

Relations between cumulants in a class of estimators

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確率変数 $X_1, X_2, \dots, X_n$ は互いに独立で, 共通の確率密度関数 $f(x, \theta)$, $\theta \in \Theta \subset \mathbb{R}$, Θ は開集合, を持つものとする. X_i 's に基づく一致推定量は, $o(n^{-1})$ まで Edgeworth-展開可能で $n^{1/2}(T - \theta)$ の第 i -cumulant $K_i(\theta)$ は

$$\begin{aligned} K_1(\theta) &= K_{10}(\theta) + n^{-1/2} K_{11}(\theta) + n^{-1} K_{12}(\theta) + o(n^{-1}) \\ K_2(\theta) &= K_{20}(\theta) + n^{-1/2} K_{21}(\theta) + n^{-1} K_{22}(\theta) + o(n^{-1}) \\ K_3(\theta) &= n^{-1/2} K_{31}(\theta) + n^{-1} K_{32}(\theta) + o(n^{-1}) \\ K_4(\theta) &= n^{-1} K_{42}(\theta) + o(n^{-1}) \\ K_i(\theta) &= o(n^{-1}) \quad (i \geq 5) \end{aligned} \tag{1}$$

の形で表われ, 3回連続微分可能であるものとする. $P_\theta(T \leq \theta) = g(\theta, n^{1/2})$ に対し, $P_\theta(\theta_n \leq \theta) = g(\theta, n^{1/2}) + o(n^{-1})$ を満たし, $o(n^{-1})$ まで Edgeworth-展開可能で, その cumulant に関して T と同様な条件を満たす θ の一致推定量の全体を $C_2(g(\theta, n^{1/2}))$ とする. このとき $f(x, \theta)$ に対する適当な正則条件の下で, $g(\theta, n^{1/2})$ によって定まる適当な $b(\theta, t, n^{1/2})$ に対し,

$$\sum_{i=1}^n \log f(x_i, \theta + n^{1/2}t) - \sum_{i=1}^n \log f(x_i, \theta) = b(\theta, t, n^{1/2})$$

を満たす θ の解を $\theta^*(t)$ とし, $\tilde{\theta}(t) = \theta^*(t) + n^{1/2}t$ とするとき, 任意の $\theta_n \in C_2(g(\theta, n^{1/2}))$ に対し,

$$t > 0 \text{ のとき } P_\theta(n^{1/2}(\theta_n - \theta) \leq t) \leq P_\theta(n^{1/2}(\tilde{\theta}(t) - \theta) \leq t) + o(n^{-1})$$

$$t < 0 \text{ のとき } P_\theta(n^{1/2}(\theta_n - \theta) \leq t) + o(n^{-1}) \geq P_\theta(n^{1/2}(\tilde{\theta}(t) - \theta) \leq t)$$

ところで

$$P(n^{1/2}(\tilde{\theta}(t) - \theta) \leq t) = \Phi(Z)$$

$$- n^{-1/2} \phi(Z) \left\{ I_{1,1}^{1/2}(\theta) K_{11}^*(\theta, t) + \frac{1}{2} I_{1,1}(\theta) K_{21}^*(\theta, t) H_1(Z) + \frac{1}{6} I_{1,1}^{3/2}(\theta) K_{31}^*(\theta, t) H_2(Z) \right\}$$

$$- n^{-1} \phi(Z) \left[I_{1,1}^{1/2}(\theta) K_{12}^*(\theta, t) + \frac{1}{2} I_{1,1}(\theta) K_{22}^*(\theta, t) H_1(Z) + \frac{1}{6} I_{1,1}^{3/2}(\theta) K_{32}^*(\theta, t) H_2(Z) \right. \\ \left. + \frac{1}{24} I_{1,1}^2(\theta) K_{42}^*(\theta, t) H_3(Z) \right.$$

$$+ \frac{1}{2} \left\{ I_{1,1}(\theta) K_{11}^{*2}(\theta, t) H_1(Z) + \frac{1}{4} I_{1,1}^2(\theta) K_{21}^{*2}(\theta, t) H_4(Z) \right. \\ \left. + \frac{1}{36} I_{1,1}^3(\theta) K_{31}^{*2}(\theta, t) H_6(Z) + I_{1,1}^{3/2}(\theta) K_{11}^*(\theta, t) K_{21}^*(\theta, t) H_3(Z) \right. \\ \left. + \frac{1}{3} I_{1,1}^2(\theta) K_{11}^*(\theta, t) K_{31}^*(\theta, t) H_4(Z) \right. \\ \left. + \frac{1}{6} I_{1,1}^{5/2}(\theta) K_{21}^*(\theta, t) K_{31}^*(\theta, t) H_5(Z) \right\} \Big] + o(n^{-1})$$

ただし

$$Z = I_{1,1}^{1/2}(\theta) (t - \tilde{K}_{10}(\theta, t)),$$

$$K_1^*(\theta, t, \frac{1}{n^{1/2}}) = E_{\theta} \{ n^{1/2}(\theta^*(t) - \theta) \} \\ = C_0(\theta, t) - t + \frac{1}{n^{1/2}} \left\{ -\frac{1}{2} I_{1,1}^{-2}(\theta) (J_{2,1}(\theta) + J_{1,1,1}(\theta)) + C_1(\theta, t) \right\} \\ + \frac{1}{n} \left\{ -\frac{1}{2} (C_0(\theta, t) - \frac{t}{2}) I_{1,1}^{-3}(\theta) J_{2,1}(\theta) (J_3(\theta) + 4J_{2,1}(\theta)) \right. \\ \left. + \frac{1}{2} (C_0(\theta, t) - \frac{t}{2}) I_{1,1}^{-2}(\theta) (K_{2,2}(\theta) - K_{2,1,1}(\theta)) \right. \\ \left. + \frac{1}{2} C_0'(\theta, t) I_{1,1}^{-2}(\theta) (J_3(\theta) + 2J_{2,1}(\theta)) \right. \\ \left. - (C_0(\theta, t) - \frac{t}{2}) + \frac{1}{2} C_0''(\theta, t) I_{1,1}^{-1}(\theta) + C_2(\theta, t) \right\} + o(n^{-1})$$

$$= K_{10}^*(\theta, t) + n^{1/2} K_{11}^*(\theta, t) + n^{-1} K_{12}^*(\theta, t), \quad (\text{say})$$

$$\tilde{K}_{10}(\theta, t) = K_{10}^*(\theta, t) + t = C_0(\theta, t)$$

$$\begin{aligned}
K_2^*(\theta, t, \frac{1}{n^{1/2}}) &= E_{\theta} \{ n^{1/2} (\theta^*(t) - \theta) - K_1^*(\theta, t) \}^2 \\
&= I_{1,1}^{-1}(\theta) + \frac{2}{n^{1/2}} I_{1,1}^{-1}(\theta) C_0'(\theta, t) \\
&\quad + \frac{1}{n} \{ -I_{1,1}^{-3}(\theta) (K_{3,1}(\theta) + 4K_{2,1,1}(\theta) + K_{1,1,1,1}(\theta) + I_{1,1}^2(\theta)) \\
&\quad + \frac{1}{2} I_{1,1}^{-4}(\theta) (7J_{2,1}^2(\theta) + 14J_{2,1}(\theta) J_{1,1,1}(\theta) + 5J_{1,1,1}^2(\theta)) \\
&\quad + (C_0(\theta, t) - \frac{t}{2})^2 I_{1,1}^{-3}(\theta) (I_{1,1}(\theta) K_{2,2}(\theta) - I_{1,1}^3(\theta) - J_{2,1}^2(\theta)) \\
&\quad + (C_0'(\theta, t))^2 I_{1,1}^{-1}(\theta) + 2C_0'(\theta, t) I_{1,1}^{-1}(\theta) \} + o(n^{-1}) \\
&= K_{20}^*(\theta, t) + n^{-1/2} K_{21}^*(\theta, t) + n^{-1} K_{22}^*(\theta, t) + o(n^{-1}), \quad (\text{say})
\end{aligned}$$

$$\begin{aligned}
K_3^*(\theta, t, \frac{1}{n^{1/2}}) &= E_{\theta} \{ n^{1/2} (\theta^*(t) - \theta) - K_1^*(\theta, t) \}^3 \\
&= -\frac{1}{n^{1/2}} I_{1,1}^{-3}(\theta) (3J_{2,1}(\theta) + 2J_{1,1,1}(\theta)) \\
&\quad + \frac{3}{n} \{ (C_0(\theta, t) - \frac{t}{2}) I_{1,1}^{-4}(\theta) (I_{1,1}(\theta) K_{2,2}(\theta) - J_{2,1}^2(\theta) - I_{1,1}^3(\theta)) \\
&\quad - C_0'(\theta, t) I_{1,1}^{-3}(\theta) (3J_{2,1}(\theta) + 2J_{1,1,1}(\theta)) \\
&\quad + C_0''(\theta, t) I_{1,1}^{-2}(\theta) \} + o(n^{-1}) \\
&= n^{-1/2} K_{31}^*(\theta, t) + n^{-1} K_{32}^*(\theta, t) + o(n^{-1}), \quad (\text{say})
\end{aligned}$$

$$\begin{aligned}
K_4^*(\theta, t, \frac{1}{n^{1/2}}) &= E_{\theta} \{ n^{1/2} (\theta^*(t) - \theta) - K_1^*(\theta, t, \frac{1}{n^{1/2}}) \}^4 - 3(K_2^*(\theta, t, \frac{1}{n^{1/2}}))^2 \\
&= \frac{1}{n} \{ -I_{1,1}^{-4}(\theta) (4K_{3,1}(\theta) + 12K_{2,1,1}(\theta) + 3K_{1,1,1,1}(\theta) + 3I_{1,1}^2(\theta)) \\
&\quad + 12I_{1,1}^{-5}(\theta) (2J_{2,1}(\theta) + J_{1,1,1}(\theta)) (J_{2,1}(\theta) + J_{1,1,1}(\theta)) \} + o(n^{-1}), \\
&= n^{-1} K_{42}^*(\theta, t) + o(n^{-1}) \quad (\text{say})
\end{aligned}$$

以後 $o(n^{-1/2})$ まで Edgeworth-展開可能で, $o(n^{-1/2})$ まで (1) と同様な形の cumulant を持ち, $K_{ij}(\theta)$ ($i=1, 2, \dots, 5; j=0, 1, 2, \dots, k$) が $(k+1)$ 回微分可能である θ の推定量の全体を $C_E(k)$ で表わそう.
 また $T^* \in C_E(k)$ に対して次の (a), (b) を満たす $\theta_n \in C_E(k)$ が存在しないとき, T^* は $(k+1)$ th order admissible であるということにしよう.

(a) すべての $x_1, x_2 \geq 0$ およびすべての $\theta \in \Theta$ に対して,

$$P_\theta(-x_1 \leq n^{1/2}(T^* - \theta) \leq x_2) \leq P_\theta(-x_1 \leq n^{1/2}(\theta_n - \theta) \leq x_2) + o(n^{-1}),$$

(b) ある $x'_1, x'_2 \geq 0$ およびある $\theta' \in \Theta$ に対して,

$$P_{\theta'}(-x_1 \leq n^{1/2}(T^* - \theta) \leq x_2) < P_{\theta'}(-x_1 \leq n^{1/2}(\theta_n - \theta) \leq x_2) + o(n^{-1})$$

ところで $\tilde{K}_{10}(\theta, t), K_{12}^*(\theta, t), K_{20}^*(\theta, t), K_{21}^*(\theta, t), K_{31}^*(\theta, t)$ は t と無関係であるから, 任意の t, x に対して

$$\tilde{\theta}(t) = \tilde{\theta}(x) + o(n^{-1}) \quad (\text{in law}).$$

従って任意の $x_1, x_2 \geq 0$ および任意の $\theta \in \Theta$ に対して

$$P_\theta(-x_1 \leq n^{1/2}(T - \theta) \leq x_2) \leq P_\theta(-x_1 \leq n^{1/2}(\tilde{\theta}(t) - \theta) \leq x_2) + o(n^{1/2})$$

このことから任意の $T \in C_E(2)$ は次の (I), (II) のいずれか 1つを満たす.

任意の $\theta \in \Theta$ に対して,

$$(I) \quad K_{20}(\theta) > I_{1,1}^{-1}(\theta)$$

$$(II) \quad K_{20}(\theta) = I_{1,1}^{-1}(\theta), \quad K_{21}(\theta) \geq 2 I_{1,1}^{-1}(\theta) K'_{10}(\theta)$$

次に任意の $x_1, x_2 \geq 0$ に対して

$$P_\theta(-x_1 \leq n^{1/2}(T - \theta) \leq x_2) = P_\theta(-x_1 \leq n^{1/2}(\tilde{\theta}(t) - \theta) \leq x_2) + o(n^{-1})$$

を満たす場合を考えよう. すなわち任意の t に対して

$$n^{1/2}(T - \theta) = n^{1/2}(\tilde{\theta}(t) - \theta) + o(n^{1/2}) \quad (\text{in law})$$

が成り立つとする. このとき

$$P_\theta(n^{1/2}(T - \theta) \leq t) - P_\theta(n^{1/2}(\tilde{\theta}(t) - \theta) \leq t)$$

$$\begin{aligned}
&= \pi^{-1} \phi(I_{1,1}^{1/2}(\theta)(t - C_0(\theta))t \\
&\quad \times I_{1,1}^{5/2}(\theta) \{-A_{31}(\theta) + (K_{42}^*(\theta) - K_{42}(\theta))I_{1,1}(\theta)\} t^2 \\
&\quad + I_{1,1}^{5/2}(\theta) [4(A_{30}(C_0'(\theta), C_0''(\theta), \theta) - K_{32}(\theta)) \\
&\quad + \{4A_{31}(\theta) - 3(K_{42}^*(\theta) - K_{42}(\theta))I_{1,1}(\theta)\} C_0(\theta)] t \\
&\quad + I_{1,1}^{3/2}(\theta) [12(A_{20}(C_0'(\theta), C_1'(\theta), \theta) - K_{22}(\theta)) - 3(K_{42}^*(\theta) - K_{42}(\theta))I_{1,1}(\theta) \\
&\quad - 8(A_{30}(C_0'(\theta), C_0''(\theta), \theta) - K_{32}(\theta))I_{1,1}(\theta)C_0(\theta) \\
&\quad + \{-4A_{31}(\theta) + 3(K_{42}^*(\theta) - K_{42}(\theta))I_{1,1}(\theta)\} I_{1,1}(\theta)C_0^2(\theta)] + o(\pi^{-1})
\end{aligned}$$

次に

$$\begin{aligned}
A_{20}(C_0'(\theta), C_1'(\theta), \theta) &= -I_{1,1}^{-3}(\theta)(K_{3,1}(\theta) + 4K_{2,1,1}(\theta) + K_{1,1,1,1}(\theta) + I_{1,1}^2(\theta)) \\
&\quad + \frac{1}{2}I_{1,1}^{-4}(\theta)(7J_{2,1}^2(\theta) + 14J_{2,1}(\theta)J_{1,1,1}(\theta) + 5J_{1,1,1}^2(\theta)) \\
&\quad + (C_0'(\theta))^2 I_{1,1}^{-1}(\theta) + 2C_1'(\theta)I_{1,1}^{-1}(\theta), \\
A_{21}(\theta) &= I_{1,1}^{-3}(\theta) \{I_{1,1}(\theta)(K_{2,2}(\theta) - I_{1,1}^2(\theta)) - J_{2,1}^2(\theta)\}, \\
A_{30}(C_0'(\theta), C_0''(\theta), \theta) &= -3C_0'(\theta)I_{1,1}^{-3}(\theta)(3J_{2,1}(\theta) + 2J_{1,1,1}(\theta)) \\
&\quad + 3C_0''(\theta)I_{1,1}^{-2}(\theta), \\
A_{31}(\theta) &= 3I_{1,1}^{-4}(\theta) \{I_{1,1}(\theta)(K_{2,2}(\theta) - I_{1,1}^2(\theta)) - J_{2,1}^2(\theta)\},
\end{aligned}$$

$$C_0(\theta, t) = C_0(\theta) \quad (\text{say}),$$

$$C_1(\theta, t) = C_1(\theta) \quad (\text{say}).$$

従って任意の $t \neq 0$ に対して

$$\begin{aligned}
&I_{1,1}^{5/2}(\theta) \{-A_{31}(\theta) + (K_{42}^*(\theta) - K_{42}(\theta))I_{1,1}(\theta)\} t^2 \\
&+ I_{1,1}^{5/2}(\theta) [4(A_{30}(C_0'(\theta), C_0''(\theta), \theta) - K_{32}(\theta)) \\
&\quad + \{4A_{31}(\theta) - 3(K_{42}^*(\theta) - K_{42}(\theta))I_{1,1}(\theta)\} C_0(\theta)] t \\
&+ I_{1,1}^{3/2}(\theta) [12(A_{20}(C_0'(\theta), C_1'(\theta), \theta) - K_{22}(\theta)) - 3(K_{42}^*(\theta) - K_{42}(\theta))I_{1,1}(\theta) \\
&\quad - 8(A_{30}(C_0'(\theta), C_0''(\theta), \theta) - K_{32}(\theta))I_{1,1}(\theta)C_0(\theta) \\
&\quad + \{-4A_{31}(\theta) + 3(K_{42}^*(\theta) - K_{42}(\theta))I_{1,1}(\theta)\} I_{1,1}(\theta)C_0^2(\theta)] \leq 0
\end{aligned}$$

上の不等式を観察することによって,

$$K_{42}^*(\theta) - K_{42}(\theta) \leq 0. \quad (2)$$

$$\begin{aligned} & I_{1.1}(\theta) \{ A_{31}(\theta) - (K_{42}^*(\theta) - K_{42}(\theta)) I_{1.1}(\theta) \} t^2(\theta) \\ & + 12(A_{20}(C_0'(\theta), C_1'(\theta), \theta) - K_{22}(\theta)) - 3(K_{42}^*(\theta) - K_{42}(\theta)) I_{1.1}(\theta) \\ & - 8(A_{30}(C_0'(\theta), C_0''(\theta), \theta) - K_{32}(\theta)) I_{1.1}(\theta) C_0(\theta) \\ & + \{ -4A_{31}(\theta) + 3(K_{42}^*(\theta) - K_{42}(\theta)) I_{1.1}(\theta) \} I_{1.1}(\theta) C_0^2(\theta) \leq 0, \end{aligned} \quad (3)$$

を得る. したがって

$$\begin{aligned} & t(\theta) \\ & = \frac{4(A_{30}(C_0'(\theta), C_0''(\theta), \theta) - K_{32}(\theta)) + (4A_{31}(\theta) - 3(K_{42}^*(\theta) - K_{42}(\theta)) I_{1.1}(\theta)) C_0(\theta)}{2 \{ A_{31}(\theta) - (K_{42}^*(\theta) - K_{42}(\theta)) I_{1.1}(\theta) \}} \end{aligned}$$

一方

$$\begin{aligned} & P_\theta(n^{1/2}(T - \theta) \leq x) - P_\theta(n^{1/2}(\tilde{\theta}(t(\theta)) - \theta) \leq x) \\ & = \frac{1}{24n} \phi(I_{1.1}^{1/2}(\theta)(x - C_0(\theta))) I_{1.1}^{5/2}(\theta) x \\ & \times [(K_{42}^*(\theta) - K_{42}(\theta)) I_{1.1}(\theta)(x - t(\theta))^2 - (K_{42}^*(\theta) - K_{42}(\theta)) I_{1.1}(\theta) t^2(\theta) \\ & + 12(A_{20}(C_0'(\theta), C_1'(\theta), \theta) - K_{22}(\theta)) I_{1.1}^{-1}(\theta) + 12A_{21}(\theta) I_{1.1}^{-1}(\theta)(C_0(\theta) \\ & - \frac{1}{2}t(\theta))^2 \\ & - 8\{A_{30}(C_0'(\theta), C_0''(\theta), \theta) - K_{32}(\theta) + A_{31}(\theta)(C_0(\theta) - \frac{1}{2}t(\theta))\} C_0(\theta) \\ & + 3(K_{42}^*(\theta) - K_{42}(\theta))(I_{1.1}(\theta) C_0^2(\theta) - 1)] + o(n^{-1}) \\ & = \frac{1}{24n} \phi(I_{1.1}^{1/2}(\theta)(x - C_0(\theta))) I_{1.1}^{5/2}(\theta) x \\ & \times [(K_{42}^*(\theta) - K_{42}(\theta)) I_{1.1}(\theta)(x - t(\theta))^2 \\ & - \{ -A_{31}(\theta) + (K_{42}^*(\theta) - K_{42}(\theta)) I_{1.1}(\theta) \} t^2(\theta) \\ & + 12(A_{20}(C_0'(\theta), C_1'(\theta), \theta) - K_{22}(\theta)) I_{1.1}^{-1}(\theta) - 3(K_{42}^*(\theta) - K_{42}(\theta)) \\ & - 8(A_{30}(C_0'(\theta), C_0''(\theta), \theta) - K_{32}(\theta)) C_0(\theta) \\ & - \{ 4A_{31}(\theta) - 3(K_{42}^*(\theta) - K_{42}(\theta)) I_{1.1}(\theta) \} C_0^2(\theta)] + o(n^{-1}) \end{aligned}$$

不等式(2), (3)を用いて,

$$x \geq 0 \text{ のとき } P_{\theta}(n^{1/2}(T - \theta) \leq x) \leq P_{\theta}(n^{1/2}\{\tilde{\theta}(t(\theta)) - \theta\} \leq x) + o(n^{-1})$$

$$x \leq 0 \text{ のとき } P_{\theta}(n^{1/2}(T - \theta) < x) + o(n^{-1}) \geq P_{\theta}(n^{1/2}\{\tilde{\theta}(t(\theta)) - \theta\} < x)$$

が成り立つ. すなわちすべての $x_1 \geq 0, x_2 \geq 0$ に対して,

$$P_{\theta}(-x_1 \leq n^{1/2}(T - \theta) \leq x_2) \leq P_{\theta}(-x_1 \leq n^{1/2}\{\tilde{\theta}(t(\theta)) - \theta\} \leq x_2) + o(n^{-1})$$

従って推定量 T が third order admissible であるためには

$$K_{42}(\theta) = K_{42}^*(\theta),$$

$$I_{1,1}(\theta) \{ A_{31}(\theta) - (K_{42}^*(\theta) - K_{42}(\theta)) I_{1,1}(\theta) \} t^2(\theta)$$

$$\begin{aligned} & [12(A_{20}(C_0'(\theta), C_1'(\theta), \theta) - K_{22}(\theta)) - 3(K_{42}^*(\theta) - K_{42}(\theta)) I_{1,1}(\theta) \\ & - 8(A_{30}(C_0'(\theta), C_0''(\theta), \theta) - K_{32}(\theta)) I_{1,1}(\theta) C_0(\theta) \\ & + \{-4A_{31}(\theta) + 3(K_{42}^*(\theta) - K_{42}(\theta)) I_{1,1}(\theta)\} I_{1,1}(\theta) C_0^2(\theta)] = 0 \end{aligned}$$

整理して,

$$K_{42}(\theta) = K_{42}^*(\theta), \quad (5)$$

$$\begin{aligned} & (A_{30}(C_0'(\theta), C_0''(\theta), \theta) - K_{32}(\theta))^2 I_{1,1}(\theta) \\ & + 3 \cdot A_{31}(\theta) (A_{20}(C_0'(\theta), C_1'(\theta), \theta) - K_{22}(\theta)) = 0 \quad (6) \end{aligned}$$

を満たすことである.

定理1. 推定量 $T \in C_E(2)$ が second order admissible ならば不等式(2), (3)を満たす.

定理2. 推定量 $T \in C_E(2)$ が third order admissible であるための条件は, $K_{20}(\theta) = I_{1,1}'(\theta)$, $K_{21}(\theta) = 2I_{1,1}'(\theta) K'_{10}(\theta)$ であって,かつ等式 (5), (6) を満たすことである.