{I} \in \mathbb{I}$ . Then  $\mathbf{I} = \text{conv}(\text{sc}(\mathbf{I}) \cup \text{sk}(\mathbf{I}))$ . In particular, if  $\mathbf{J} \in \mathbb{I}$  and  $\text{sc}(\mathbf{I}) \cup \text{sk}(\mathbf{I}) \subseteq \mathbf{J}$ , then  $\mathbf{I} \leq \mathbf{J}$ .*

*Proof.* The assertion follows immediately from Lemma 3.2.2.  $\square$

**Proposition 5.2.11.** *Let  $\xi = (\xi_{\mathbf{I}})_{\mathbf{I} \in \mathbb{I}}$  be a compression system for  $\mathbf{P}$ , and let  $\mathbf{I}, \mathbf{J} \in \mathbb{I}$ . Then*

$$c_{V_{\mathbf{J}}}^{\xi}(\mathbf{I}) = \begin{cases} 1, & \text{if } \mathbf{I} \leq \mathbf{J}, \\ 0, & \text{otherwise.} \end{cases}$$

*Proof.* If  $\mathbf{I} \leq \mathbf{J}$ , then  $c_{V_{\mathbf{J}}}^{\xi}(\mathbf{I}) = d_{R_{\mathbf{I}}(V_{\mathbf{J}})}(R_{\mathbf{I}}(V_{\mathbf{I}})) = d_{R_{\mathbf{I}}(V_{\mathbf{I}})}(R_{\mathbf{I}}(V_{\mathbf{I}})) = 1$  by noticing Lemma 5.2.9. Otherwise, Lemma 5.2.10 ensures the existence of a vertex  $x \in \text{sc}(\mathbf{I}) \cup \text{sk}(\mathbf{I})$  that is not in  $\mathbf{J}$ . By assumption on  $\xi_{\mathbf{I}}$ , there exists  $x' \in \mathbf{I}^\xi$  such that  $\xi_{\mathbf{I}}(x') = x$ . By definition,  $x'$  satisfies  $R_{\mathbf{I}}(V_{\mathbf{I}})(x') = \mathbb{k}$  and  $R_{\mathbf{I}}(V_{\mathbf{J}})(x') = 0$ . Hence in particular,  $R_{\mathbf{I}}(V_{\mathbf{I}})$  is not a direct summand of  $R_{\mathbf{I}}(V_{\mathbf{J}})$ , so  $c_{V_{\mathbf{J}}}^{\xi}(\mathbf{I}) = 0$ .  $\square$

**Proposition 5.2.12.** *Let  $M, N \in \text{mod } A$ , and  $\mathbf{I} \in \mathbb{I}$ . Then*

$$c_{M \oplus N}^{\xi}(\mathbf{I}) = c_M^{\xi}(\mathbf{I}) + c_N^{\xi}(\mathbf{I}).$$

*Proof.* This is a direct consequence of the additivity of  $R_{\mathbf{I}}$  and the uniqueness of  $d_{M \oplus N}$  in Theorem 2.5.15.  $\square$

Under certain conditions, the compression multiplicity has a monotonically decreasing property.

**Proposition 5.2.13.** *Let  $\xi = (\xi_{\mathbf{I}})_{\mathbf{I} \in \mathbb{I}}$  be a compression system for  $\mathbf{P}$ . Suppose that  $\xi$  further satisfies the following condition:*

- *For all intervals  $\mathbf{I}, \mathbf{J} \in \mathbb{I}$  with  $\mathbf{I} \leq \mathbf{J}$ , there exists an order-preserving map  $\Phi: \mathbf{I}^\xi \rightarrow \mathbf{J}^\xi$  such that  $\xi_{\mathbf{I}} = \xi_{\mathbf{J}} \circ \Phi$ .*

*Then for any intervals  $\mathbf{I}, \mathbf{J} \in \mathbb{I}$  such that  $\mathbf{I} \leq \mathbf{J}$  and for every  $M \in \text{mod } A$ , we have*

$$c_M^{\xi}(\mathbf{I}) \geq c_M^{\xi}(\mathbf{J}).$$

*Proof.* We write  $s = c_M^{\xi}(\mathbf{J})$  and  $r = c_M^{\xi}(\mathbf{I})$ . Recall the notation  $R_{\mathbf{I}}$  given in Notation 5.2.7. By Definition 5.2.8, we have the decomposition

$$R_{\mathbf{J}}(M) \cong [R_{\mathbf{J}}(V_{\mathbf{J}})]^s \oplus N \text{ in } \text{mod } \mathbb{k}[\mathbf{J}^\xi], \quad (5.2.3)$$

where  $N$  has no direct summand that is isomorphic to  $R_{\mathbf{J}}(V_{\mathbf{J}})$ . Applying the restriction functor  $R_{\Phi}$  induced by  $\Phi$  to (5.2.3) yields

$$R_{\mathbf{I}}(M) \cong [R_{\mathbf{I}}(V_{\mathbf{J}})]^s \oplus R_{\Phi}(N) \text{ in } \text{mod } \mathbb{k}[\mathbf{I}^{\xi}].$$

By noticing Lemma 5.2.9 and  $\mathbf{I} \leq \mathbf{J}$ ,  $R_{\mathbf{I}}(M) \cong [R_{\mathbf{I}}(V_{\mathbf{I}})]^s \oplus R_{\Phi}(N)$ . Hence  $r \geq s$ .  $\square$

**Corollary 5.2.14.** *Let  $\xi = \text{tot}$ . Then for all intervals  $\mathbf{I}, \mathbf{J} \in \mathbb{I}$  with  $\mathbf{I} \leq \mathbf{J}$  and for every  $M \in \text{mod } A$ , we have*

$$c_M^{\text{tot}}(\mathbf{I}) \geq c_M^{\text{tot}}(\mathbf{J}).$$

*Proof.* The assertion follows by noticing  $\mathbf{I}^{\text{tot}} = \mathbf{I}$ ,  $\mathbf{J}^{\text{tot}} = \mathbf{J}$  and taking  $\Phi$  in Proposition 5.2.13 to be the inclusion map  $\mathbf{I} \hookrightarrow \mathbf{J}$ .  $\square$

The total compression multiplicity is the least among all compression systems, which we conclude as follows.

**Proposition 5.2.15.** *Let  $\text{tot}$  be the total compression system for  $\mathbf{P}$ . Then for any compression system  $\xi$ , any interval  $\mathbf{I} \in \mathbb{I}$ , and any  $M \in \text{mod } A$ , we have*

$$c_M^{\xi}(\mathbf{I}) \geq c_M^{\text{tot}}(\mathbf{I}).$$

*Proof.* We write  $s = c_M^{\text{tot}}(\mathbf{I})$  and  $r = c_M^{\xi}(\mathbf{I})$ . By the definition of compression multiplicity under  $\text{tot}$ , we have the decomposition

$$R(M) \cong [R(V_{\mathbf{I}})]^s \oplus N \text{ in } \text{mod } \mathbb{k}[\mathbf{I}], \quad (5.2.4)$$

where  $N$  has no direct summand that is isomorphic to  $R(V_{\mathbf{I}})$ . Applying the functor  $R'$  (given in Notation 5.2.7) to (5.2.4) yields

$$R_{\mathbf{I}}(M) \cong [R_{\mathbf{I}}(V_{\mathbf{I}})]^s \oplus R'(N) \text{ in } \text{mod } \mathbb{k}[\mathbf{I}^{\xi}].$$

Therefore  $r \geq s$ , and the assertion follows.  $\square$

**Example 5.2.16.** Let  $\mathbf{P} = \mathbf{G}_{3,2}$  and let  $M \in \text{mod } A$  given by

$$\begin{array}{ccccc} \mathbb{k} & \xrightarrow{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} & \mathbb{k}^2 & \xrightarrow{\begin{bmatrix} 0 & 1 \end{bmatrix}} & \mathbb{k} \\ \uparrow & & \uparrow \begin{bmatrix} 0 \\ 1 \end{bmatrix} & & \uparrow 1 \\ 0 & \longrightarrow & \mathbb{k} & \xrightarrow{1} & \mathbb{k} \end{array}$$

Take the following interval  $\mathbf{I}$  of  $\mathbf{P}$ :

$$\begin{array}{ccccc} \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \\ & & \uparrow & & \uparrow \\ & & \bullet & \longrightarrow & \bullet \end{array}.$$

Then  $c_M^{\text{ss}}(\mathbf{I}) = 1$  while  $c_M^{\text{tot}}(\mathbf{I}) = 0$ . This shows that  $c_M^{\text{ss}}(\mathbf{I}) \geq c_M^{\text{tot}}(\mathbf{I})$ .

### 5.3 Interval rank invariant: a generalization of rank invariant

In this section, we first explain why the compression multiplicity can be viewed as a generalization of the rank invariant (Example 5.1.6 (2)). This motivates us to define an additive invariant of  $A$ . Then we provide explicit formulas to compute the defined invariant.

#### 5.3.1 Definition of interval rank invariants

**Proposition 5.3.1.** *Let  $\xi$  be a compression system for  $\mathbf{P}$ . Then for all  $M \in \text{mod } A$  and  $[x, y] \in \text{Seg}(\mathbf{P})$ , we have*

$$c_M^\xi([x, y]) = \text{rank } M(p_{y,x}).$$

*In particular,  $c_M^\xi([x, y])$  does not depend on  $\xi$ . Therefore,  $c_M|_{\text{Seg}(\mathbf{P})} = \mathbf{rank } M$ .*

*Proof.* For simplicity, we put  $\mathbf{I} := [x, y]$ . We first compute  $c_M^{\text{tot}} = d_{R(M)}(R(V_{\mathbf{I}}))$ . Note that both  $R(M)$  and  $R(V_{\mathbf{I}})$  are  $\mathbf{I}$ -persistence modules, and

$$R(V_{\mathbf{I}}) \cong I_y \cong P_x \text{ in } \text{mod } \mathbb{k}[\mathbf{I}].$$

Here  $I_y$  (resp.  $P_x$ ) is the injective (resp. projective) indecomposable  $\mathbf{I}$ -persistence module at  $y$  (resp.  $x$ ). By the formula (3.1.15) and because  $\text{sc}_1(\mathbf{I}) = \text{sc}(\uparrow \mathbf{I}) = \text{sk}_1(\mathbf{I}) = \text{sk}(\downarrow \mathbf{I}) = \emptyset$ , we obtain

$$d_{R(M)}(R(V_{\mathbf{I}})) = \text{rank } M(p_{y,x}),$$

and by Proposition 5.2.15,  $c_M^\xi(\mathbf{I}) \geq c_M^{\text{tot}}(\mathbf{I}) = d_{R(M)}(R(V_{\mathbf{I}})) = \text{rank } M(p_{y,x})$  follows.

Next, we show the converse inequality. Set  $c := c_M^\xi(\mathbf{I}) = d_{R_{\mathbf{I}}(M)}(R_{\mathbf{I}}(V_{\mathbf{I}}))$ . Then we have an isomorphism

$$R_{\mathbf{I}}(M) \cong [R_{\mathbf{I}}(V_{\mathbf{I}})]^c \oplus N \text{ in } \text{mod } \mathbb{k}[\mathbf{I}^\xi]. \quad (5.3.5)$$

By assumption on  $\xi_{\mathbf{I}}$ , we have  $x, y \in \xi_{\mathbf{I}}(\mathbf{I}^\xi)$ , and there exists a morphism  $q$  in  $\mathbb{k}[\mathbf{I}^\xi]$  with  $F_{\mathbf{I}}(q) = p_{y,x} \in A(x, y)$ . Then from (5.3.5), we have  $R_{\mathbf{I}}(M)(q) \cong [R_{\mathbf{I}}(V_{\mathbf{I}})(q)]^c \oplus N(q)$  and hence

$$\text{rank } M(p_{y,x}) = \text{rank } M(F_{\mathbf{I}}(q)) = \text{rank } R_{\mathbf{I}}(M)(q) \geq c \cdot \text{rank } R_{\mathbf{I}}(V_{\mathbf{I}})(q) = c$$

because  $\text{rank } R_{\mathbf{I}}(V_{\mathbf{I}})(q) = \text{rank } V_{\mathbf{I}}(p_{y,x}) = 1$ . Thus,  $\text{rank } M(p_{y,x}) \geq c_M^\xi(\mathbf{I})$ . Therefore, the assertion follows.  $\square$

It is interesting to see that even for the compression system provided in Remark 5.2.6, the proposition above still holds because the segment  $\mathbf{I}$  has a unique source and a unique sink. The following is an immediate consequence of Proposition 5.3.1.

**Corollary 5.3.2.** *Let  $M \in \text{mod } A$  and  $\mathbf{I} = [x, y] \in \text{Seg}(\mathbf{P})$ . If  $R(M) \in \text{mod } \mathbb{k}[\mathbf{I}]$  is indecomposable and not isomorphic to  $V_{\mathbf{I}}$ , then  $M(p_{y,x}) = 0$ .*

*Proof.* Here we take the total compression system  $\text{tot}$  provided in Example 5.2.2. Then by assumption, we have  $c_M^{\text{tot}}(\mathbf{I}) = 0$ . Hence by Proposition 5.3.1 we obtain  $\text{rank } M(p_{y,x}) = c_M^{\text{tot}}(\mathbf{I}) = 0$ , which shows the assertion.  $\square$

**Example 5.3.3.** Let  $\mathbf{P} := \mathbf{G}_{5,2}$  and put  $\mathbf{I} := \mathbf{P}$ . It is known that there exists an indecomposable  $\mathbb{k}[\mathbf{I}]$ -module  $M$  with dimension vector  $\begin{bmatrix} 2 & 3 & 3 & 2 & 1 \\ 1 & 2 & 3 & 3 & 2 \end{bmatrix}$ . Then since  $M \not\cong V_{\mathbf{I}}$ , we have to have  $M(p) = 0$  for any path  $p$  from  $(1,1)$  to  $(5,2)$  by the corollary above.

Indeed, it is straightforward to check that the following module  $M$  is indecomposable with this dimension vector (thus this dimension vector is realized as this  $M$ ):

$$\begin{array}{ccccccc} \mathbb{k}^2 & \xrightarrow{\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}} & \mathbb{k}^3 & \xrightarrow{\mathbf{1}} & \mathbb{k}^3 & \xrightarrow{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}} & \mathbb{k}^2 & \xrightarrow{\begin{bmatrix} 1 & 0 \end{bmatrix}} & \mathbb{k} \\ \uparrow \begin{bmatrix} -1 & 1 \end{bmatrix} & & \uparrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & & \uparrow \mathbf{1} & & \uparrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} & & \uparrow \begin{bmatrix} 1 & 0 \end{bmatrix} \\ \mathbb{k} & \xrightarrow{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} & \mathbb{k}^2 & \xrightarrow{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} & \mathbb{k}^3 & \xrightarrow{\mathbf{1}} & \mathbb{k}^3 & \xrightarrow{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}} & \mathbb{k}^2 \end{array}$$

It is certain that  $M$  satisfies the condition  $M(p) = 0$  stated above. We note that this  $M$  is obtained as the Auslander–Reiten translation  $\tau M_{\lambda}$  of the indecomposable module  $M_{\lambda}$  with  $\lambda = 1$  provided later in Example 6.3.3.

Proposition 5.3.1 suggests us to define the following.

**Definition 5.3.4.** Let  $\xi = (\xi_{\mathbf{I}})_{\mathbf{I} \in \mathbb{I}}$  be a compression system for  $\mathbf{P}$  and let  $M \in \text{mod } A$ .

- (1) We set

$$\mathbf{rank}_{\mathbf{I}}^{\xi} M := c_M^{\xi}(\mathbf{I})$$

and call it the  $\mathbf{I}$ -rank of  $M$  under  $\xi$ .

- (2) The *interval rank invariant under  $\xi$* , denoted by  $\mathbf{rank}_{\mathbb{I}}^{\xi}$ , is an additive invariant of  $A$  given by

$$\mathbf{rank}_{\mathbb{I}}^{\xi}: (\text{mod } A)_0 \rightarrow \mathbb{Z}^{\mathbb{I}}, \quad M \mapsto c_M^{\xi}.$$

To extend the definition starting from  $K^{\oplus}(A)$ , for each  $\llbracket M \rrbracket \in K^{\oplus}(A)$  with  $\llbracket M \rrbracket = \llbracket M_+ \rrbracket - \llbracket M_- \rrbracket$  and  $M_+, M_- \in \text{mod } A$ , we set

$$\mathbf{rank}_{\mathbb{I}}^{\xi} \llbracket M \rrbracket := \mathbf{rank}_{\mathbb{I}}^{\xi} M_+ - \mathbf{rank}_{\mathbb{I}}^{\xi} M_-. \quad (5.3.6)$$

Note that since for each  $\mathbf{J} \in \mathbb{I}$ ,  $\mathbf{rank}_{\mathbf{I}}^{\xi} V_{\mathbf{J}}$  does not depend on  $\xi$  by Proposition 5.2.11, we see that for every interval-decomposable module  $M$ ,  $\mathbf{rank}_{\mathbf{I}}^{\xi} M$  does not depend on  $\xi$ , and hence we may write it  $\mathbf{rank}_{\mathbf{I}} M$ .

### 5.3.2 Formulas for interval rank invariants

Now we are in the position to give formulas for computing the interval rank invariant. Notice that the induced restriction functor  $R_{\mathbf{I}}: \text{mod } A \rightarrow \text{mod } \mathbb{k}[\mathbf{I}^{\xi}]$  carries persistence modules over  $\mathbf{P}$  to those over  $\mathbf{I}^{\xi}$ , and recall that the interval rank invariant is defined as the multiplicity inside  $\text{mod } \mathbb{k}[\mathbf{I}^{\xi}]$ , we thus obtain an explicit formula for computing the interval rank invariant under  $\xi$ . We first show the following.

**Lemma 5.3.5.** *For every  $\mathbf{I} \in \mathbb{I}$ , we have  $R_{\mathbf{I}}(V_{\mathbf{I}}) = V_{\mathbf{I}^\xi}$  as  $\mathbf{I}^\xi$ -persistence modules. we slightly abuse notation by using the same symbol  $V_{\cdot}$  for interval modules in different categories:  $V_{\mathbf{I}}$  denotes the interval module in  $\text{mod } A$ , whereas  $V_{\mathbf{I}^\xi}$  denotes the interval module in  $\text{mod } \mathbb{k}[\mathbf{I}^\xi]$ .*

*Proof.* Let  $x \in \mathbf{I}^\xi$ . Then since  $\xi_{\mathbf{I}}(x) \in \mathbf{I}$  holds by definition of  $\xi$ , we have  $R_{\mathbf{I}}(V_{\mathbf{I}})(x) = V_{\mathbf{I}}(F_{\mathbf{I}}(x)) = V_{\mathbf{I}}(\xi_{\mathbf{I}}(x)) = \mathbb{k}$ . Now let  $p_{y,x} \in \mathbb{k}[\mathbf{I}^\xi]_1$  for  $(x, y) \in [\leq]_{\mathbf{I}^\xi}$ . Then  $\xi_{\mathbf{I}}(x) \leq \xi_{\mathbf{I}}(y)$  in  $\mathbf{P}$ , where  $\xi_{\mathbf{I}}(x), \xi_{\mathbf{I}}(y) \in \mathbf{I}$ . Hence  $R_{\mathbf{I}}(V_{\mathbf{I}})(p_{y,x}) = V_{\mathbf{I}}(F_{\mathbf{I}}(p_{y,x})) = V_{\mathbf{I}}(p_{\xi_{\mathbf{I}}(y), \xi_{\mathbf{I}}(x)}) = \mathbb{k}$ . As a consequence, we have  $R_{\mathbf{I}}(V_{\mathbf{I}}) = V_{\mathbf{I}^\xi}$ .  $\square$

Definition 5.2.8 and Lemma 5.3.5 indicate that the compression multiplicity is indeed the interval multiplicity in  $\text{mod } \mathbb{k}[\mathbf{I}^\xi]$ . Thus we shall apply Theorem 3.1.26 to the setting where  $\mathbf{P} = \mathbf{I}^\xi$ , and substitute the modules  $M$  and  $V_{\mathbf{I}}$  appearing in (3.1.27) by  $R_{\mathbf{I}}(M)$  and  $R_{\mathbf{I}}(V_{\mathbf{I}}) = V_{\mathbf{I}^\xi}$ , respectively.

We recall notations provided in Notation 3.1.3 and rewrite some notations here for the case where we are now discussing.

**Notation 5.3.6.** We recall the symbol  $C_2\mathbf{T}$  for a totally ordered set  $\mathbf{T}$ , and for any  $\mathbf{a} \in C_2\mathbf{T}$  we can write  $\mathbf{a} = \{\underline{\mathbf{a}}, \overline{\mathbf{a}}\}$ . Let  $\mathbf{P} = \mathbf{I}^\xi$ , and  $\mathbf{I}^\xi$  is the full subposet of itself. Thus the following notations are given with respect to  $\mathbf{P} = \mathbf{I}^\xi$ .

- (1) Suppose  $|\text{sc}(\mathbf{I}^\xi)| = n$  (resp.  $|\text{sk}(\mathbf{I}^\xi)| = m$ ). We give a total order on the set  $\text{sc}(\mathbf{I}^\xi)$  (resp.  $\text{sk}(\mathbf{I}^\xi)$ ) by giving a poset isomorphism  $a: [n] \rightarrow \text{sc}(\mathbf{I}^\xi)$ ,  $i \mapsto a_i$  (resp.  $b: [m] \rightarrow \text{sk}(\mathbf{I}^\xi)$ ,  $i \mapsto b_i$ ), and apply the notation above for  $\mathbf{T} = \text{sc}(\mathbf{I}^\xi)$  and  $\text{sk}(\mathbf{I}^\xi)$ .
- (2) For any  $\mathbf{a} \in C_2\text{sc}(\mathbf{I}^\xi)$  (resp.  $\mathbf{b} \in C_2\text{sk}(\mathbf{I}^\xi)$ ), we set

$$\vee' \mathbf{a} := \text{sc}(\uparrow_{\mathbf{I}^\xi} \underline{\mathbf{a}} \cap \uparrow_{\mathbf{I}^\xi} \overline{\mathbf{a}}) \quad (\text{resp. } \wedge' \mathbf{b} := \text{sk}(\downarrow_{\mathbf{I}^\xi} \underline{\mathbf{b}} \cap \downarrow_{\mathbf{I}^\xi} \overline{\mathbf{b}})).$$

Note that because  $\mathbf{I}^\xi$  is both an up-set and a down-set of itself (or according to the fact that  $\mathbf{I}^\xi$  is the ambient poset), then  $\vee' \mathbf{a} \subseteq \mathbf{I}^\xi$  (resp.  $\wedge' \mathbf{b} \subseteq \mathbf{I}^\xi$ ) by the definition of up-sets (resp. down-sets). In particular, if  $|\text{sc}(\uparrow_{\mathbf{I}^\xi} \underline{\mathbf{a}} \cap \uparrow_{\mathbf{I}^\xi} \overline{\mathbf{a}})| = 1$  (resp.  $|\text{sk}(\downarrow_{\mathbf{I}^\xi} \underline{\mathbf{b}} \cap \downarrow_{\mathbf{I}^\xi} \overline{\mathbf{b}})| = 1$ ), then the unique element is the join (resp. meet) of  $\mathbf{a}$  (resp.  $\mathbf{b}$ ).

- (3) We set

$$\begin{aligned} \text{sc}_1(\mathbf{I}^\xi) &:= \bigsqcup_{\mathbf{a} \in C_2\text{sc}(\mathbf{I}^\xi)} \vee' \mathbf{a} = \{\mathbf{a}_c := (\mathbf{a}, c) \mid \mathbf{a} \in C_2\text{sc}(\mathbf{I}^\xi), c \in \vee' \mathbf{a}\}, \\ \text{sk}_1(\mathbf{I}^\xi) &:= \bigsqcup_{\mathbf{b} \in C_2\text{sk}(\mathbf{I}^\xi)} \wedge' \mathbf{b} = \{\mathbf{b}_d := (\mathbf{b}, d) \mid \mathbf{b} \in C_2\text{sk}(\mathbf{I}^\xi), d \in \wedge' \mathbf{b}\}. \end{aligned}$$

Note here that the family  $(\vee' \mathbf{a})_{\mathbf{a} \in C_2\text{sc}(\mathbf{I}^\xi)}$  (resp.  $(\wedge' \mathbf{b})_{\mathbf{b} \in C_2\text{sk}(\mathbf{I}^\xi)}$ ) does not need to be disjoint.

Furthermore, we equip  $\text{sc}_1(\mathbf{I}^\xi)$  with another total order  $\preceq_{\text{lex}}$ , defined by  $\mathbf{a}_c \preceq_{\text{lex}} \mathbf{a}'_{c'}$  if and only if  $(\underline{\mathbf{a}}, \overline{\mathbf{a}}, c) \leq_{\text{lex}} (\underline{\mathbf{a}'}, \overline{\mathbf{a}'}, c')$ , where  $\leq_{\text{lex}}$  denotes the lexicographic order from left to right. Similarly, we give a total order to  $\text{sk}_1(\mathbf{I}^\xi)$ . These total orders will be used to express matrices having  $\text{sc}_1(\mathbf{I}^\xi)$  or  $\text{sk}_1(\mathbf{I}^\xi)$  as an index set later.

We illustrate the notations in Figure 5.1. Note that there are two elements  $x$  and  $y$  in  $\vee'\{a_2, a_3\}$ , labeled as  $\{a_2, a_3\}_x$  and  $\{a_2, a_3\}_y$  in  $\text{sc}_1(\mathbf{I}^\xi)$ , respectively. The element  $w \notin \vee'\{a_2, a_3\}$  since  $w$  is not minimal in  $\uparrow_{\mathbf{I}^\xi} a_2 \cap \uparrow_{\mathbf{I}^\xi} a_3$ , but  $w$  is in both  $\vee'\{a_1, a_2\}$  and  $\vee'\{a_1, a_3\}$ , thus we write  $\{a_1, a_2\}_w$  and  $\{a_1, a_3\}_w$  in  $\text{sc}_1(\mathbf{I}^\xi)$ , standing for  $w \in \mathbf{I}^\xi$ .

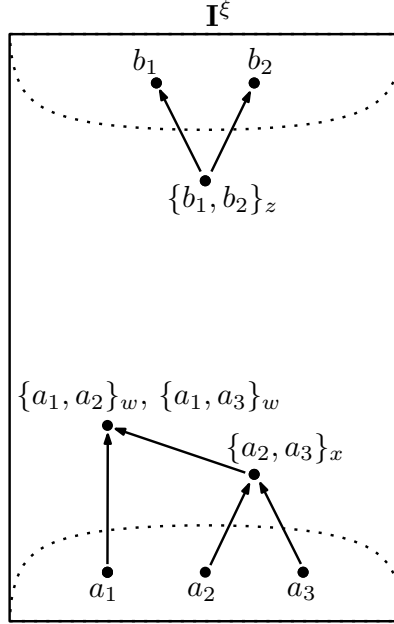


Figure 5.1: Illustration of notations in Notation 5.3.6

**Theorem 5.3.7.** *Let  $\xi = (\xi_{\mathbf{I}}: \mathbf{I}^\xi \rightarrow \mathbf{P})_{\mathbf{I} \in \mathbb{I}}$  be a compression system for  $\mathbf{P}$ , and  $M \in \text{mod } A$ . Then*

$$\text{rank}_{\mathbf{I}}^\xi M = \text{rank}_{\substack{\text{sc}_1(\mathbf{I}^\xi) \\ \text{sk}(\mathbf{I}^\xi)}}^{\substack{\text{sc}(\mathbf{I}^\xi) \quad \text{sk}_1(\mathbf{I}^\xi)} \begin{pmatrix} \tilde{M}_1 & \mathbf{0} \\ \tilde{M}_3 & \tilde{M}_5 \end{pmatrix} - \text{rank} \begin{pmatrix} \tilde{M}_1 & \mathbf{0} \\ \mathbf{0} & \tilde{M}_5 \end{pmatrix} \quad (5.3.7)$$

holds. Here the  $(\mathbf{a}_c, a)$ -entry of  $\tilde{M}_1$  (resp. the  $(b, \mathbf{b}_d)$ -entry of  $\tilde{M}_5$ ) is given by

$$(\tilde{M}_1)_{\mathbf{a}_c, a} := \begin{cases} M_{\xi_{\mathbf{I}}(c), \xi_{\mathbf{I}}(a)}, & \text{if } a = \underline{\mathbf{a}}, \\ -M_{\xi_{\mathbf{I}}(c), \xi_{\mathbf{I}}(a)}, & \text{if } a = \overline{\mathbf{a}}, \\ \mathbf{0}, & \text{if } a \notin \mathbf{a}. \end{cases} \quad \left( \text{resp. } (\tilde{M}_5)_{b, \mathbf{b}_d} := \begin{cases} M_{\xi_{\mathbf{I}}(b), \xi_{\mathbf{I}}(d)}, & \text{if } b = \underline{\mathbf{b}}, \\ -M_{\xi_{\mathbf{I}}(b), \xi_{\mathbf{I}}(d)}, & \text{if } b = \overline{\mathbf{b}}, \\ \mathbf{0}, & \text{if } b \notin \mathbf{b}. \end{cases} \right)$$

$\tilde{M}_3$  is given by a choice of pair  $(b_j, a_i) \in \text{sk}(\mathbf{I}^\xi) \times \text{sc}(\mathbf{I}^\xi)$  with  $(a_i, b_j) \in [\leq]_{\mathbf{I}^\xi}$ . Namely, the  $(b_j, a_i)$ -entry is the only nonzero entry of  $\tilde{M}_3$  and it is given by  $M_{\xi_{\mathbf{I}}(b_j), \xi_{\mathbf{I}}(a_i)}$ . Index sets of matrices in formula (5.3.7) are allowed to be empty.

*Proof.* Note that  $\text{sc}(\uparrow \mathbf{I}^\xi) = \text{sk}(\downarrow \mathbf{I}^\xi) = \emptyset$  because  $R_{\mathbf{I}}(M)$  and  $V_{\mathbf{I}^\xi}$  are modules over  $\mathbf{I}^\xi$ , and  $\mathbf{I}^\xi$  is the maximum interval of itself. Also, we note that for every  $p_{y,x} \in \mathbb{k}[\mathbf{I}^\xi]_1$  ( $((x, y) \in [\leq]_{\mathbf{I}^\xi})$ ),  $R_{\mathbf{I}}(M)(p_{y,x}) = M(F_{\mathbf{I}}(p_{y,x})) = M(p_{\xi_{\mathbf{I}}(y), \xi_{\mathbf{I}}(x)}) = M_{\xi_{\mathbf{I}}(y), \xi_{\mathbf{I}}(x)}$  holds. The assertion therefore follows immediately by Theorem 3.1.26.  $\square$

By adding some assumptions on  $\mathbf{I}^\xi$ , we will make the obtained formula (5.3.7) simpler. From now on, we set  $\text{sc}(\mathbf{I}^\xi) = \{a_1, \dots, a_n\}$  and  $\text{sk}(\mathbf{I}^\xi) = \{b_1, \dots, b_m\}$ .

**Definition 5.3.8.** The poset  $\mathbf{I}^\xi$  is said to *satisfy the existence condition of pairwise joins* in  $\text{sc}(\mathbf{I}^\xi)$  (resp. *meets* in  $\text{sk}(\mathbf{I}^\xi)$ ) if  $a_i \vee a_j$  (resp.  $b_i \wedge b_j$ ) exists in  $\mathbf{I}^\xi$  for every  $i \neq j$ . If this is the case, to shorten notation, we set  $a_{ij} := a_i \vee a_j$  (resp.  $b_{ij} := b_i \wedge b_j$ ). Note that we have  $a_{ij} = a_{ji}$  for all  $i \neq j$  in  $[n]$  (resp.  $b_{ij} = b_{ji}$  for all  $i \neq j$  in  $[m]$ ).

Under Notation 5.3.6, we have the following.

**Remark 5.3.9.** If  $\mathbf{I}^\xi$  satisfies the existence condition of pairwise join in  $\text{sc}(\mathbf{I}^\xi)$  (resp. meets in  $\text{sk}(\mathbf{I}^\xi)$ ), then we have  $\vee'\{a_i, a_j\} = \{a_{ij}\}$  for all  $i \neq j$  in  $[n]$  (resp.  $\wedge'\{b_i, b_j\} = \{b_{ij}\}$  for all  $i \neq j$  in  $[m]$ ). Hence in this case, we may write

$$\begin{aligned} \text{sc}_1(\mathbf{I}^\xi) &:= \{a_{i_1 i_2} \mid i_1, i_2 \in [n] \text{ with } i_1 < i_2\}, \\ \text{sk}_1(\mathbf{I}^\xi) &:= \{b_{i_1 i_2} \mid i_1, i_2 \in [m] \text{ with } i_1 < i_2\}. \end{aligned}$$

Namely, the ordered pair  $(i_1, i_2)$  with  $i_1 < i_2$  in the subscripts stands for the subset  $\{i_1, i_2\}$  of  $\mathbf{I}^\xi$  with cardinality 2.

**Remark 5.3.10.** If  $\mathbf{I}^\xi$  satisfies the existence condition of pairwise joins in  $\text{sc}(\mathbf{I}^\xi)$  and meets in  $\text{sk}(\mathbf{I}^\xi)$ , then the matrices  $\tilde{\mathbf{M}}_1, \tilde{\mathbf{M}}_5$  have the following forms, where we denote  $M_{\xi_I(b), \xi_I(a)}$  simply by  $M_{b,a}^{\xi_I}$  for all  $(a, b) \in [\leq]_{\mathbf{I}^\xi}$ :

$$\begin{array}{c} \begin{matrix} 12 \\ 13 \\ \vdots \\ 1n \\ 23 \\ \vdots \\ 2n \\ \vdots \\ n-1, n \end{matrix} \end{array} \begin{pmatrix} \begin{matrix} 1 & 2 & 3 & \cdots & n-1 & n \end{matrix} \\ \begin{matrix} M_{a_{12}, a_1}^{\xi_I} & -M_{a_{12}, a_2}^{\xi_I} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ M_{a_{13}, a_1}^{\xi_I} & \mathbf{0} & -M_{a_{13}, a_3}^{\xi_I} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ M_{a_{1n}, a_1}^{\xi_I} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & -M_{a_{1n}, a_n}^{\xi_I} \\ \mathbf{0} & M_{a_{23}, a_2}^{\xi_I} & -M_{a_{23}, a_3}^{\xi_I} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \mathbf{0} & M_{a_{2n}, a_2}^{\xi_I} & \mathbf{0} & \cdots & \mathbf{0} & -M_{a_{2n}, a_n}^{\xi_I} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & M_{a_{n-1}, n, a_{n-1}}^{\xi_I} & -M_{a_{n-1}, n, a_n}^{\xi_I} \end{matrix} \end{pmatrix}$$

and

$$\begin{array}{c} \begin{matrix} 1 \\ 2 \\ 3 \\ \vdots \\ m-1 \\ m \end{matrix} \end{array} \begin{pmatrix} \begin{matrix} 12 & \cdots & 1m & 23 & \cdots & 2m & \cdots & m-1, m \end{matrix} \\ \begin{matrix} M_{b_1, b_{12}}^{\xi_I} & \cdots & M_{b_1, b_{1m}}^{\xi_I} & \mathbf{0} & \cdots & \mathbf{0} & \cdots & \mathbf{0} \\ -M_{b_2, b_{12}}^{\xi_I} & \cdots & \mathbf{0} & M_{b_2, b_{23}}^{\xi_I} & \cdots & M_{b_2, b_{2m}}^{\xi_I} & \cdots & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & -M_{b_3, b_{23}}^{\xi_I} & \cdots & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \cdots & M_{b_{m-1}, b_{m-1}, m}^{\xi_I} \\ \mathbf{0} & \cdots & -M_{b_m, b_{1m}}^{\xi_I} & \mathbf{0} & \cdots & -M_{b_m, b_{2m}}^{\xi_I} & \cdots & -M_{b_m, b_{m-1}, m}^{\xi_I} \end{matrix} \end{pmatrix}$$

respectively.

**Remark 5.3.11.** Even in the case where  $\mathbf{I}^\xi$  satisfies the existence condition of pairwise joins in  $\text{sc}(\mathbf{I}^\xi)$  and meets in  $\text{sk}(\mathbf{I}^\xi)$ , the matrices above are still redundant for the formula (5.3.7) because the projective presentations appearing in the proof of formulas are not minimal in general if there are order relations between pairwise joins in  $\mathbf{I}^\xi$ . We provide the following lemma and corollary to explain this redundancy.

Let  $\{i, j\} \in C_2[n]$ . For the next lemma, we note again the fact that  $a_{ij} = a_{ji}$ . Thus for the notation  $a_{ij}$ , we do not care about the order relation between  $i$  and  $j$ , and just assume that  $i \neq j$ .

**Lemma 5.3.12.** *We keep the setting of Theorem 5.3.7 and assume the existing condition of pairwise joins in  $\text{sc}(\mathbf{I}^\xi)$  and meets in  $\text{sk}(\mathbf{I}^\xi)$ . Let  $\{i, j, k\} \subseteq [n]$ . For any distinct  $\mathbf{S}$  and  $\mathbf{T}$  in  $C_2\{i, j, k\}$ , the intersection  $\mathbf{S} \cap \mathbf{T}$  has cardinality 1. Without loss of generality, we may set  $\mathbf{S} := \{i, j\}$  and  $\mathbf{T} := \{i, k\}$  with  $\mathbf{S} \cap \mathbf{T} = \{i\}$ . Keeping this in mind, we now consider  $a_{ij}$  and  $a_{ik}$  in  $\mathbf{I}^\xi$ . Then the following are equivalent:*

- (1)  $a_{ij} \leq a_{ik}$ ;
- (2)  $a_j \leq a_{ik}$ ;
- (3)  $a_i, a_j, a_k \leq a_{ik}$ .

If one of the above holds, then the formula (5.3.7) remains valid even when  $\tilde{\mathbf{M}}_1$  in Remark 5.3.10 is replaced by the matrix obtained by deleting the  $\{i, k\}$ -th row of  $\tilde{\mathbf{M}}_1$ . The dual statement can be applied to  $\tilde{\mathbf{M}}_5$  similarly.

*Proof.* The equivalence of the three statements is trivial. Now assume that one of them holds. Then all of them hold. By (3), we have  $a_{ij} \leq a_{ik}$  and  $a_{jk} \leq a_{ik}$ . Thus there exist morphisms  $p_{a_{ik}, a_{ij}}$  and  $p_{a_{ik}, a_{jk}}$  in  $\mathbb{k}[\mathbf{I}^\xi]_1$ . The following row operations on  $\tilde{\mathbf{M}}_1$  can be done without changing ranks (to understand these operations easily, look at the  $\tilde{\mathbf{M}}_1$  in Remark 5.3.10 for  $(i, j, k) = (1, 2, 3)$ ):

- To the  $\{i, k\}$ -th row, add the row obtained from the  $\{i, j\}$ -th row by the left multiplication with  $-M_{a_{ik}, a_{ij}}^{\xi_1}$ .
- To the  $\{i, k\}$ -th row, add the row obtained from the  $\{j, k\}$ -th row by the left multiplication with  $-M_{a_{ik}, a_{jk}}^{\xi_1}$ .

By these operations, the  $\{i, k\}$ -th row of  $\tilde{\mathbf{M}}_1$  becomes zero, and thus we can delete the  $\{i, k\}$ -th row without changing the value of the right-hand side of (5.3.7).  $\square$

Utilizing Lemma 5.3.12, the formula in the 2D-grid case becomes much simpler. As an example, we exhibit the formula in the case where  $\xi = \text{tot}$ .

**Corollary 5.3.13** (Specialization to 2D-grids). *Let  $\mathbf{P}$  be a 2D-grid and let  $\xi = \text{tot}$  be the total compression system for  $\mathbf{P}$ . Take an interval  $\mathbf{I} \in \mathbb{I}$  with  $\text{sc}(\mathbf{I}) = \{a_1, \dots, a_n\}$  and  $\text{sk}(\mathbf{I}) = \{b_1, \dots, b_m\}$ . Without loss of generality, we assume that the first coordinate (i.e., the  $x$ -coordinate in Example 2.2.4 (2)) of  $a_i$  (resp.  $b_j$ ) is strictly less than that of  $a_{i+1}$  ( $i \in [n-1]$ ) (resp.  $b_{j+1}$  ( $j \in [m-1]$ )). Then we have*

$$\text{rank}_{\mathbf{I}}^{\text{tot}} M = \text{rank} \begin{bmatrix} \tilde{\mathbf{M}}_1 & \mathbf{0} \\ \tilde{\mathbf{M}}_3 & \tilde{\mathbf{M}}_5 \end{bmatrix} - \text{rank} \begin{bmatrix} \tilde{\mathbf{M}}_1 & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{M}}_5 \end{bmatrix}, \quad (5.3.8)$$

where  $\tilde{M}_1$  has the form:

$$\begin{matrix} & 1 & 2 & 3 & \cdots & n-1 & n \\ \begin{matrix} 12 \\ 23 \\ \vdots \\ n-2, n-1 \\ n-1, n \end{matrix} & \begin{pmatrix} M_{a_{12}, a_1} & -M_{a_{12}, a_2} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & M_{a_{23}, a_2} & -M_{a_{23}, a_3} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & -M_{a_{n-2}, n-1, a_{n-1}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & M_{a_{n-1}, n, a_{n-1}} & -M_{a_{n-1}, n, a_n} \end{pmatrix} \end{matrix},$$

$\tilde{M}_3$  has the form:

$$\begin{matrix} & 1 & 2 & 3 & \cdots & n-1 & n \\ \begin{matrix} 1 \\ 2 \\ \vdots \\ m-1 \\ m \end{matrix} & \begin{pmatrix} M_{b_1, a_1} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \end{pmatrix} \end{matrix},$$

and  $\tilde{M}_5$  has the form:

$$\begin{matrix} & 12 & \cdots & 1m & 23 & \cdots & 2m & \cdots & m-1, m \\ \begin{matrix} 1 \\ 2 \\ 3 \\ \vdots \\ m-1 \\ m \end{matrix} & \begin{pmatrix} M_{b_1, b_{12}} & \cdots & M_{b_1, b_{1m}} & \mathbf{0} & \cdots & \mathbf{0} & \cdots & \mathbf{0} \\ -M_{b_2, b_{12}} & \cdots & \mathbf{0} & M_{b_2, b_{23}} & \cdots & M_{b_2, b_{2m}} & \cdots & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & -M_{b_3, b_{23}} & \cdots & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \cdots & M_{b_{m-1}, b_{m-1}, m} \\ \mathbf{0} & \cdots & -M_{b_m, b_{1m}} & \mathbf{0} & \cdots & -M_{b_m, b_{2m}} & \cdots & -M_{b_m, b_{m-1}, m} \end{pmatrix} \end{matrix}.$$

**Example 5.3.14.** Let  $\xi = (\xi_I: \mathbf{I}^\xi \rightarrow \mathbf{P})_{I \in \mathbb{I}}$  be a compression system for  $\mathbf{P}$ . Fix an interval  $\mathbf{I}$  and assume that the associated poset  $\mathbf{I}^\xi$  has  $\text{sc}(\mathbf{I}^\xi) = \{a_1, a_2\}$  and  $\text{sk}(\mathbf{I}^\xi) = \{b_1, b_2\}$ . Further, assume that both the join  $x := a_1 \vee a_2$  and the meet  $y := b_1 \wedge b_2$  exist in  $\mathbf{I}^\xi$ . Since  $\text{sk}(\mathbf{I}^\xi) = \{b_1, b_2\}$ , we have  $a_1 \leq b_1$  or  $a_1 \leq b_2$ , and hence we may assume that  $a_1 \leq b_1$  without loss of generality. Then for any  $M \in \text{mod } A$  we have

$$\begin{aligned} \text{rank}_I^\xi(M) &= \text{rank} \begin{bmatrix} M_{x, a_1}^{\xi_I} & -M_{x, a_2}^{\xi_I} & \mathbf{0} \\ M_{b_1, a_1}^{\xi_I} & \mathbf{0} & M_{b_1, y}^{\xi_I} \\ \mathbf{0} & \mathbf{0} & -M_{b_2, y}^{\xi_I} \end{bmatrix} - \text{rank} [M_{x, a_1}^{\xi_I}, -M_{x, a_2}^{\xi_I}] - \text{rank} \begin{bmatrix} M_{b_1, y}^{\xi_I} \\ -M_{b_2, y}^{\xi_I} \end{bmatrix} \\ &= \text{rank} \begin{bmatrix} M_{\xi_I(x), \xi_I(a_1)} & M_{\xi_I(x), \xi_I(a_2)} & \mathbf{0} \\ M_{\xi_I(b_1), \xi_I(a_1)} & \mathbf{0} & M_{\xi_I(b_1), \xi_I(y)} \\ \mathbf{0} & \mathbf{0} & M_{\xi_I(b_2), \xi_I(y)} \end{bmatrix} - \text{rank} \begin{bmatrix} M_{\xi_I(x), \xi_I(a_1)} & M_{\xi_I(x), \xi_I(a_2)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & M_{\xi_I(b_1), \xi_I(y)} \\ \mathbf{0} & \mathbf{0} & M_{\xi_I(b_2), \xi_I(y)} \end{bmatrix}. \end{aligned}$$

### 5.3.3 Essential covers relative to compression systems

Recall that  $R_I: \text{mod } A \rightarrow \text{mod } \mathbb{k}[\mathbf{I}^\xi]$  is the restriction functor induced by  $\xi_I: \mathbf{I}^\xi \rightarrow \mathbf{P}$ , which is given in Notation 5.2.7. For every  $M \in \text{mod } A$ ,  $R_I(M) = M \circ F_I = M \circ \mathbb{k}[\xi_I]$ .

By Notation 4.1.6, Theorem 5.3.7 can be restated as follows.

**Theorem 5.3.15.** Let  $\xi = (\xi_{\mathbf{I}}: \mathbf{I}^\xi \rightarrow \mathbf{P})_{\mathbf{I} \in \mathbb{I}}$  be a compression system for  $\mathbf{P}$ . Fix an interval  $\mathbf{I}$  of  $\mathbf{P}$ . Choose any  $(b, a) \in \text{sk}(\mathbf{I}^\xi) \times \text{sc}(\mathbf{I}^\xi)$  with  $(a, b) \in [\leq]_{\mathbf{I}^\xi}$ , and set

$$\mathbf{g} := \mathbf{g}(\mathbf{c}, \mathbf{d}, (b, a)) := \left[ \begin{array}{c|c} \mathbf{g}_1 & \mathbf{0} \\ \hline \mathbf{g}_3 & \mathbf{g}_2 \end{array} \right],$$

where  $\mathbf{g}_1 := [\tilde{p}_{a, \mathbf{a}_c}]_{(\mathbf{a}_c, a) \in \text{sc}_1(\mathbf{I}^\xi) \times \text{sc}(\mathbf{I}^\xi)}$  with the entries given by

$$\tilde{p}_{a, \mathbf{a}_c} := \begin{cases} p_{c, a}, & \text{if } a = \underline{\mathbf{a}}, \\ -p_{c, a}, & \text{if } a = \overline{\mathbf{a}}, \\ \mathbf{0}, & \text{if } a \notin \mathbf{a}, \end{cases}$$

for all  $\mathbf{a}_c \in \text{sc}_1(\mathbf{I}^\xi)$  and  $a \in \text{sc}(\mathbf{I}^\xi)$ ; and  $\mathbf{g}_2 := [\hat{p}_{b, \mathbf{b}_d}]_{(\mathbf{b}_d, b) \in \text{sk}(\mathbf{I}^\xi) \times \text{sk}_1(\mathbf{I}^\xi)}$  with the entries given by

$$\hat{p}_{b, \mathbf{b}_d} := \begin{cases} p_{b, d}, & \text{if } b = \underline{\mathbf{b}}, \\ -p_{b, d}, & \text{if } b = \overline{\mathbf{b}}, \\ \mathbf{0}, & \text{if } b \notin \mathbf{b}, \end{cases}$$

for all  $b \in \text{sk}(\mathbf{I}^\xi)$  and  $\mathbf{b}_d \in \text{sk}_1(\mathbf{I}^\xi)$ ; and  $\mathbf{g}_3$  is the block matrix with the size  $|\text{sk}(\mathbf{I}^\xi)| \times |\text{sc}(\mathbf{I}^\xi)|$ , the  $(b, a)$ -entry of  $\mathbf{g}_3$ , given by  $p_{b, a}$ , is the only nonzero entry. Then for any  $M \in \text{mod } A$  we have

$$\text{rank}_{\mathbf{I}}^\xi M = \text{rank } R_{\mathbf{I}}(M)(\mathbf{g}) - \text{rank } R_{\mathbf{I}}(M)(\mathbf{g}_1) - \text{rank } R_{\mathbf{I}}(M)(\mathbf{g}_2). \quad (5.3.9)$$

We recall Definition 4.1.10 here. Let  $\zeta: \mathbf{Z} \rightarrow \mathbf{P}$  be an order-preserving map, and  $\alpha: x \rightarrow y$  a morphism in  $\bigoplus \mathbb{k}[\mathbf{P}]$ . We say that  $\zeta$  covers  $\alpha$  if there exists a morphism  $\alpha': x' \rightarrow y'$  in  $\bigoplus \mathbb{k}[\mathbf{Z}]$  such that  $\zeta(\alpha') = \alpha$  (see Example 4.1.2 for  $\zeta(\alpha')$ ).

**Definition 5.3.16.** Let  $\xi = (\xi_{\mathbf{I}}: \mathbf{I}^\xi \rightarrow \mathbf{P})_{\mathbf{I} \in \mathbb{I}}$  be a compression system for  $\mathbf{P}$ . Fix an interval  $\mathbf{I}$  of  $\mathbf{P}$  and let  $\zeta: \mathbf{Z} \rightarrow \mathbf{P}$  be an order-preserving map. Then we say that  $\zeta$  *essentially covers  $\mathbf{I}$  relative to  $\xi$*  if there exists some morphism  $\mathbf{g} := \begin{bmatrix} \mathbf{g}_1 & \mathbf{0} \\ \mathbf{g}_3 & \mathbf{g}_2 \end{bmatrix}$  in  $\bigoplus \mathbb{k}[\mathbf{I}^\xi]$  such that for any  $M \in \text{mod } A$  we have a formula

$$\text{rank}_{\mathbf{I}}^\xi M = \text{rank} \begin{bmatrix} R_{\mathbf{I}}(M)(\mathbf{g}_1) & \mathbf{0} \\ R_{\mathbf{I}}(M)(\mathbf{g}_3) & R_{\mathbf{I}}(M)(\mathbf{g}_2) \end{bmatrix} - \text{rank} \begin{bmatrix} R_{\mathbf{I}}(M)(\mathbf{g}_1) & \mathbf{0} \\ \mathbf{0} & R_{\mathbf{I}}(M)(\mathbf{g}_2) \end{bmatrix}, \quad (5.3.10)$$

and there exists an order-preserving map  $\zeta_{\mathbf{I}}: \mathbf{Z} \rightarrow \mathbf{I}^\xi$  covering  $\mathbf{g}$  and making the following diagram commute

$$\begin{array}{ccc} \mathbf{Z} & \xrightarrow{\zeta} & \mathbf{P} \\ & \searrow \zeta_{\mathbf{I}} & \nearrow \xi_{\mathbf{I}} \\ & \mathbf{I}^\xi & \end{array}.$$

We remark here that Theorem 5.3.15 guarantees that such a morphism  $\mathbf{g}$  in  $\bigoplus \mathbb{k}[\mathbf{I}^\xi]$  can always be taken such that the first condition (5.3.10) is always satisfied. We also caution the reader that  $\mathbf{g}$  is not unique in general, for example, due to the redundancy explained in Remark 5.3.11 and Lemma 5.3.12.

Now, we are in the position to provide the main theorem of this subsection, which enables one to compute the interval rank invariants by computing the multiplicity in some essential poset. From now on, the restriction functor induced by  $\zeta$  will be denoted by  $R_\zeta$ .

**Theorem 5.3.17.** *Let  $\xi = (\xi_{\mathbf{I}}: \mathbf{I}^\xi \rightarrow \mathbf{P})_{\mathbf{I} \in \mathbb{I}}$  be a compression system for  $\mathbf{P}$ . Fix an interval  $\mathbf{I}$  of  $\mathbf{P}$  and let  $\zeta: \mathbf{Z} \rightarrow \mathbf{P}$  be an order-preserving map that essentially covers  $\mathbf{I}$  relative to  $\xi$ . Then for every  $M \in \text{mod } A$  we have*

$$\text{rank}_{\mathbf{I}}^\xi M = \bar{d}_{R_\zeta(M)}(R_\zeta(V_{\mathbf{I}})), \quad (5.3.11)$$

where  $R_\zeta$  is the restriction functor induced by  $\zeta$ .

*Proof.* We set  $r = \text{rank}_{\mathbf{I}}^\xi M$  and  $s = \bar{d}_{R_\zeta(M)}(R_\zeta(V_{\mathbf{I}}))$  for convenience. By the definition of the  $\mathbf{I}$ -rank, we have the isomorphism  $R_{\mathbf{I}}(M) \cong [R_{\mathbf{I}}(V_{\mathbf{I}})]^r \oplus N$ . By Definition 5.3.16 there exists an order-preserving map  $\zeta_{\mathbf{I}}: \mathbf{Z} \rightarrow \mathbf{I}^\xi$  such that  $\zeta = \xi_{\mathbf{I}} \circ \zeta_{\mathbf{I}}$ . Applying the restriction functor  $R_{\zeta_{\mathbf{I}}}$  induced by  $\zeta_{\mathbf{I}}$  to the isomorphism above yields  $R_\zeta(M) \cong [R_\zeta(V_{\mathbf{I}})]^r \oplus R_{\zeta_{\mathbf{I}}}(N)$ . By Notation 4.1.14,  $s$  is the maximal number of copies of  $R_\zeta(V_{\mathbf{I}})$  that can be taken as a direct summand of  $R_\zeta(M)$  such that no further copies of  $R_\zeta(V_{\mathbf{I}})$  remain in the complement. This implies  $s \geq r$ .

On the other hand, since  $s = \bar{d}_{R_\zeta(M)}(R_\zeta(V_{\mathbf{I}}))$ , we can write  $R_\zeta(M) \cong [R_\zeta(V_{\mathbf{I}})]^s \oplus L$  for some module  $L$  in  $\text{mod } \mathbb{k}[\mathbf{Z}]$ . Take  $\mathbf{g} = \begin{bmatrix} \mathbf{g}_1 & \mathbf{0} \\ \mathbf{g}_3 & \mathbf{g}_2 \end{bmatrix}$  provided in Theorem 5.3.15. Because  $\zeta_{\mathbf{I}}$  covers  $\mathbf{g}$ , there exists a morphism  $\mathbf{g}^{\mathbf{Z}} = \begin{bmatrix} \mathbf{g}_1^{\mathbf{Z}} & \mathbf{0} \\ \mathbf{g}_3^{\mathbf{Z}} & \mathbf{g}_2^{\mathbf{Z}} \end{bmatrix}$  in  $\bigoplus \mathbb{k}[\mathbf{Z}]$  such that  $\zeta_{\mathbf{I}}(\mathbf{g}^{\mathbf{Z}}) := \mathbb{k}[\zeta_{\mathbf{I}}](\mathbf{g}^{\mathbf{Z}}) = \mathbf{g}$ . Then by applying Lemma 4.1.13 to the isomorphism  $R_\zeta(M) \cong [R_\zeta(V_{\mathbf{I}})]^s \oplus L$ , we have the following equalities:

$$\begin{aligned} \text{rank} \begin{bmatrix} R_\zeta(M)(\mathbf{g}_1^{\mathbf{Z}}) & \mathbf{0} \\ R_\zeta(M)(\mathbf{g}_3^{\mathbf{Z}}) & R_\zeta(M)(\mathbf{g}_2^{\mathbf{Z}}) \end{bmatrix} &= s \text{rank} \begin{bmatrix} R_\zeta(V_{\mathbf{I}})(\mathbf{g}_1^{\mathbf{Z}}) & \mathbf{0} \\ R_\zeta(V_{\mathbf{I}})(\mathbf{g}_3^{\mathbf{Z}}) & R_\zeta(V_{\mathbf{I}})(\mathbf{g}_2^{\mathbf{Z}}) \end{bmatrix} \\ &+ \text{rank} \begin{bmatrix} L(\mathbf{g}_1^{\mathbf{Z}}) & \mathbf{0} \\ L(\mathbf{g}_3^{\mathbf{Z}}) & L(\mathbf{g}_2^{\mathbf{Z}}) \end{bmatrix}, \end{aligned} \quad (5.3.12)$$

and

$$\begin{aligned} \text{rank} \begin{bmatrix} R_\zeta(M)(\mathbf{g}_1^{\mathbf{Z}}) & \mathbf{0} \\ \mathbf{0} & R_\zeta(M)(\mathbf{g}_2^{\mathbf{Z}}) \end{bmatrix} &= s \text{rank} \begin{bmatrix} R_\zeta(V_{\mathbf{I}})(\mathbf{g}_1^{\mathbf{Z}}) & \mathbf{0} \\ \mathbf{0} & R_\zeta(V_{\mathbf{I}})(\mathbf{g}_2^{\mathbf{Z}}) \end{bmatrix} \\ &+ \text{rank} \begin{bmatrix} L(\mathbf{g}_1^{\mathbf{Z}}) & \mathbf{0} \\ \mathbf{0} & L(\mathbf{g}_2^{\mathbf{Z}}) \end{bmatrix}. \end{aligned} \quad (5.3.13)$$

Note that  $R_\zeta(M) = R_{\zeta_{\mathbf{I}}}(R_{\mathbf{I}}(M)) = R_{\mathbf{I}}(M) \circ \mathbb{k}[\zeta_{\mathbf{I}}]$  for all  $M \in \text{mod } A$ . Then (5.3.12) and (5.3.13) become

$$\begin{aligned} \text{rank} \begin{bmatrix} R_{\mathbf{I}}(M)(\mathbf{g}_1) & \mathbf{0} \\ R_{\mathbf{I}}(M)(\mathbf{g}_3) & R_{\mathbf{I}}(M)(\mathbf{g}_2) \end{bmatrix} &= s \text{rank} \begin{bmatrix} R_{\mathbf{I}}(V_{\mathbf{I}})(\mathbf{g}_1) & \mathbf{0} \\ R_{\mathbf{I}}(V_{\mathbf{I}})(\mathbf{g}_3) & R_{\mathbf{I}}(V_{\mathbf{I}})(\mathbf{g}_2) \end{bmatrix} \\ &+ \text{rank} \begin{bmatrix} L(\mathbf{g}_1^{\mathbf{Z}}) & \mathbf{0} \\ L(\mathbf{g}_3^{\mathbf{Z}}) & L(\mathbf{g}_2^{\mathbf{Z}}) \end{bmatrix}, \end{aligned} \quad (5.3.14)$$

and

$$\begin{aligned} \text{rank} \begin{bmatrix} R_{\mathbf{I}}(M)(\mathbf{g}_1) & \mathbf{0} \\ \mathbf{0} & R_{\mathbf{I}}(M)(\mathbf{g}_2) \end{bmatrix} &= s \text{rank} \begin{bmatrix} R_{\mathbf{I}}(V_{\mathbf{I}})(\mathbf{g}_1) & \mathbf{0} \\ \mathbf{0} & R_{\mathbf{I}}(V_{\mathbf{I}})(\mathbf{g}_2) \end{bmatrix} \\ &+ \text{rank} \begin{bmatrix} L(\mathbf{g}_1^{\mathbf{Z}}) & \mathbf{0} \\ \mathbf{0} & L(\mathbf{g}_2^{\mathbf{Z}}) \end{bmatrix}. \end{aligned} \quad (5.3.15)$$

Put  $M = V_{\mathbf{I}}$  in the formula (5.3.9). Thus we have

$$d_{R_{\mathbf{I}}(V_{\mathbf{I}})}(R_{\mathbf{I}}(V_{\mathbf{I}})) = \text{rank} \begin{bmatrix} R_{\mathbf{I}}(V_{\mathbf{I}})(\mathbf{g}_1) & \mathbf{0} \\ R_{\mathbf{I}}(V_{\mathbf{I}})(\mathbf{g}_3) & R_{\mathbf{I}}(V_{\mathbf{I}})(\mathbf{g}_2) \end{bmatrix} - \text{rank} \begin{bmatrix} R_{\mathbf{I}}(V_{\mathbf{I}})(\mathbf{g}_1) & \mathbf{0} \\ \mathbf{0} & R_{\mathbf{I}}(V_{\mathbf{I}})(\mathbf{g}_2) \end{bmatrix}. \quad (5.3.16)$$

Noticing the equalities (5.3.9) and (5.3.16), the formula (5.3.14) minus the formula (5.3.15) implies

$$\begin{aligned} r &= s \cdot d_{R_{\mathbf{I}}(V_{\mathbf{I}})}(R_{\mathbf{I}}(V_{\mathbf{I}})) + \text{rank} \begin{bmatrix} L(\mathbf{g}_1^{\mathbf{Z}}) & \mathbf{0} \\ L(\mathbf{g}_3^{\mathbf{Z}}) & L(\mathbf{g}_2^{\mathbf{Z}}) \end{bmatrix} - \text{rank} \begin{bmatrix} L(\mathbf{g}_1^{\mathbf{Z}}) & \mathbf{0} \\ \mathbf{0} & L(\mathbf{g}_2^{\mathbf{Z}}) \end{bmatrix} \\ &= s + \text{rank} \begin{bmatrix} L(\mathbf{g}_1^{\mathbf{Z}}) & \mathbf{0} \\ L(\mathbf{g}_3^{\mathbf{Z}}) & L(\mathbf{g}_2^{\mathbf{Z}}) \end{bmatrix} - \text{rank} \begin{bmatrix} L(\mathbf{g}_1^{\mathbf{Z}}) & \mathbf{0} \\ \mathbf{0} & L(\mathbf{g}_2^{\mathbf{Z}}) \end{bmatrix} \geq s. \end{aligned}$$

Hence we have  $r = s$ , and the proof is completed.  $\square$

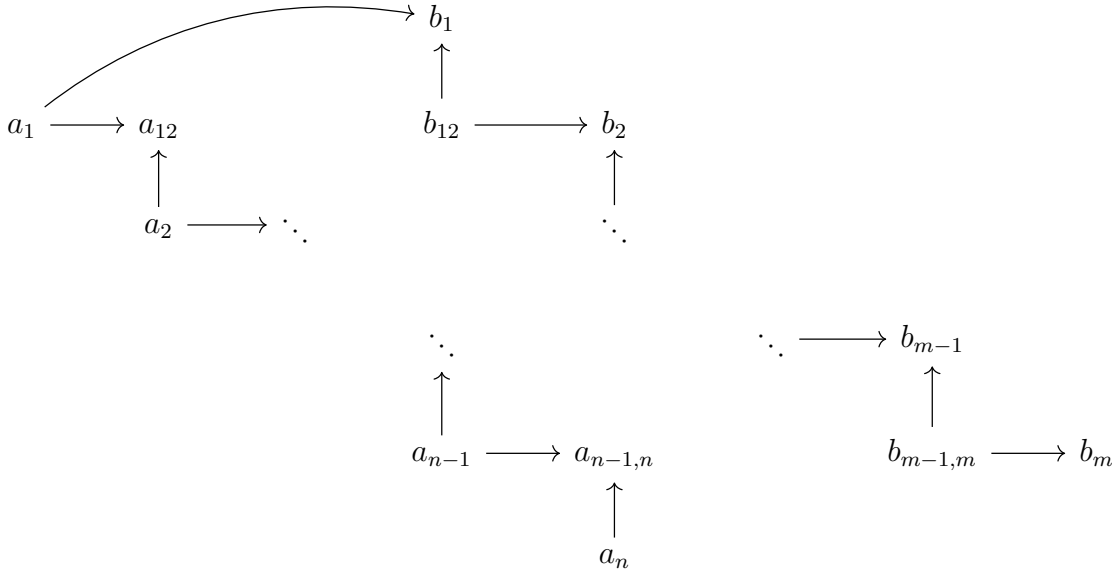
By using the essential cover relative to the total compression system on the 2D-grid, we can easily find zigzag posets essentially covering all intervals of the 2D-grid. Indeed, we first recall the notations for pairwise joins and meets in Definition 5.3.8. For each  $\mathbf{I}^{\text{tot}} = \mathbf{I} \in \mathbb{I}$  with  $\text{sc}(\mathbf{I}) = \{a_1, \dots, a_n\}$  and  $\text{sk}(\mathbf{I}) = \{b_1, \dots, b_m\}$ , we assume that the first coordinate of  $a_i$  (resp.  $b_j$ ) is strictly less than that of  $a_{i+1}$  ( $i \in [n-1]$ ) (resp.  $b_{j+1}$  ( $j \in [m-1]$ )), and we assign a (not full) subposet  $\mathbf{I}^{\text{zz}}$  of  $\mathbf{I}$  with elements

$$\text{sc}(\mathbf{I}) \cup \{a_{i,i+1} \mid i \in [n-1]\} \cup \text{sk}(\mathbf{I}) \cup \{b_{j,j+1} \mid j \in [m-1]\}$$

and the order relation is partially inherited from  $\mathbf{I}$ :

$$a_i \leq a_{i,i+1}, a_{i+1} \leq a_{i,i+1}, b_j \leq b_{j,j+1}, b_{j+1} \leq b_{j,j+1}, \text{ and } a_1 \leq b_1 \ (i \in [n-1], j \in [m-1]).$$

It is clear that the poset  $\mathbf{I}^{\text{zz}}$  has the following Hasse quiver:



Set  $\text{zz}_{\mathbf{I}}: \mathbf{I}^{\text{zz}} \hookrightarrow \mathbf{I} \hookrightarrow \mathbf{P}$  to be the usual inclusion map. It is not difficult to check that the family  $(\text{zz}_{\mathbf{I}}: \mathbf{I}^{\text{zz}} \hookrightarrow \mathbf{P})_{\mathbf{I} \in \mathbb{I}}$  is a compression system. We denote this compression system by  $\text{zz}$ .

By Theorem 5.3.17 we can show the following.

**Corollary 5.3.18.** *Let  $\mathbf{P}$  be the 2D-grid, and we let  $zz = (zz_{\mathbf{I}}: \mathbf{I}^{zz} \hookrightarrow \mathbf{P})_{\mathbf{I} \in \mathbb{I}}$  be the compression system defined above, and  $\text{tot}$  the total compression system. Then interval rank invariants under  $zz$  and  $\text{tot}$  coincide, i.e.,*

$$\mathbf{rank}_{\mathbb{I}}^{\text{tot}} = \mathbf{rank}_{\mathbb{I}}^{zz}.$$

*Proof.* We show that for every  $\mathbf{I} \in \mathbb{I}$ ,  $zz_{\mathbf{I}}$  essentially covers  $\mathbf{I}$  relative to  $\text{tot}$ . By Corollary 5.3.13 and Theorem 5.3.15, there exists a morphism  $\mathbf{g} = \begin{bmatrix} \mathbf{g}_1 & \mathbf{0} \\ \mathbf{g}_3 & \mathbf{g}_2 \end{bmatrix}$  in  $\bigoplus \mathbb{k}[\mathbf{I}]$  such that (5.3.10) holds. Here  $\mathbf{g}_1$  has the form:

$$\begin{array}{c} 12 \\ 23 \\ \vdots \\ n-2, n-1 \\ n-1, n \end{array} \begin{pmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ p_{a_{12}, a_1} & -p_{a_{12}, a_2} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & p_{a_{23}, a_2} & -p_{a_{23}, a_3} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & -p_{a_{n-2}, n-1, a_{n-1}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & p_{a_{n-1}, n, a_{n-1}} & -p_{a_{n-1}, n, a_n} \end{pmatrix},$$

$\mathbf{g}_3$  has the form:

$$\begin{array}{c} 1 \\ 2 \\ \vdots \\ m-1 \\ m \end{array} \begin{pmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ p_{b_1, a_1} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \end{pmatrix},$$

and  $\mathbf{g}_2$  has the form:

$$\begin{array}{c} 1 \\ 2 \\ 3 \\ \vdots \\ m-1 \\ m \end{array} \begin{pmatrix} 12 & \cdots & 1m & 23 & \cdots & 2m & \cdots & m-1, m \\ p_{b_1, b_{12}} & \cdots & p_{b_1, b_{1m}} & \mathbf{0} & \cdots & \mathbf{0} & \cdots & \mathbf{0} \\ -p_{b_2, b_{12}} & \cdots & \mathbf{0} & p_{b_2, b_{23}} & \cdots & p_{b_2, b_{2m}} & \cdots & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & -p_{b_3, b_{23}} & \cdots & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \cdots & p_{b_{m-1}, b_{m-1}, m} \\ \mathbf{0} & \cdots & -p_{b_m, b_{1m}} & \mathbf{0} & \cdots & -p_{b_m, b_{2m}} & \cdots & -p_{b_m, b_{m-1}, m} \end{pmatrix}.$$

From the definition of  $zz_{\mathbf{I}}$ , we naturally have the following commutative diagram:

$$\begin{array}{ccc} \mathbf{I}^{zz} & \xrightarrow{zz_{\mathbf{I}}} & \mathbf{P} \\ & \searrow \zeta_{\mathbf{I}} & \nearrow \text{tot}_{\mathbf{I}} = \iota_{\mathbf{I}} \\ & \mathbf{I}^{\text{tot}} = \mathbf{I} & \end{array}$$

for the inclusion map  $\zeta_{\mathbf{I}}: \mathbf{I}^{zz} \rightarrow \mathbf{I}$ , and  $\zeta_{\mathbf{I}}$  covers  $\mathbf{g}$  because  $\zeta_{\mathbf{I}}(\mathbf{g}) = \mathbf{g}$ . We abuse the notation  $\mathbf{g}$  since  $\zeta_{\mathbf{I}}$  is the inclusion, and we remark that the first  $\mathbf{g}$  is a morphism in  $\bigoplus \mathbb{k}[\mathbf{I}^{zz}]$ . Thus for every  $M \in \text{mod } A$ ,

$$\mathbf{rank}_{\mathbf{I}}^{\text{tot}} M = \bar{d}_{R_{zz_{\mathbf{I}}}(M)}(R_{zz_{\mathbf{I}}}(V_{\mathbf{I}}))$$

by Theorem 5.3.17. Notice that  $R_{zz\mathbf{I}}(V_{\mathbf{I}}) = V_{zz\mathbf{I}}$  is an interval module in  $\text{mod } \mathbb{k}[\mathbf{I}^{zz}]$ , hence an indecomposable module by Lemma 5.3.5. It follows that

$$\bar{d}_{R_{zz\mathbf{I}}(M)}(R_{zz\mathbf{I}}(V_{\mathbf{I}})) = d_{R_{zz\mathbf{I}}(M)}(R_{zz\mathbf{I}}(V_{\mathbf{I}})) = \mathbf{rank}_{\mathbf{I}}^{zz} M.$$

Therefore, the assertion follows.  $\square$

**Remark 5.3.19.** Corollary 5.3.18 above gives an alternative proof of Theorem in [31, Theorem 3.12] by Dey–Kim–Mémoli for the case where  $\mathbf{P}$  is a 2D-grid because the interval rank invariant  $\mathbf{rank}_{\mathbf{I}}^{\text{tot}}$  coincides with their generalized rank invariant **GRI**. The latter statement follows by [24, Lemma 3.1], but the description of the proof was imprecise, and in the process of making it accurate we found a small gap in the proof. Therefore, we give a precise proof of it by filling the gap below.

We first review the definition of the generalized rank.

**Definition 5.3.20.** Let  $\mathbf{I}$  be a finite connected poset, and  $M \in \text{mod } \mathbb{k}[\mathbf{I}]$ . Since  $\mathbf{I}$  is finite and  $M \in \text{mod } \mathbb{k}[\mathbf{I}]$ , both  $\varprojlim M$  and  $\varinjlim M$  are easily constructed in  $\text{mod } \mathbb{k}$ . By definition, we have a commutative diagram

$$\begin{array}{ccccc} & & M(x) & & \\ & \nearrow \pi_x & \downarrow M(p_{y,x}) & \searrow \sigma_x & \\ \varprojlim M & & & & \varinjlim M \\ & \searrow \pi_y & \downarrow & \nearrow \sigma_y & \\ & & M(y) & & \end{array}$$

for any  $(x, y) \in [\leq]_{\mathbf{I}}$ , which shows that for any  $x, y \in \mathbf{I}$ , we have  $\sigma_x \pi_x = \sigma_y \pi_y$  if  $x$  and  $y$  are in the same connected component of  $\mathbf{I}$ . But since  $\mathbf{I}$  is connected, the equality above holds for all  $x, y \in \mathbf{I}$ . The common linear map is denoted by  $\Theta_M: \varprojlim M \rightarrow \varinjlim M$ .

Now, for a (locally finite) poset  $\mathbf{P}$ , a finite interval subposet  $\mathbf{I}$  of  $\mathbf{P}$ , and  $M \in \text{mod } \mathbb{k}[\mathbf{P}]$ , the rank of the linear map  $\Theta_{R(M)}$  for the module  $R(M) \in \text{mod } \mathbb{k}[\mathbf{I}]$  is called the *generalized rank* of  $M$  at  $\mathbf{I}$ .

We now give a proof of [24, Lemma 3.1] below.

**Lemma 5.3.21.** *Let  $\mathbf{I}$  be a connected poset, and  $M \in \text{mod } \mathbb{k}[\mathbf{I}]$ . Then  $M$  has a direct sum decomposition*

$$M \cong V_{\mathbf{I}}^s \oplus N$$

*as  $\mathbb{k}[\mathbf{I}]$ -modules for some  $N$ , where  $\text{rank } \Theta_M = s$ ,  $\text{rank } \Theta_N = 0$ . Hence in particular, we have  $d_M(V_{\mathbf{I}}) = \text{rank } \Theta_M$ .*

*Proof.* There exist some vector subspaces  $P \subseteq \varprojlim M$  and  $T \subseteq \varinjlim M$  such that  $P \oplus \text{Ker } \Theta_M = \varprojlim M$  and  $\text{Im } \Theta_M \oplus T = \varinjlim M$ . Let  $\sigma: P \rightarrow \varinjlim M$  be the inclusion and  $\pi: \varinjlim M \rightarrow \text{Im } \Theta_M$  the projection with respect to this decomposition. We

set  $\phi_x := \pi_x \sigma$ ,  $\rho_x := \pi \sigma_x$  for all  $x \in \mathbf{I}$ . Then we have the following commutative diagram:

$$\begin{array}{ccccccc}
 & & & M(x) & & & \\
 & \nearrow \phi_x & & \downarrow \pi_x & \searrow \sigma_x & \nearrow \rho_x & \\
 P & \xrightarrow{\sigma} & \varprojlim M & \xrightarrow{\Theta_M} & \varinjlim M & \xrightarrow{\pi} & \text{Im } \Theta_M \\
 & \searrow \phi_y & & \downarrow \pi_y & \nearrow \sigma_y & \searrow \rho_y & \\
 & & & M(y) & & & 
 \end{array}$$

Since  $\Theta_M$  restricts to an isomorphism  $\mu': P \rightarrow \text{Im } \Theta_M$ , we have  $\mu' = \pi \Theta_M \sigma$ . Thus  $\dim P = \dim \text{Im } \Theta_M = \text{rank } \Theta_M$ . Set  $s := \text{rank } \Theta_M$ , the common value. Then there exists an isomorphism  $\alpha: \mathbb{k}^s \rightarrow P$ , which gives an isomorphism  $\beta := (\mu' \alpha)^{-1}: \text{Im } \Theta_M \rightarrow \mathbb{k}^s$ . Set  $\phi'_x := \phi_x \alpha: \mathbb{k}^s \rightarrow M(x)$  and  $\rho'_x := \beta \rho_x: M(x) \rightarrow \mathbb{k}^s$ . Then  $\text{Coker } \phi'_x = \text{Coker } \phi_x$ ,  $\text{Ker } \rho'_x = \text{Ker } \rho_x$  for all  $x \in \mathbf{I}$ , and we have the following two commutative diagrams with exact rows:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathbb{k}^s & \xrightarrow{\phi'_x} & M(x) & \xrightarrow{\psi_x} & \text{Coker } \phi_x \longrightarrow 0 \\
 & & \parallel & & \downarrow M(p_{y,x}) & & \downarrow f_{y,x} \\
 0 & \longrightarrow & \mathbb{k}^s & \xrightarrow{\phi'_y} & M(y) & \xrightarrow{\psi_y} & \text{Coker } \phi_y \longrightarrow 0
 \end{array} , \text{ and} \tag{5.3.17}$$

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \text{Ker } \rho_x & \xrightarrow{\tau_x} & M(x) & \xrightarrow{\rho'_x} & \mathbb{k}^s \longrightarrow 0 \\
 & & \downarrow g_{y,x} & & \downarrow M(p_{y,x}) & & \parallel \\
 0 & \longrightarrow & \text{Ker } \rho_y & \xrightarrow{\tau_y} & M(y) & \xrightarrow{\rho'_y} & \mathbb{k}^s \longrightarrow 0
 \end{array} ,$$

where the vertical map  $f_{y,x}$  (resp.  $g_{y,x}$ ) is the unique linear map making the diagram commutative, and  $\tau_z: \text{Ker } \rho_z \rightarrow M(z)$  is the inclusion map for all  $z \in \mathbf{I}$ . The uniqueness of  $f_{y,x}$  (resp.  $g_{y,x}$ ) for all  $(x, y) \in [\leq]_{\mathbf{I}}$  defines a  $\mathbb{k}[\mathbf{I}]$ -module  $C$  (resp.  $K$ ) by setting  $C(x) := \text{Coker } \phi_x$ ,  $C(p_{y,x}) := f_{y,x}$  (resp.  $K(x) := \text{Ker } \rho_x$ ,  $K(p_{y,x}) := g_{y,x}$ ) for all  $x \in \mathbf{I}$ ,  $(x, y) \in [\leq]_{\mathbf{I}}$ . Set  $\gamma := (\gamma_x)_{x \in \mathbf{I}}$  for all  $\gamma \in \{\phi', \psi, \tau, \rho'\}$ . Then the commutative diagrams above show that  $\phi', \psi, \tau, \rho'$  are morphisms of  $\mathbb{k}[\mathbf{I}]$ -modules, and give us the following short exact sequences of  $\mathbb{k}[\mathbf{I}]$ -modules:

$$0 \rightarrow V_{\mathbf{I}}^s \xrightarrow{\phi'} M \xrightarrow{\psi} C \rightarrow 0 \text{ and } 0 \rightarrow K \xrightarrow{\tau} M \xrightarrow{\rho'} V_{\mathbf{I}}^s \rightarrow 0. \tag{5.3.18}$$

We claim  $\rho' \phi' = \mathbb{1}_{V_{\mathbf{I}}}$ . Indeed, for each  $x \in \mathbf{I}$ , we have  $\rho'_x \phi'_x = \beta \rho_x \phi_x \alpha = \beta \mu' \alpha = \mathbb{1}_{\mathbb{k}^s}$ . As a consequence, the short exact sequences above split, and hence  $M$  has direct sum decompositions  $V_{\mathbf{I}}^s \oplus C \cong M (\cong K \oplus V_{\mathbf{I}}^s)$  as  $\mathbb{k}[\mathbf{I}]$ -modules. By the additivity of both  $\varprojlim$  and  $\varinjlim$ , we have  $\text{rank } \Theta_M = s \text{rank } \Theta_{V_{\mathbf{I}}} + \text{rank } \Theta_C$ . Here note that  $\Theta_{V_{\mathbf{I}}}$  is given by the identity  $\mathbb{1}_{\mathbb{k}}: \mathbb{k} \rightarrow \mathbb{k}$ , thus  $\text{rank } \Theta_{V_{\mathbf{I}}} = 1$ , which together with  $\text{rank } \Theta_M = s$  shows that  $\text{rank } \Theta_C = 0$ . Therefore the assertion holds for  $N := C$ .

Note that  $N$  does not have direct summand isomorphic to  $V_{\mathbf{I}}$  because  $\text{rank } \Theta_N = 0$ . Hence we have  $d_M(V_{\mathbf{I}}) = s = \text{rank } \Theta_M$ .  $\square$

**Remark 5.3.22.** In the proof of [24, Lemma 3.1], the authors said that the decomposition  $M(x) \cong P \oplus \text{Coker } \phi_x$  is preserved by  $M(p_{y,x})$ , which is equivalent

to the existence of the commutative diagram with exact rows with a unique morphism  $f_{y,x}$  in (5.3.17). They continued to say that this fact establishes a direct sum  $M \cong V_{\mathbf{I}}^s \oplus C$ . This assertion is obvious as vector spaces, but as  $\mathbb{k}[\mathbf{I}]$ -modules it is not clear. This fact was not proved in their paper. Namely, the missing part is to show that the exact sequence in (5.3.18) on the left splits over  $\mathbb{k}[\mathbf{I}]$ . For this, we need one more exact sequence in (5.3.18) on the right that serves us the necessary retraction  $\rho'$  for  $\phi'$ .

**Example 5.3.23.** Consider a poset  $\mathbf{P}$  having the following Hasse quiver:

$$\begin{array}{ccccc} & & 4 & & \\ & & \downarrow & & \\ 1 & \longrightarrow & 2 & \longrightarrow & 3 \end{array}.$$

Let  $M_1$  and  $M_2$  be two  $\mathbf{P}$ -persistence modules given by

$$M_1 := \begin{array}{ccccc} & & \mathbb{k} & & \\ & & \downarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix} & & \\ \mathbb{k} & \xrightarrow{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} & \mathbb{k}^2 & \xrightarrow{\begin{bmatrix} 1, 0 \end{bmatrix}} & \mathbb{k} \end{array} \quad \text{and} \quad M_2 := \begin{array}{ccccc} & & \mathbb{k} & & \\ & & \downarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix} & & \\ \mathbb{k} & \xrightarrow{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} & \mathbb{k}^2 & \xrightarrow{\begin{bmatrix} 1, 1 \end{bmatrix}} & \mathbb{k} \end{array}.$$

We compute the  $\mathbf{P}$ -rank of both modules under  $\xi = \text{tot}$ . In this case,  $\mathbf{I} = \mathbf{P} = \mathbf{I}^{\text{tot}}$ .

By Theorem 5.3.15, there exists a morphism  $\mathbf{g} = \begin{bmatrix} \mathbf{g}_1 & \mathbf{0} \\ \mathbf{g}_3 & \mathbf{g}_2 \end{bmatrix}$  in  $\bigoplus \mathbb{k}[\mathbf{P}]$  such that (5.3.10) holds. Here  $\mathbf{g}$  has the form:

$$\mathbf{g} := \left[ \begin{array}{c|c} \mathbf{g}_1 & \mathbf{0} \\ \hline \mathbf{g}_3 & \mathbf{g}_2 \end{array} \right] = \left[ \begin{array}{c|c} p_{2,1} & -p_{2,4} \\ \hline p_{3,1} & \mathbf{0} \end{array} \right].$$

Hence, it is now clear that if we take the (not full) subposet  $\mathbf{Z}$  of  $\mathbf{P}$  given by

$$\mathbf{Z} := \begin{array}{ccccc} & & 4 & & \\ & & \downarrow & & \\ 1 & \longrightarrow & 2 & \longrightarrow & 3 \end{array},$$

then the inclusion map  $\iota: \mathbf{Z} \hookrightarrow \mathbf{P}$  essentially covers  $\mathbf{P}$  relative to  $\text{tot}$ . By Theorem 5.3.17 it suffices to compute  $\bar{d}_{R_\iota(M_j)}(R_\iota(V_{\mathbf{I}})) = d_{R_\iota(M_j)}(V_{\mathbf{Z}})$  for  $j \in \{1, 2\}$ . Now, because

$$R_\iota(M_1) = \begin{array}{ccccc} & & \mathbb{k} & & \\ & & \downarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix} & & \\ \mathbb{k} & \xrightarrow{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} & \mathbb{k}^2 & \xrightarrow{\begin{bmatrix} 1, 0 \end{bmatrix}} & \mathbb{k} \end{array} \cong \begin{array}{ccccc} & & \mathbb{k} & & \\ & & \downarrow 1 & & \\ \mathbb{k} & \xrightarrow{1} & \mathbb{k} & \xrightarrow{\quad} & \mathbb{k} \end{array} \oplus \begin{array}{ccccc} & & 0 & & \\ & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{k} & \longrightarrow & 0 \end{array}$$

and

$$R_\iota(M_2) = \begin{array}{ccccc} & & \mathbb{k} & & \\ & & \downarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix} & & \\ \mathbb{k} & \xrightarrow{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} & \mathbb{k}^2 & \xrightarrow{\begin{bmatrix} 1, 1 \end{bmatrix}} & \mathbb{k} \end{array} \cong \begin{array}{ccccc} & & 0 & & \\ & & \downarrow & & \\ \mathbb{k} & \xrightarrow{1} & \mathbb{k} & \xrightarrow{\quad} & \mathbb{k} \end{array} \oplus \begin{array}{ccccc} & & \mathbb{k} & & \\ & & \downarrow 1 & & \\ 0 & \longrightarrow & \mathbb{k} & \longrightarrow & 0 \end{array},$$

we conclude that  $\mathbf{rank}_{\mathbf{P}}^{\text{tot}} M_1 = 1$ , but  $\mathbf{rank}_{\mathbf{P}}^{\text{tot}} M_2 = 0$ .

**Example 5.3.24.** Consider a poset  $\mathbf{P}$  having the following Hasse quiver:

$$\begin{array}{ccccc} & & 4 & & \\ & & \uparrow & & \\ 1 & \longleftarrow & 2 & \longrightarrow & 3 \end{array}.$$

Let  $M$  be a  $\mathbf{P}$ -persistence module given by

$$M := \begin{array}{ccccc} & & \mathbb{k} & & \\ & & \uparrow & & \\ & [1, 0, 0] & & & \\ \mathbb{k} & \xleftarrow{[1, 0, 0]} & \mathbb{k}^3 & \xrightarrow{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}} & \mathbb{k}^2 \end{array}.$$

We compute the  $\mathbf{P}$ -rank of  $M$  under  $\xi = \text{tot}$ . Again in this case,  $\mathbf{I} = \mathbf{P} = \mathbf{I}^{\text{tot}}$ .

By Theorem 5.3.15, there exists a morphism  $\mathbf{g}$  in  $\bigoplus \mathbb{k}[\mathbf{P}]$  such that (5.3.10) holds. Here we first take  $\mathbf{g}$  to be:

$$\mathbf{g} := [\mathbf{g}_3 \mid \mathbf{g}_2] = \left[ \begin{array}{c|ccc} \mathbf{0} & p_{1,2} & p_{1,2} & \mathbf{0} \\ p_{3,2} & -p_{3,2} & \mathbf{0} & p_{3,2} \\ \mathbf{0} & \mathbf{0} & -p_{4,2} & -p_{4,2} \end{array} \right]. \quad (5.3.19)$$

Notice that the last column of  $\mathbf{g}_2$  is the linear combination of its first two columns, hence we may take another morphism  $\tilde{\mathbf{g}}$  in  $\bigoplus \mathbb{k}[\mathbf{P}]$  given by

$$\tilde{\mathbf{g}} := [\mathbf{g}_3 \mid \tilde{\mathbf{g}}_2] = \left[ \begin{array}{c|cc} \mathbf{0} & p_{1,2} & p_{1,2} \\ p_{3,2} & -p_{3,2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -p_{4,2} \end{array} \right],$$

such that  $\text{rank } M(\mathbf{g}) - \text{rank } M(\mathbf{g}_2) = \text{rank } M(\tilde{\mathbf{g}}) - \text{rank } M(\tilde{\mathbf{g}}_2)$ . This shows that the new morphism  $\tilde{\mathbf{g}}$  satisfies the formula (5.3.10) in Definition 5.3.16.

Now, let us take the following finite zigzag poset

$$\mathbf{Z} := \begin{array}{ccccccc} & 2 & & 2' & & 2'' & \\ & \searrow & & \swarrow & \searrow & \swarrow & \searrow \\ & 3 & & & 1 & & 4 \end{array}$$

and define the order-preserving map  $\zeta: \mathbf{Z} \rightarrow \mathbf{P}$  by

$$\zeta(x) := \begin{cases} 2, & \text{if } x \in \{2, 2', 2''\}, \\ x, & \text{if } x \in \{1, 3, 4\}. \end{cases}$$

Then  $\zeta$  essentially covers  $\mathbf{P}$  relative to  $\text{tot}$ . Indeed, we have the following equality:

$$\mathbb{k}[\zeta] \left( \left[ \begin{array}{c|ccc} \mathbf{0} & p_{1,2'} & p_{1,2''} & \\ p_{3,2} & -p_{3,2'} & \mathbf{0} & \\ \mathbf{0} & \mathbf{0} & -p_{4,2''} & \end{array} \right] \right) = \left[ \begin{array}{c|ccc} \mathbf{0} & p_{1,2} & p_{1,2} & \\ p_{3,2} & -p_{3,2} & \mathbf{0} & \\ \mathbf{0} & \mathbf{0} & -p_{4,2} & \end{array} \right]$$

Hence by Theorem 5.3.17 it suffices to compute  $\bar{d}_{R_\zeta(M)}(R_\zeta(V_{\mathbf{I}})) = d_{R_\iota(M_j)}(V_{\mathbf{Z}})$ . Now, because

$$\begin{aligned}
 R_\zeta(M) &= \begin{array}{ccccc}
 \mathbb{k}^3 & & \mathbb{k}^3 & & \mathbb{k}^3 \\
 & \searrow^{[1 \ 0 \ 0]} & \swarrow_{[0 \ 1 \ 1]} & \searrow^{[1, 0, 0]} & \swarrow_{[1, 0, 0]} \\
 [1 \ 0 \ 0] & & \mathbb{k}^2 & & \mathbb{k} & & \mathbb{k} \\
 [0 \ 1 \ 1] & & & & & & \\
 & \swarrow_{[1, 0, 0]} & & \swarrow_{[1, 0, 0]} & & \swarrow_{[1, 0, 0]} & \\
 & \mathbb{k} & & \mathbb{k} & & \mathbb{k} & 
 \end{array} \\
 &\cong [1 \ 1 \ 1 \ 1 \ 1] \oplus [1 \ 1 \ 1 \ 0 \ 0] \oplus [1 \ 0 \ 0 \ 0 \ 0] \oplus [0 \ 1 \ 0 \ 0 \ 0] \oplus [0 \ 0 \ 0 \ 1 \ 0]^2,
 \end{aligned}$$

we conclude that  $\mathbf{rank}_{\mathbf{P}}^{\text{tot}} M = 1$ .

We highlight that in the example above, finding a new morphism  $\tilde{\mathbf{g}}$  such that the formula (4.1.2) holds is crucial for finding the zigzag poset  $\mathbf{Z}$ . Indeed, we first notice that  $\zeta$  does not cover the original choice of  $\mathbf{g}$  given in (5.3.19). Next, it is straightforward to verify that the following order-preserving map  $\zeta': \mathbf{Z}' \rightarrow \mathbf{P}$  covers both  $\mathbf{g}$  and  $\tilde{\mathbf{g}}$ :

$$\mathbf{Z}' := \begin{array}{ccccc}
 & 2' & & 2'' & \\
 & \swarrow & \searrow & \swarrow & \searrow \\
 2 & \rightarrow & 3 & & 1 & & 4 \\
 & \swarrow & & \swarrow & & \swarrow & \\
 & 2''' & & & & & 
 \end{array}, \text{ and } \zeta'(x) := \begin{cases} 2, & \text{if } x \in \{2, 2', 2'', 2'''\}, \\ x, & \text{if } x \in \{1, 3, 4\}. \end{cases}$$

However,  $\mathbf{Z}'$  is not the poset of Dynkin type  $\mathbb{A}$ .

# Chapter 6

## Interval replacements

In one-parameter persistent homology, the (interval) multiplicities can be computed by the ranks. We address the formula (4.1.3) as follows.

$$d_M(V_{[s,t]}) = \text{rank } M_{t,s} + \text{rank } M_{t+1,s-1} - \text{rank } M_{t+1,s} - \text{rank } M_{t,s-1}$$

holds for all  $M \in \text{mod } \mathbb{k}[\mathbb{A}_n]$ . Conversely, the ranks can be obtained from the (interval) multiplicities. So far, we have generalized the rank invariant to the interval rank invariant, but we do not have the corresponding invariant that generalizes interval multiplicities.

This chapter is devoted to generalizing the standard theory to persistent homology over any finite poset  $\mathbf{P}$ , regarding the relation between the interval ranks and the generalized interval multiplicities. We will then propose a new additive invariant in terms of the compression systems. Throughout the chapter, the multiplicity function  $d_M$  also stands for a function  $\mathbb{I}(\mathbf{P}) \rightarrow \mathbb{Z}$  by identifying  $d_M(\mathbf{I})$  and  $d_M(V_{\mathbf{I}})$  for all  $\mathbf{I} \in \mathbb{I}(\mathbf{P})$ . As before, we write  $A$  for  $\mathbb{k}[\mathbf{P}]$  and  $\mathbb{I}$  for  $\mathbb{I}(\mathbf{P})$  for short.

### 6.1 Signed interval multiplicities

We begin with the interval-decomposable case. Suppose  $M \in \text{mod } A$  is interval-decomposable, it is possible to express the compression multiplicities of interval modules by multiplicities of interval modules and vice-versa. Recall that  $\mathbb{I}$ , endowed with the set-inclusion relation, is a finite poset (see Definition 2.2.15 (6)). Since we will mostly apply the Möbius inversion formula to the poset  $\mathbb{I}$  later, we concentrate on the incidence algebra of  $\mathbb{I}$  with the field  $\mathbb{R}$ , denoted by  $\mathbb{RI}$ .

**Proposition 6.1.1.** *Let  $\xi$  be a compression system for  $\mathbf{P}$ . Take  $M \in \text{mod } A$  and  $\mathbf{I} \in \mathbb{I}$ . If  $M$  is interval-decomposable, then*

$$c_M^\xi(\mathbf{I}) = \sum_{\substack{\mathbf{I}, \mathbf{J} \in \mathbb{I} \\ \mathbf{I} \leq \mathbf{J}}} d_M(\mathbf{J}).$$

*This can be rewritten by (2.3.5) as*

$$c_M^\xi = d_M|_{\mathbb{I}} \cdot \zeta.$$

*Proof.* By assumption,  $M$  can be decomposed as a direct sum of interval modules, i.e.,  $M \cong \bigoplus_{\mathbf{J} \in \mathbb{I}} V_{\mathbf{J}}^{d_M(\mathbf{J})}$ . Now, Proposition 5.2.12 yields

$$c_M^\xi(\mathbf{I}) = \sum_{\mathbf{J} \in \mathbb{I}} d_M(\mathbf{J}) c_{V_{\mathbf{J}}}^\xi(\mathbf{I}).$$

Proposition 5.2.11 leads to the desired formula. □

By the Möbius inversion formula, we immediately obtain the following

**Corollary 6.1.2.** *Let  $\xi$  be a compression system for  $\mathbf{P}$  and let  $M \in \text{mod } A$ . If  $M$  is interval-decomposable, then*

$$d_M|_{\mathbb{I}} = c_M^\xi \cdot \boldsymbol{\mu}.$$

*Proof.* This follows directly from Theorem 2.3.4.  $\square$

By adopting the argument used in [6, Theorem 4.23], it is possible to write  $\boldsymbol{\mu}$  explicitly. We first define the following by recalling Definition 2.2.7.

**Definition 6.1.3.** Let  $\mathbf{I} \in \mathbb{I}$ . The *cover* of  $\mathbf{I}$  is defined as

$$\text{Cov}(\mathbf{I}) := \{\mathbf{L} \in \mathbb{I} \mid \mathbf{L} \text{ covers } \mathbf{I}\}.$$

**Proposition 6.1.4.** *Let  $\mathcal{U}$  be a nonempty full subposet of  $\mathbb{I}$ . If  $\mathcal{U}$  has a lower bound, then the join of  $\mathcal{U}$  exists.*

*Proof.* Let  $\mathbf{I}$  be a lower bound of  $\mathcal{U}$ . Let us write  $\mathcal{U} := \{\mathbf{I}_1, \dots, \mathbf{I}_n\}$  with  $n \geq 1$ . Then the subset of  $\mathbf{P}$  defined by  $\bigcup_{k \in [n]} \mathbf{I}_k$  is connected since  $\mathbf{I} \leq \mathbf{I}_k$  for all  $k \in [n]$ . It follows that  $\text{conv}\left(\bigcup_{k \in [n]} \mathbf{I}_k\right)$  is connected, convex, and thus  $\text{conv}\left(\bigcup_{k \in [n]} \mathbf{I}_k\right) \in \mathbb{I}$ . Hence  $\text{conv}\left(\bigcup_{k \in [n]} \mathbf{I}_k\right)$  is an upper bound of  $\mathcal{U}$  since  $\mathbf{I}_k \leq \text{conv}\left(\bigcup_{k \in [n]} \mathbf{I}_k\right)$  for all  $k \in [n]$ .

Now we show  $\text{conv}\left(\bigcup_{k \in [n]} \mathbf{I}_k\right)$  is the least upper bound of  $\mathcal{U}$ . Let  $\mathbf{W}$  be an upper bound of  $\mathcal{U}$ , i.e.,  $\mathbf{I}_k \leq \mathbf{W}$  for all  $k \in [n]$ . Since  $\bigcup_{k \in [n]} \mathbf{I}_k \subseteq \mathbf{W}$  and  $\mathbf{W}$  is convex, we have  $\text{conv}\left(\bigcup_{k \in [n]} \mathbf{I}_k\right) \subseteq \mathbf{W}$ . Thus  $\text{conv}\left(\bigcup_{k \in [n]} \mathbf{I}_k\right) = \bigvee \mathcal{U}$ .  $\square$

**Theorem 6.1.5.** *Let us define  $\boldsymbol{\mu}' \in \mathbb{RI}$  by*

$$\boldsymbol{\mu}'([\mathbf{I}, \mathbf{J}]) := \sum_{\mathcal{S} \in \mathfrak{E}} (-1)^{|\mathcal{S}|},$$

for  $\mathbf{I}, \mathbf{J} \in \mathbb{I}$  with  $\mathbf{I} \leq \mathbf{J}$ , and where  $\mathfrak{E}$  is the set of all full posets  $\mathcal{S}$  of  $\mathbb{I}$  such that  $\mathcal{S} \subseteq \text{Cov}(\mathbf{I})$  and  $\bigvee \mathcal{S} = \mathbf{J}$ . Note that if  $\mathcal{S}$  is nonempty, then  $\bigvee \mathcal{S}$  is well defined by Proposition 6.1.4. On the other hand, we set  $\bigvee \emptyset := \mathbf{I}$  to simplify notations if  $\mathcal{S}$  is empty. Then we have

$$\boldsymbol{\mu} = \boldsymbol{\mu}'.$$

*Proof.* Let us prove that  $\zeta \boldsymbol{\mu}' = 1_{\mathbb{RI}}$ , where  $1_{\mathbb{RI}}$  denotes the identity of  $\mathbb{RI}$ . Let  $\mathbf{I}, \mathbf{J} \in \mathbb{I}$  with  $\mathbf{I} \leq \mathbf{J}$ . We have

$$\begin{aligned} (\zeta \boldsymbol{\mu}')([\mathbf{I}, \mathbf{J}]) &= \sum_{\mathbf{I} \leq \mathbf{L} \leq \mathbf{J}} \boldsymbol{\mu}'([\mathbf{I}, \mathbf{L}]) \\ &= \sum_{\mathbf{I} \leq \mathbf{L} \leq \mathbf{J}} \sum_{\mathcal{S} \in \mathfrak{E}} (-1)^{|\mathcal{S}|} \\ &= \sum_{\substack{\mathcal{S} \subseteq \text{Cov}(\mathbf{I}) \\ \bigvee \mathcal{S} \leq \mathbf{J}}} (-1)^{|\mathcal{S}|} \\ &= 1 - \sum_{\substack{\emptyset \neq \mathcal{S} \subseteq \text{Cov}(\mathbf{I}) \\ \bigvee \mathcal{S} \leq \mathbf{J}}} (-1)^{|\mathcal{S}|-1} \end{aligned}$$

Now define the function  $f$  as follows:

$$\begin{aligned} f: 2^{\uparrow \mathbf{I}} &\rightarrow \mathbb{R} \\ \mathcal{Z} &\mapsto \sum_{\mathbf{L} \in \mathcal{Z}} d_{V_{\mathbf{J}}}(\mathbf{L}), \end{aligned}$$

where  $2^{\uparrow \mathbf{I}}$  denotes the power set of  $\uparrow \mathbf{I}$ . Note that by definition

$$\bigcap_{\mathbf{L} \in \mathcal{S}} \uparrow \mathbf{L} = \uparrow \mathbf{I} \bigvee \mathcal{S}.$$

Therefore, we have

$$\begin{aligned} f\left(\bigcap_{\mathbf{L} \in \mathcal{S}} \uparrow \mathbf{L}\right) &= f\left(\uparrow \mathbf{I} \bigvee \mathcal{S}\right) \\ &= \sum_{\bigvee \mathcal{S} \leq \mathbf{L}} d_{V_{\mathbf{J}}}(\mathbf{L}) \\ &= c_{V_{\mathbf{J}}}^{\xi}\left(\bigvee \mathcal{S}\right) \\ &= \begin{cases} 1, & \text{if } \bigvee \mathcal{S} \leq \mathbf{J}, \\ 0, & \text{otherwise,} \end{cases} \end{aligned}$$

where the last two equalities come from Propositions 5.2.11 and 6.1.1 respectively. Thus we can write:

$$(\zeta \mu')([\mathbf{I}, \mathbf{J}]) = 1 - \sum_{\emptyset \neq \mathcal{S} \subseteq \text{Cov}(\mathbf{I})} (-1)^{|\mathcal{S}|-1} f\left(\bigcap_{\mathbf{L} \in \mathcal{S}} \uparrow \mathbf{L}\right).$$

It is easily seen that  $(\uparrow \mathbf{I}, 2^{\uparrow \mathbf{I}}, f)$  is a finite measure space. So, by the inclusion-exclusion principle

$$\begin{aligned} (\zeta \mu')([\mathbf{I}, \mathbf{J}]) &= 1 - f\left(\bigcup_{\mathbf{L} \in \text{Cov}(\mathbf{I})} \uparrow \mathbf{L}\right) \\ &= 1 - \sum_{\mathbf{I} < \mathbf{L}} d_{V_{\mathbf{J}}}(\mathbf{L}) \\ &= 1 - \left(\sum_{\mathbf{I} \leq \mathbf{L}} d_{V_{\mathbf{J}}}(\mathbf{L}) - d_{V_{\mathbf{J}}}(\mathbf{I})\right) \\ &= 1 - \left(c_{V_{\mathbf{J}}}^{\xi}(\mathbf{I}) - d_{V_{\mathbf{J}}}(\mathbf{I})\right) \\ &= d_{V_{\mathbf{J}}}(\mathbf{I}). \end{aligned}$$

The last two equalities hold by Propositions 5.2.11 and 6.1.1 respectively. Finally, since  $d_{V_{\mathbf{J}}}(\mathbf{I}) = 1$  if and only if  $\mathbf{I} = \mathbf{J}$ , we have  $(\zeta \mu')([\mathbf{I}, \mathbf{J}]) = 1_{\mathbb{R}\mathbf{I}}([\mathbf{I}, \mathbf{J}])$ , so  $\zeta \mu' = 1_{\mathbb{R}\mathbf{I}}$  and we deduce that  $\mu = \mu'$ .  $\square$

It is now possible to rewrite Corollary 6.1.2 in the following way:

**Corollary 6.1.6.** *Let  $M \in \text{mod } A$  and  $\mathbf{I} \in \mathbb{I}$ . If  $M$  is interval-decomposable, then*

$$d_M(\mathbf{I}) = \sum_{\mathcal{S} \subseteq \text{Cov}(\mathbf{I})} (-1)^{|\mathcal{S}|} c_M^\xi \left( \bigvee \mathcal{S} \right).$$

Now we are in the position to generalize interval multiplicities for any persistence modules over finite posets.

**Definition 6.1.7.** Let  $M \in \text{mod } A$  and  $\mathbf{I} \in \mathbb{I}$ . We define the *signed interval multiplicity*  $\delta_M^\xi$  of  $M$  under  $\xi$  as the function  $\delta_M^\xi: \mathbb{I} \rightarrow \mathbb{Z}$  by setting

$$\delta_M^\xi(\mathbf{I}) := \sum_{\mathcal{S} \subseteq \text{Cov}(\mathbf{I})} (-1)^{|\mathcal{S}|} c_M^\xi \left( \bigvee \mathcal{S} \right)$$

for all  $\mathbf{I} \in \mathbb{I}$ . By Theorem 6.1.5, this can be rewritten as

$$\delta_M^\xi := c_M^\xi \cdot \mu.$$

The signed interval multiplicities give an invariant of  $A$  called the *interval approximation invariant under  $\xi$*  (see [39]), given by

$$\mathbf{appx}_{\mathbb{I}}^\xi: (\text{mod } A)_0 \rightarrow \mathbb{Z}^{\mathbb{I}}, \quad M \mapsto \delta_M^\xi.$$

Note that  $\mathbf{appx}_{\mathbb{I}}^\xi$  is an additive invariant of  $A$ . Indeed, by Proposition 5.2.12 we have

$$\begin{aligned} \delta_{M \oplus N}^\xi(\mathbf{I}) &= \sum_{\mathcal{S} \subseteq \text{Cov}(\mathbf{I})} (-1)^{|\mathcal{S}|} c_{M \oplus N}^\xi \left( \bigvee \mathcal{S} \right) \\ &= \sum_{\mathcal{S} \subseteq \text{Cov}(\mathbf{I})} (-1)^{|\mathcal{S}|} \left[ c_M^\xi \left( \bigvee \mathcal{S} \right) + c_N^\xi \left( \bigvee \mathcal{S} \right) \right] \\ &= \delta_M^\xi(\mathbf{I}) + \delta_N^\xi(\mathbf{I}) \end{aligned}$$

for all  $\mathbf{I} \in \mathbb{I}$ . Notably, in the case where  $\xi = \text{tot}$ , the pair  $(\mathbb{I}, \delta_M^{\text{tot}})$  coincides with the *generalized persistence diagram* of  $M$ , proposed by Kim and Mémoli [40].

**Remark 6.1.8.** Note that in Definition 6.1.7,  $M$  is not necessarily interval-decomposable anymore. If  $M$  is interval-decomposable, it is clear that  $\delta_M^\xi = d_M|_{\mathbb{I}}$  as functions on  $\mathbb{I}$  by Corollary 6.1.6.

**Proposition 6.1.9.** *Let  $M \in \text{mod } A$ . For all  $\mathbf{I} \in \mathbb{I}$ , we have*

$$c_M^\xi(\mathbf{I}) = \sum_{\substack{\mathbf{I}, \mathbf{J} \in \mathbb{I} \\ \mathbf{I} \leq \mathbf{J}}} \delta_M^\xi(\mathbf{J}),$$

$$\text{i.e., } c_M^\xi = \delta_M^\xi \cdot \zeta.$$

*Proof.* This is a direct consequence of Theorem 2.3.4. □

**Remark 6.1.10.** Proposition 6.1.9 gives an alternative definition of the signed interval multiplicity  $\delta_M^\xi$  without using the Möbius inversion formula. Indeed, it is possible to define  $\delta_M^\xi$  by induction in the following way: first, define  $\delta_M^\xi(\mathbf{I}) := c_M^\xi(\mathbf{I})$  for all maximal (in terms of the poset  $(\mathbb{I}, \leq)$ ) intervals  $\mathbf{I}$ . Then one can define inductively  $\delta_M^\xi(\mathbf{I}) := c_M^\xi(\mathbf{I}) - \sum_{\mathbf{I} < \mathbf{J}} \delta_M^\xi(\mathbf{J})$ .

## 6.2 Interval replacement: an additional invariant

Noting that for each  $\mathbf{I} \in \mathbb{I}$ ,  $\delta_M^\xi(\mathbf{I})$  can be defined even for modules  $M$  that are not necessarily interval-decomposable, we introduce the following.

**Definition 6.2.1.** Let  $M \in \text{mod } A$ . We set

$$\begin{aligned} \delta^\xi(M)_+ &:= \bigoplus_{\substack{\mathbf{I} \in \mathbb{I} \\ \delta_M^\xi(\mathbf{I}) > 0}} V_{\mathbf{I}}^{\delta_M^\xi(\mathbf{I})}, & \delta^\xi(M)_- &:= \bigoplus_{\substack{\mathbf{I} \in \mathbb{I} \\ \delta_M^\xi(\mathbf{I}) < 0}} V_{\mathbf{I}}^{(-\delta_M^\xi(\mathbf{I}))}, \text{ and} \\ \delta^\xi(M) &:= [\delta^\xi(M)_+] - [\delta^\xi(M)_-], \end{aligned}$$

where  $[X]$  is the element of the split Grothendieck group  $K^\oplus(A)$  of  $A$  corresponding to a module  $X$ . We call  $\delta^\xi(M)$  the *interval replacement of  $M$  under  $\xi$* , and we call  $\delta^\xi(M)_+$ ,  $\delta^\xi(M)_-$  the *positive part* and the *negative part* of  $\delta^\xi(M)$ , respectively. Note that  $\delta^\xi(M)$  is not a module, just an element of the split Grothendieck group, while both  $\delta^\xi(M)_+$  and  $\delta^\xi(M)_-$  are interval-decomposable modules, and that  $\delta^\xi(M)$  can be presented by the pair of these interval-decomposable modules.

**Remark 6.2.2.** The interval replacement is an additive invariant of  $A$ . Indeed, put  $G = K^\oplus(A)$  in Definition 5.1.2 and set the map to be

$$\mathbf{q} := \delta^\xi: (\text{mod } A)_0 \rightarrow K^\oplus(A), \quad M \mapsto \delta^\xi(M).$$

It is easy to check that  $\delta^\xi$  satisfies  $\delta^\xi(M) = \delta^\xi(N)$  if  $M \cong N$  in  $\text{mod } A$ , and  $\delta^\xi(M \oplus N) = \delta^\xi(M) + \delta^\xi(N)$ . Alternatively, we can set the map to be

$$\mathbf{q} := \delta^\xi: (\text{mod } A)_0 \rightarrow K^\oplus(\text{add } \mathcal{I}), \quad M \mapsto \delta^\xi(M),$$

where we recall that  $\mathcal{I}$  denotes the complete set of isoclasses of interval modules in  $\text{mod } A$  and  $\text{add } \mathcal{I}$  denotes the additive closure of  $\mathcal{I}$ . We also remark that such a map  $\mathbf{q}: (\text{mod } A)_0 \rightarrow K^\oplus(\text{add } \mathcal{I})$  (more accurately, the group homomorphism  $K^\oplus(A) \rightarrow K^\oplus(\text{add } \mathcal{I})$  explained in Remark 5.1.5) is recently called the *interval-barcoding invariant* by Escolar and Kim [37].

Recalling Definition 5.3.4, we give the definition of interval ranks of the interval replacement.

**Definition 6.2.3.** For each  $\mathbf{I} \in \mathbb{I}$ , we define the  $\mathbf{I}$ -rank of  $\delta^\xi(M)$  under  $\xi$  to be

$$\begin{aligned} \mathbf{rank}_{\mathbf{I}}^\xi \delta^\xi(M) &:= \mathbf{rank}_{\mathbf{I}}^\xi \delta^\xi(M)_+ - \mathbf{rank}_{\mathbf{I}}^\xi \delta^\xi(M)_- \\ &= (\mathbf{rank}_{\mathbf{I}} \delta^\xi(M)_+ - \mathbf{rank}_{\mathbf{I}} \delta^\xi(M)_-). \end{aligned}$$

Note that  $\mathbf{rank}_{\mathbf{I}}^\xi \delta^\xi(M)$  may depend on  $\xi$  because  $\delta^\xi(M)_\pm$  depend on  $\xi$ . Thus by Definition 6.2.1, we have

$$\mathbf{rank}_{\mathbf{I}}^\xi \delta^\xi(M) = \sum_{\mathbf{J} \in \mathbb{I}} \delta_M^\xi(\mathbf{J}) \cdot \mathbf{rank}_{\mathbf{I}} V_{\mathbf{J}}.$$

Using the notations given above, we obtain the following.

**Theorem 6.2.4.** *Let  $M \in \text{mod } A$ , and  $\mathbf{I} \in \mathbb{I}$ . Then*

$$\text{rank}_{\mathbf{I}}^{\xi} \delta^{\xi}(M) = \text{rank}_{\mathbf{I}}^{\xi} M.$$

*In particular, for any  $[x, y] \in \text{Seg}(\mathbf{P})$ , we have*

$$\begin{aligned} \text{rank}_{[x,y]}^{\xi} \delta^{\xi}(M) &= \text{rank}_{[x,y]} M, \\ \underline{\dim} \delta^{\xi}(M) &= \underline{\dim} M. \end{aligned}$$

Thus,  $\delta^{\xi}$  preserves the interval rank invariants of all persistence modules  $M$ . In this sense, we called  $\delta^{\xi}(M)$  an *interval replacement* of  $M$ .

*Proof.* It is enough to show the first equality because the second (resp. third) one follows by considering the case that  $\mathbf{I} = [x, y]$  (resp. the cases that  $[x, x]$  for all  $x \in \mathbf{P}$ ). By Definition 5.3.4, to show the first one, it suffices to show the following:

$$\sum_{\mathbf{J} \in \mathbb{I}} \delta_M^{\xi}(\mathbf{J}) \cdot \text{rank}_{\mathbf{I}} V_{\mathbf{J}} = \text{rank}_{\mathbf{I}}^{\xi} M.$$

Let  $\mathbf{I}, \mathbf{J} \in \mathbb{I}$ . Then by Proposition 5.2.11, we have

$$\text{rank}_{\mathbf{I}} V_{\mathbf{J}} = c_{V_{\mathbf{J}}}^{\xi}(\mathbf{I}) = \begin{cases} 1, & \text{if } \mathbf{I} \leq \mathbf{J}, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore,

$$\sum_{\mathbf{J} \in \mathbb{I}} \delta_M^{\xi}(\mathbf{J}) \cdot \text{rank}_{\mathbf{I}} V_{\mathbf{J}} = \sum_{\substack{\mathbf{I}, \mathbf{J} \in \mathbb{I} \\ \mathbf{I} \leq \mathbf{J}}} \delta_M^{\xi}(\mathbf{J}) = c_M^{\xi}(\mathbf{I}) = \text{rank}_{\mathbf{I}}^{\xi} M,$$

where the second equality holds by Proposition 6.1.9, and the last one by definition.  $\square$

**Remark 6.2.5.** In this research, we developed a computational project hosted on GitHub for computing interval rank invariant and interval replacement under the total and source-sink compression systems of persistence modules over any  $d$ D-grid ( $d \geq 1$ ), mainly based on Theorem 5.3.7 and Remark 5.3.10. We believe that this project will be useful and can be integrated into the topological data analysis pipeline to provide algebraic descriptors from data. For more details on the implementation and to access the code, we refer the reader to the project repository at <https://github.com/GauthierE/interval-replacement>.

**Example 6.2.6.** We exhibit an application of utilizing the interval replacement to distinguish two filtrations  $\mathcal{F}_1$  and  $\mathcal{F}_2$  over the 2D-grid (Figure 6.1). We consider the 1st homology  $H_1(-; \mathbb{Z}/2\mathbb{Z})$  and denote  $M_j := H_1(-; \mathbb{Z}/2\mathbb{Z}) \circ \mathcal{F}_j$  ( $j \in \{1, 2\}$ ). By implementing our computational project mentioned above, the interval replacements of  $M_j$  under tot are given as follows (interval modules are written as their dimension vectors).

$$\begin{aligned} \delta^{\text{tot}}(M_1) &= [[\delta^{\text{tot}}(M_1)_+]] - [[\delta^{\text{tot}}(M_1)_-]] \\ &= [[\begin{smallmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{smallmatrix}]] \oplus [[\begin{smallmatrix} 1 & 0 & 1 & 1 & 1 \end{smallmatrix}]] \oplus [[\begin{smallmatrix} 0 & 1 & 1 & 1 & 1 \end{smallmatrix}]] \oplus [[\begin{smallmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{smallmatrix}]] - [[\begin{smallmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{smallmatrix}]] \end{aligned} \quad (6.2.1)$$

and

$$\begin{aligned}\delta^{\text{tot}}(M_2) &= [[\delta^{\text{tot}}(M_2)_+]] - [[\delta^{\text{tot}}(M_2)_-]] \\ &= [[\begin{smallmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{smallmatrix}]] \oplus [\begin{smallmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{smallmatrix}] \oplus [\begin{smallmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{smallmatrix}]].\end{aligned}\quad (6.2.2)$$

By observing (6.2.1) and (6.2.2), one can easily tell that filtrations  $\mathcal{F}_1$  and  $\mathcal{F}_2$  are essentially different due to their distinct interval replacement invariants. Moreover,  $M_1$  is not interval-decomposable because of the existence of negative part  $\delta^{\text{tot}}(M_1)_-$ . In comparison,  $M_2$  is interval-decomposable because there is no negative part of its replacement, and  $M_2 \cong \delta^{\text{tot}}(M_2)_+$ .

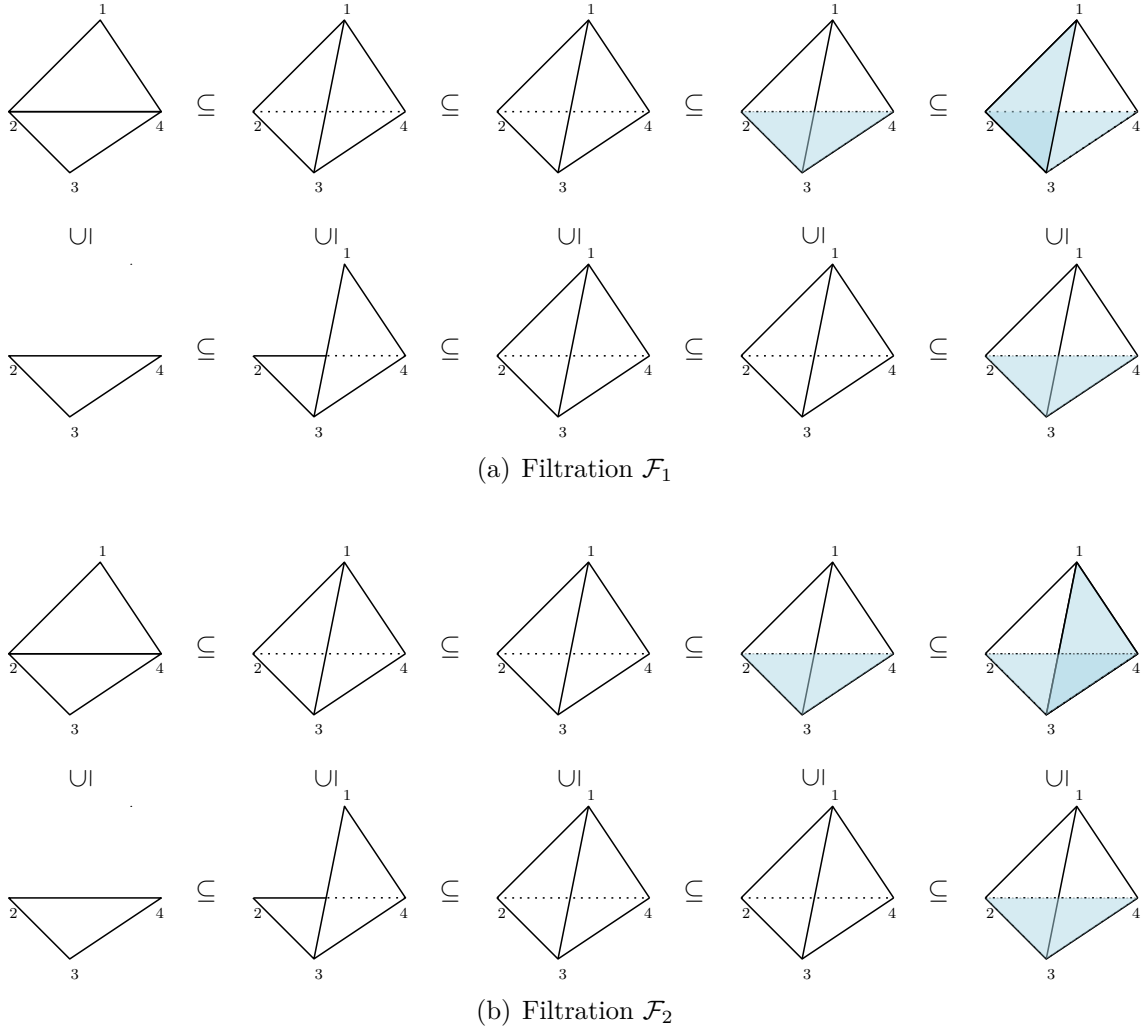


Figure 6.1: Two  $\mathbf{G}_{5,2}$ -filtrations in Example 6.2.6

## 6.3 Examples

### 6.3.1 Computing the maximal interval-decomposable summand

Let  $M$  be a persistence module over  $\mathbf{P}$ . The first application of the general formula (3.1.27) is to compute the maximal interval-decomposable summand of  $M$  that is defined by

$$M_{\mathbb{I}} := \bigoplus_{\mathbf{I} \in \mathbb{I}} V_{\mathbf{I}}^{d_M(\mathbf{I})}. \quad (6.3.3)$$

The maximal interval-decomposable summand  $M_{\mathbb{I}}$  determines the interval decomposability of  $M$  trivially, in the following sense.

**Lemma 6.3.1.** *For any module  $M \in \text{mod } A$  the following are equivalent:*

- (1)  $M$  is interval-decomposable.
- (2)  $\underline{\dim} M = \sum_{\mathbf{I} \in \mathbb{I}} d_M(\mathbf{I}) \cdot \underline{\dim} V_{\mathbf{I}}$ .
- (3)  $\dim M = \sum_{\mathbf{I} \in \mathbb{I}} d_M(\mathbf{I}) \cdot \dim V_{\mathbf{I}}$ .

We would remark on the difference between our proposed method here and the algorithm provided in [31] checking the interval decomposability. The key idea of the algorithm in [31] is depending on [6, Theorem 5.10], that is, a persistence module  $M$  is interval-decomposable if and only if  $M$  is isomorphic to the positive part of its interval placement  $\delta^\xi(M)_+$ . Thus, they suggest picking up the interval  $\mathbf{I}$  appearing in  $\delta^\xi(M)_+$  and then checking whether the interval multiplicity  $d_M(\mathbf{I})$  is zero or not by utilizing the [5, Algorithm 3]. However, no explicit formula is provided in [5] and one has to compute everything involving computing the almost split sequence of  $V_{\mathbf{I}}$  by the computer program, causing a high computation cost. On the contrary, we provide a direct way to compute the interval multiplicity  $d_M(\mathbf{I})$ . This eliminates the procedure to compute the almost split sequence that is required to perform [5, Algorithm 3] because this has already been computed theoretically to give the formula, and then the interval decomposability can be easily verified by checking the dimension equality in Lemma 6.3.1. On the other hand, another advantage of our method is that we can find out the maximal interval-decomposable summand of a given persistence module  $M$  (over any finite poset  $\mathbf{P}$ ) that is not interval-decomposable, while the algorithm in [31] can not.

### 6.3.2 Interval replacement is incomplete

Although the interval rank invariant of a persistence module  $M$  under a compression system  $\xi$  captures more information than the rank invariant, it can still not retrieve all the information contained in  $M$  in general. Namely, it is possible to construct  $\xi$  and two objects  $M, N \in \text{mod } A$  not isomorphic to each other such that  $\delta_M^\xi(\mathbf{I}) = \delta_N^\xi(\mathbf{I})$  for all  $\mathbf{I} \in \mathbb{I}$ . We now give such examples.

**Example 6.3.2.** (1) Define a poset  $\mathbf{P}_1$  and  $\mathbf{P}_1$ -persistence modules  $M(\theta)$  by

$$\mathbf{P}_1 := \begin{array}{ccc} 1 & \longrightarrow & 2 \\ \downarrow & & \uparrow \\ 3 & \longleftarrow & 4 \end{array}, \quad M(\theta) := \begin{array}{ccc} \mathbb{R} & \xrightarrow{1} & \mathbb{R} \\ \downarrow 1 & & \uparrow \theta \\ \mathbb{R} & \xleftarrow{1} & \mathbb{R} \end{array}$$

for  $\theta \in \mathbb{R} \setminus \{0, 1\}$ . We take  $\xi := \text{tot}$  (see Example 5.2.2). Let  $\theta_1, \theta_2 \in \mathbb{R} \setminus \{0, 1\}$  such that  $\theta_1 \neq \theta_2$ . Then  $M(\theta_1)$  and  $M(\theta_2)$  are clearly not isomorphic to each other but they have the same interval replacement. One can compute the interval replacement of  $M(\theta)$  for  $\theta = \theta_1, \theta_2$  by using Remark 6.1.10:

Table 6.1: Computation of  $\delta_{M(\theta)}^{\text{tot}}(\mathbf{I})$  for  $\theta \in \mathbb{R} \setminus \{0, 1\}$ .

| Interval<br>$\mathbf{I}$ | I-rank<br>$\text{rank}_{\mathbf{I}}^{\text{tot}} M(\theta)$ | Signed interval multiplicity<br>$\delta_{M(\theta)}^{\text{tot}}(\mathbf{I})$ |
|--------------------------|-------------------------------------------------------------|-------------------------------------------------------------------------------|
| $\{1, 2, 3, 4\}$         | 0                                                           | 0                                                                             |
| $\{1, 2, 3\}$            | 1                                                           | 1                                                                             |
| $\{1, 2, 4\}$            | 1                                                           | 1                                                                             |
| $\{1, 3, 4\}$            | 1                                                           | 1                                                                             |
| $\{2, 3, 4\}$            | 1                                                           | 1                                                                             |
| $\{1, 2\}$               | 1                                                           | -1                                                                            |
| $\{1, 3\}$               | 1                                                           | -1                                                                            |
| $\{2, 4\}$               | 1                                                           | -1                                                                            |
| $\{3, 4\}$               | 1                                                           | -1                                                                            |
| $\{1\}$                  | 1                                                           | 0                                                                             |
| $\{2\}$                  | 1                                                           | 0                                                                             |
| $\{3\}$                  | 1                                                           | 0                                                                             |
| $\{4\}$                  | 1                                                           | 0                                                                             |

(2) Define a poset  $\mathbf{P}_2$  and  $\mathbf{P}_2$ -persistence modules  $N(\theta)$  by

$$\mathbf{P}_2 := \begin{array}{ccccccc} & & 7 & \longrightarrow & 8 & & \\ & \nearrow & \uparrow & & \uparrow & \nearrow & \\ 3 & \longrightarrow & & & 4 & \longrightarrow & \\ \uparrow & & \uparrow & & \uparrow & & \\ & \nearrow & 5 & \longrightarrow & 6 & \nearrow & \\ 1 & \longrightarrow & & & 2 & \longrightarrow & \end{array}, \quad N(\theta) := \begin{array}{ccccccc} & & \mathbb{R} & \longrightarrow & 0 & & \\ & \nearrow 1 & \uparrow & & \uparrow & \nearrow & \\ \mathbb{R} & \longrightarrow & & & \mathbb{R} & \longrightarrow & \\ \uparrow & & \uparrow 1 & & \uparrow & & \\ & \nearrow & \mathbb{R} & \longrightarrow & \mathbb{R} & \nearrow 1 & \\ 0 & \longrightarrow & & & \mathbb{R} & \longrightarrow & \end{array}$$

for  $\theta \in \mathbb{R} \setminus \{0, 1\}$ .

(3) Define a poset  $\mathbf{P}_3$  and  $\mathbf{P}_3$ -persistence modules  $L(\theta)$  by

$$\mathbf{P}_3 := \begin{array}{ccccc} 1 & \longleftarrow & 2 & \longrightarrow & 3 \\ \uparrow & & \uparrow & & \uparrow \\ 4 & \longleftarrow & 5 & \longrightarrow & 6 \\ \downarrow & & \downarrow & & \downarrow \\ 7 & \longleftarrow & 8 & \longrightarrow & 9 \end{array}, \quad L(\theta) := \begin{array}{ccccc} \mathbb{R} & \xleftarrow{1} & \mathbb{R} & \xrightarrow{1} & \mathbb{R} \\ \uparrow_1 & & \uparrow & & \uparrow_1 \\ \mathbb{R} & \xleftarrow{\quad} & 0 & \longrightarrow & \mathbb{R} \\ \downarrow_1 & & \downarrow & & \downarrow_1 \\ \mathbb{R} & \xleftarrow{1} & \mathbb{R} & \xrightarrow{\theta} & \mathbb{R} \end{array}$$

for  $\theta \in \mathbb{R} \setminus \{0, 1\}$ .

We give another example satisfying commutativity relations nontrivially that shows the incompleteness of the interval rank invariant under the compression system tot.

**Example 6.3.3.** Let  $\lambda \in \mathbb{k}$  and  $M_\lambda$  be the following persistence modules over  $\mathbf{P} := \mathbf{G}_{5,2}$ :

$$\begin{array}{ccccccc} \mathbb{k} & \xrightarrow{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} & \mathbb{k}^2 & \xrightarrow{1} & \mathbb{k}^2 & \xrightarrow{[\lambda, -1]} & \mathbb{k} \longrightarrow 0 \\ \uparrow & & \uparrow \begin{bmatrix} 0 \\ 1 \end{bmatrix} & & \uparrow 1 & & \uparrow [\lambda, -1] \\ 0 & \longrightarrow & \mathbb{k} & \xrightarrow{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} & \mathbb{k}^2 & \xrightarrow{1} & \mathbb{k}^2 \xrightarrow{[1, -1]} \mathbb{k} \end{array}.$$

Then it is easy to see that the endomorphism algebra of  $M_\lambda$  is isomorphic to  $\mathbb{k}$ , and hence  $M_\lambda$  is indecomposable, and if  $\lambda \neq \mu$  in  $\mathbb{k}$ , then  $\text{Hom}_A(M_\lambda, M_\mu) = 0$ . Thus  $M_\lambda \cong M_\mu$  if and only if  $\lambda = \mu$ . Let  $\lambda \neq \mu$  in  $\mathbb{k} \setminus \{0, 1\}$ . It is easy to check that for any interval  $\mathbf{I} \in \mathbb{I}$ ,  $\text{rank}_{\mathbf{I}}^{\text{tot}} M_\lambda = \text{rank}_{\mathbf{I}}^{\text{tot}} M_\mu$ .

The dimension vector of  $M_\lambda$  is taken from [46, A2. The frames of the tame concealed algebras] for  $\tilde{E}_7$ , and the representation  $M_\lambda$  is constructed by modifying a homogeneous representation of  $\tilde{D}_4$  in [33, Chapter 6 Tables].

### 6.3.3 Fibered bar and arc codes versus interval replacements

Lesnick and Wright [41] introduced the *fibered barcode*, an invariant of persistence modules over  $\mathbb{R}^2$  given by the collection of barcodes of all 1-D affine slices. In detail, denoting by  $L$  the collection of all (unparametrized) lines in  $\mathbb{R}^2$  with nonnegative slope and for a fixed  $\mathbb{R}^2$ -persistence module  $M$ , each line  $l \in L$  induces a one-parameter persistence module  $M|_l = M \circ i_L$  along the inclusion  $i_L: l \hookrightarrow \mathbb{R}^2$ . The *fibered barcode* of  $M$  is then the assignment of each line  $l \in L$  to the barcode of  $M|_l$ .

Closely following the construction of the fibered barcode, and due to bipath modules having arc codes, given a bipath poset  $\mathbf{B}$  and an  $\mathbb{R}^2$ -persistence module  $M$ , an order-preserving map  $f: \mathbf{B} \rightarrow \mathbb{R}^2$  induces a bipath module  $M|_f = M \circ f$  and we can define the *fibered arc code* as follows.

**Definition 6.3.4.** The *fibered arc code* of an  $\mathbb{R}^2$ -persistence module  $M$  is the assignment of each order-preserving map  $f: \mathbf{B} \rightarrow \mathbb{R}^2$ , where  $\mathbf{B}$  is a bipath poset, to the arc code of  $M|_f$ . Note that the image of  $f$  is a subposet of  $\mathbb{R}^2$ .

As we already know, the fibered barcode is an incomplete invariant. The same applies to the fibered arc code. Still, the fibered arc code gives more information than the fibered barcode: in Example 6.3.5 below we show two non-isomorphic persistence modules with equal fibered barcode, but different fibered arc code. At the end of this subsection, we provide an example demonstrating two filtrations that cannot be distinguished by the fibered arc code but can be distinguished using the interval rank invariant or interval replacement.

**Example 6.3.5.** We take an example in Example 6.3.3. Let  $\mathbf{P} = \mathbf{G}_{5,2}$  and let  $M_\lambda$  ( $\lambda \in \mathbb{k}$ ) be the following persistence module over  $\mathbf{P}$ :

$$M_\lambda := \begin{array}{ccccccccc} \mathbb{k} & \xrightarrow{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} & \mathbb{k}^2 & \xrightarrow{1} & \mathbb{k}^2 & \xrightarrow{[\lambda, -1]} & \mathbb{k} & \longrightarrow & 0 \\ \uparrow & & \uparrow \begin{bmatrix} 0 \\ 1 \end{bmatrix} & & \uparrow 1 & & \uparrow [\lambda, -1] & & \uparrow \\ 0 & \longrightarrow & \mathbb{k} & \xrightarrow{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} & \mathbb{k}^2 & \xrightarrow{1} & \mathbb{k}^2 & \xrightarrow{[1, -1]} & \mathbb{k} \end{array} .$$

We focus on the two non-isomorphic modules  $M_1$  and  $M_{-1}$ . Let  $\mathbf{B} = \mathbf{B}_{2,1}$  be the bipath poset defined in Section 4.2.1 and consider the order-preserving map  $f: \mathbf{B} \rightarrow \mathbf{P}$  whose image is

$$\begin{array}{ccccc} (3, 2) & \rightarrow & (4, 2) & \rightarrow & (5, 2) \\ \uparrow & & & & \uparrow \\ (3, 1) & \longrightarrow & & \longrightarrow & (5, 1) \end{array} .$$

Then the arc code  $\mathcal{B}(M_1|_f)$  consists of two intervals of  $\mathbf{B}$  that we depict by

$$\begin{array}{ccc} (3, 2) \rightarrow (4, 2) & & (3, 2) \\ \uparrow & , \text{ and } & \uparrow \\ (3, 1) \longrightarrow (5, 1) & & (3, 1) \end{array} ,$$

while the arc code  $\mathcal{B}(M_{-1}|_f)$  consists of two other intervals of  $\mathbf{B}$  that we depict by

$$\begin{array}{ccc} (3, 2) & & (3, 2) \rightarrow (4, 2) \\ \uparrow & , \text{ and } & \uparrow \\ (3, 1) \longrightarrow (5, 1) & & (3, 1) \end{array} .$$

We conclude that the fibered arc codes of  $M_1$  and  $M_{-1}$  are different.

In comparison,  $M_1$  and  $M_{-1}$  have the same fibered barcodes. It suffices to check three affine lines in red, as illustrated in Figure 6.2. For the red line  $l_0$  with slope 0,  $M_1|_{l_0}$  and  $M_{-1}|_{l_0}$  have the same barcodes:

$$(1, 2) \longrightarrow (2, 2) \longrightarrow (3, 2) \longrightarrow (4, 2) , \text{ and } (2, 2) \longrightarrow (3, 2) .$$

The same happens for the red line  $l_1$  with slope 1, where the barcodes are given by

$$(3, 1) \longrightarrow (4, 2) , \text{ and } (3, 1) ,$$

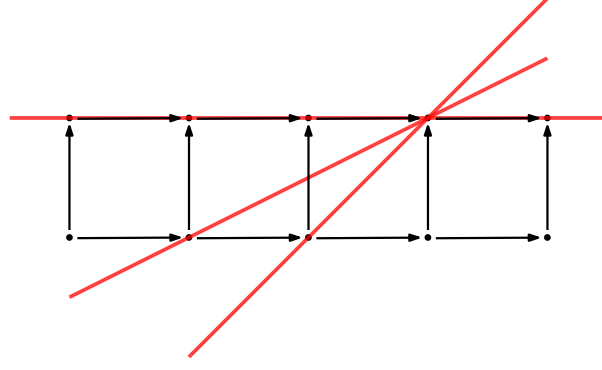


Figure 6.2: Illustration of the three affine lines used in Example 6.3.5

and for the red line  $l_{1/2}$  with slope  $1/2$ , where the barcodes are

$$(2, 1) \longrightarrow (4, 2) .$$

**Example 6.3.6.** Consider two filtrations over  $\mathbf{G}_{4,2}$  illustrated in Figure 6.3. We

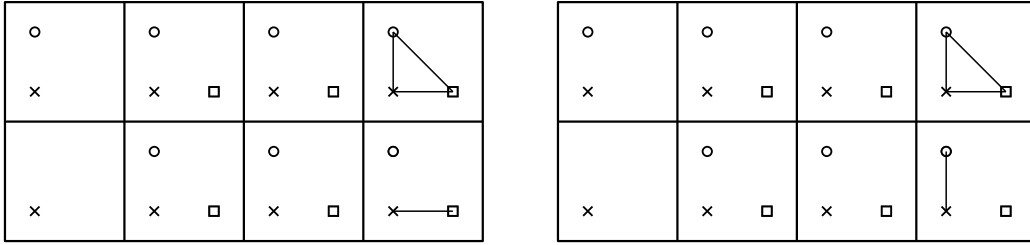
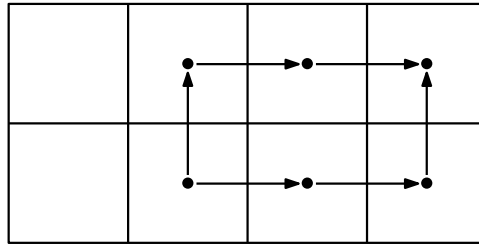


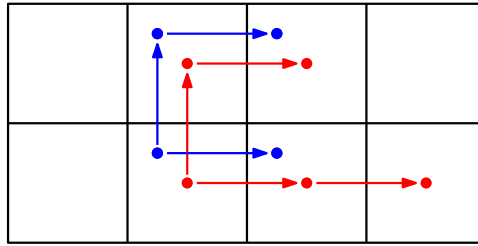
Figure 6.3: Two different filtrations

apply the 0-th reduced homology<sup>1</sup> with the coefficient field  $\mathbb{k} := \mathbb{Z}/2\mathbb{Z}$  on both. The persistent homology obtained from the left (resp. right) filtration is denoted by  $M$  (resp.  $N$ ). It is not hard to verify that  $M$  and  $N$  share the same fibered arc code. For instance, restricting to the order-preserving map  $f: \mathbf{B}_{2,2} \rightarrow \mathbf{G}_{4,2}$ , depicted by



yields that arc codes  $\mathcal{B}(M|_f)$  and  $\mathcal{B}(N|_f)$  share the same intervals as illustrated below:

<sup>1</sup>Explicitly, the 0-th reduced homology  $\tilde{H}_0(\mathcal{K}; \mathbb{k}) := \text{Ker}(\text{aug}) / \text{Im } \partial_1$ , where  $\text{aug}: C_0(\mathcal{K}; \mathbb{k}) \rightarrow \mathbb{k}$ ,  $\sum_i n_i \sigma_i \mapsto \sum_i n_i$  ( $n_i \in \mathbb{k}$ ) is the augmentation map, and  $\partial_1: C_1(\mathcal{K}; \mathbb{k}) \rightarrow C_0(\mathcal{K}; \mathbb{k})$  denotes the usual boundary map. For a detailed account, we refer the reader to [44, Chapter 1, Section 7].



On the other hand, we can distinguish these two filtrations by using the interval rank invariant. More concretely, if we take the interval  $\mathbf{J}$  of the grid illustrated in Figure 6.4 and use the source-sink compression system, then the  $\mathbf{J}$ -rank of  $M$  is 1 while that of  $N$  is 0.

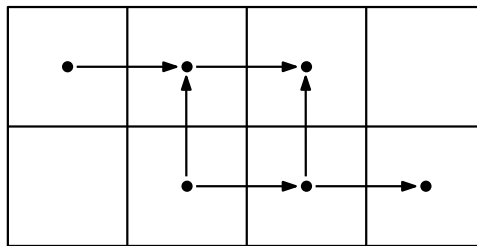


Figure 6.4: The interval  $\mathbf{J}$  of the grid



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