

REMARKS ON ISOMORPHISMS OF REGRESSIVE TRANSFORMATION SEMIGROUPS

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For a (finite or infinite) set X , let $T(X)$ be the full transformation semigroup on X , i.e. the set of all maps from X to X , the semigroup operation being composition of maps.

When X is a partially ordered set, we let

$$T_{RE}(X) = \{ f \in T(X) \mid f(x) \leq x \text{ for all } x \in X \},$$

$$T_{OP}(X) = \{ f \in T(X) \mid f(x) \leq f(y) \text{ if } x \leq y \text{ for } x, y \in X \}.$$

Then, both of them are subsemigroups of $T(X)$ with the identity $id_{T(X)}$. We call $T_{RE}(X)$ the full regressive transformation semigroup on X , and $T_{OP}(X)$ the full order-preserving transformation semigroup on X .

Recently, some interesting results on $T_{RE}(X)$ have been obtained (cf. [1], [4], [5]).

It is known that, for partially ordered sets X, Y , if $T_{OP}(X)$ and $T_{OP}(Y)$ are isomorphic as semigroups, then X and Y are isomorphic or anti-isomorphic as ordered sets (see [3], Theorem V.8.9).

It is natural to ask whether the above result holds or not for regressive transformation semigroups. In general, it does not hold. However, we obtain a necessary and sufficient condition on partially ordered sets X and Y for $T_{RE}(X)$ and $T_{RE}(Y)$ to be isomorphic.

Umar showed in [6] that, when X and Y are totally ordered sets, any idempotent in $T_{RE}(X)$ whose image is an order-ideal is mapped to an idempotent in $T_{RE}(Y)$ with the same property by isomorphisms from $T_{RE}(X)$ to $T_{RE}(Y)$, and he considered the above problem through this result. If the result holds even if "an order-ideal" in it is replaced by "a principal order-ideal", then it can be shown that if $T_{RE}(X) \cong T_{RE}(Y)$ as semigroups then $X \cong Y$ as ordered sets. At the present time, this is unsolved.

In here, we achieve our purpose by showing that any idempotent of defect 1 in $T_{RE}(X)$ is mapped to an idempotent of defect 1 in $T_{RE}(Y)$ by isomorphisms from $T_{RE}(X)$ to $T_{RE}(Y)$, where the defect of α in $T_{RE}(X)$ means the cardinality of the set of idempotents in X which do not belong to the image of α .

For partially ordered set X , an element in X is said to be *isolated* if it is incomparable with every element in X except itself. Let $Is(X)$ be the set of all isolated elements in X . Then, it is easy to see that $T_{RE}(X)$ and $T_{RE}(X \setminus Is(X))$ are isomorphic. Therefore, we may assume that every partially ordered set, treated in this paper, does not contain any isolated elements.

Let X be a partially ordered set under the order relation $\leq$.

For $a \in X$, the set of (resp. *strict*) upper bounds of a is denoted by $U(a)$ (resp. $SU(a)$), i. e.

$$U(a) = \{ x \in X \mid x \geq a \} \text{ and } SU(a) = \{ x \in X \mid x > a \},$$

and the set of all minimal elements in X is denoted by $Min(X)$,

This is an abstract and the details will be published in Semigroup Forum.

b , we have that $j'(j(a, b), k(a, b)) = a$ and $k'(j(a, b), k(a, b)) = b$.

Lemma 2. (1) $k(a, b) = k(c, d)$ if and only if $b = d$.

(2) If $a < c < b$, then $k(a, c) = j(c, b)$ and $j(a, b) = j(a, c)$.

(3) $j(a, b) = j(c, d)$ if and only if $a = c$.

Proof. (1) It is easy to see that

$$b = d \Leftrightarrow \lambda^d_c \circ \lambda^b_a = \lambda^b_a \Leftrightarrow \lambda^{k(c, d)}_{j(c, d)} \circ \lambda^{k(a, b)}_{j(a, b)} = \lambda^{k(a, b)}_{j(a, b)} \Leftrightarrow k(a, b) = k(c, d).$$

This assertion means that $k(a, b)$ depends only on b .

(2) The proof is omitted.

(3) To show the assertion, we need that X and Y are adjusted. Let $a = c$. If a is not minimal in X , then $e < a$ for some $e \in X$. From (2), we have that $j(a, b) = k(e, a) = j(a, d) = j(c, d)$.

If a is minimal in X , then b and d are connected in $SU(a)$, since X is adjusted, so that there exist $e_1, e_2, \dots, e_n \in SU(a)$ such that $b = e_1 \leq^s e_2 \leq^s \dots \leq^s e_n = d$. Since e_i and e_{i+1} are comparable, by (2) we have that $j(a, e_i) = j(a, e_{i+1})$ ($i = 1, 2, \dots, e_{n-1}$). Thus, we have that $j(a, b) = j(a, d) = j(c, d)$.

Let $j(a, b) = j(c, d)$. If we apply the above fact to j' , then we have that $a = j'(j(a, b), k(a, b)) = j'(j(c, d), k(c, d)) = c$.

This assertion means that $j(a, b)$ depends only on a .

We write $j(a, b) = j(a)$ and $k(a, b) = k(b)$ for $a, b \in X$ with $a < b$. In this case, if a is maximal in X , then $j(a)$ is undefined, and if b is minimal in X , then $k(b)$ is undefined. Since $j(a) < k(b)$ if $a < b$, we have that if a is not maximal in X , then neither is $j(a)$ in Y . By (2) of Lemma 2, if c is neither maximal nor minimal in X , then $j(c) = k(c)$.

Similarly, we write $j'(a', b') = j'(a')$ and $k'(a', b') = k'(b')$ for $a', b' \in Y$ with $a' < b'$. Then, we have that $j'(j(a)) = a$, $k'(k(b)) = b$, $j(j'(a')) = a'$ and $k(k'(b')) = b'$.

Let a be maximal in X . Then, we can show that $k(a)$ is maximal in Y .

Define a map $h : X \rightarrow Y$ by $h(a) = j(a)$ if a is not maximal in X , and $h(a) = k(a)$ if a is maximal in X . Then, we can show that the h is an order-isomorphism of X onto Y .

Since any totally ordered set is clearly adjusted, we obtain :

Corollary 3.

Let X and Y be totally ordered sets. Then, $T_{RE}(X)$ and $T_{RE}(Y)$ are isomorphic as semigroups if and only if X and Y are isomorphic as ordered sets.

Let X, Y be partially ordered sets. From Theorem 1 and Theorem 2, we have that

$$T_{RE}(X) \cong T_{RE}(Y) \Leftrightarrow T_{RE}(A(X)) \cong T_{RE}(A(Y)) \Leftrightarrow A(X) \cong A(Y).$$

Thus, we obtain the following main theorem :

Theorem 4.

Let X and Y be partially ordered sets. Then, $T_{RE}(X)$ and $T_{RE}(Y)$ are isomorphic as semigroups if and only if their adjusted sets $A(X)$ and $A(Y)$ are isomorphic as ordered sets.

and the set of all minimal elements in X is denoted by $\text{Min}(X)$.

Let $\leq^s$ be the symmetric relation generated by $\leq$, i.e. $a \leq^s b$ if and only if $a \leq b$ or $b \leq a$, and let $\leq^e$ be the equivalent relation generated by $\leq$, i.e. $a \leq^e b$ if and only if there exist $c_1, c_2, \dots, c_n \in X$ such that $a = c_1 \leq^s c_2 \leq^s \dots \leq^s c_n = b$ (see [2], I). In this case, we say that a and b are *connected* in X . A subset Y of X is *connected* if every $a, b \in Y$ are connected in Y , i. e. there exist $c_1, c_2, \dots, c_n \in Y$ such that $a = c_1 \leq^s c_2 \leq^s \dots \leq^s c_n = b$.

A partially ordered set X is said to be *adjusted* if it does not contain any minimal elements, or for every $m \in \text{Min}(X)$, $SU(m)$ is connected.

Theorem 1.

Let X be a partially ordered set. Then, there exists an adjusted partially ordered set A such that $T_{RE}(A)$ is isomorphic to $T_{RE}(X)$ as semigroups.

We can construct an adjusted partially ordered set $A(X)$ from X such that $T_{RE}(A(X))$ is isomorphic to $T_{RE}(X)$. In this case, the $A(X)$ is called *the adjusted partially ordered set of X* .

Theorem 2.

Let X, Y be adjusted partially ordered sets. Then, $T_{RE}(X)$ and $T_{RE}(Y)$ are isomorphic as semigroups if and only if X and Y are isomorphic as ordered sets.

Suppose that X and Y are isomorphic. Let h be an isomorphism from X onto Y . Then, it is easy to show that the map $i : T_{RE}(X) \rightarrow T_{RE}(Y), f \rightarrow i(f)$ defined by $i(f)(h(x)) = h(y)$ if $f(x) = y$, is an isomorphism.

To show the only if-part, we need two lemmas (Lemmas 1 and 2).

For each pair $a, b \in Z$ with $a < b$, where Z is a partially ordered set, we define λ_a^b in $T_{RE}(Z)$ by

$$\lambda_a^b(b) = a, \lambda_a^b(x) = x \text{ if } x \neq b.$$

From now until the end of the proof of Theorem 2, X and Y will denote adjusted partially ordered sets, and i will denote an isomorphism from $T_{RE}(X)$ onto $T_{RE}(Y)$.

Lemma 1. *For each pair $a, b \in X$ with $a < b$, there exist $a', b' \in Y$ such that $i(\lambda_a^b) = \lambda_{a'}^{b'}$.*

The assertion can be easily shown by using the following facts :

For $g \in T_{RE}(X)$,

$$\begin{aligned} \lambda_a^b \circ g &= \lambda_a^b \text{ if and only if } g = \text{id}_{T(X)} \text{ or } g = \lambda_a^b, \\ g \circ \lambda_a^b &= g \text{ if and only if } g(a) = g(b) \text{ and } a < b. \end{aligned}$$

For each pair $a, b \in X$ with $a < b$, the pair a', b' in Lemma 1 is clearly unique. So we write

$$a' = j(a, b) \text{ and } b' = k(a, b), \text{ namely } i(\lambda_a^b) = \lambda_{j(a,b)}^{k(a,b)}.$$

We similarly have that for each pair a', b' in Y with $a' < b'$, there exist unique elements $j'(a', b')$, $k'(a', b')$ in X such that $i^{-1}(\lambda_{a'}^{b'}) = \lambda_{j'(a',b')}^{k'(a',b')}$. Then, for each a, b in X with $a < b$

Corollary 5.

Let X and A be as in Theorem 1. Then A is uniquely determined by X up to isomorphisms.

We next aim to refine Theorem 2 to the following :

Theorem 6.

Let X and Y be as in Theorem 2, and let i be a semigroup isomorphism from $T_{RE}(X)$ onto $T_{RE}(Y)$. Then, there exists an order isomorphism h from X onto Y such that $h(f(a)) = i(f)(h(a))$ for all $f \in T_{RE}(X)$ and all $a \in X$.

Let h be an isomorphism from X onto Y determined by i in Theorem 6 as in the proof of Theorem 2. Thus, $i(\lambda_a^b) = \lambda_{h(a)}^{h(b)}$ for each $a, b \in X$ with $a < b$. We show that this h serves as a desired h in Theorem 6. To show the theorem, again we need two lemmas (Lemmas 3 and 4).

For each $f \in T_{RE}(X)$ and each $a \in X$, we define f^a and f_a , as follows :

$f^a(x) = x$ if $x \geq a$, $f^a(x) = f(x)$ otherwise, and $f_a(x) = f(x)$ if $x > a$, $f_a(x) = x$ otherwise.

Then, it is easy to check that $f = f_a \circ \lambda_{f(a)}^a \circ f^a$ for all $a \in X$, where $\lambda_{f(a)}^a = id_{T(X)}$ if $a = f(a)$.

Lemma 3. For any $f \in T_{RE}(X)$ and any $b, c \in X$ with $c \leq b$,

- (1) $f(b) = f(c)$ if and only if $i(f)(h(b)) = i(f)(h(c))$. In particular,
- (2) if $a \leq b$, then $i(f^a)(h(b)) = i(f^a)(h(c))$ implies that $h(b) = h(c)$, and
- (3) if $b \not\leq a$, then $i(f_a)(h(b)) = i(f_a)(h(c))$ implies that $h(b) = h(c)$, where $b \not\leq a$ means that $b \leq a$ or a and b are incomparable.

From Lemma 3, we have :

Lemma 4. For every $a, b \in X$,

- (1) if $h(b) \geq h(a)$, then $i(f^a)(h(b)) = h(b)$,
- (2) if $h(b) \not\geq h(a)$, then $i(f_a)(h(b)) = h(b)$.

Since $f = f_a \circ \lambda_{f(a)}^a \circ f^a$ for all $a \in X$, and since $h(f(a)) \leq h(a)$, we have that

$$\begin{aligned} i(f)(h(a)) &= i(f_a) \circ i(\lambda_{f(a)}^a) \circ i(f^a)(h(a)) = i(f_a) \circ \lambda_{h(f(a))}^{h(a)}(h(a)) \\ &= i(f_a) \circ \lambda_{h(f(a))}^{h(a)}(h(a)) = i(f_a)(h(f(a))) = h(f(a)). \end{aligned}$$

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