

Absolute Norms on $\mathbb{C}^n$

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It is known that for every absolute normalized norm on $\mathbb{C}^2$ there corresponds a unique convex function on $[0, 1]$ satisfying certain suitable conditions (see Bonsall-Duncan [1], also [2]). Recently the authors [3] extended this result to the n -dimensional case. In this note we shall present a brief introduction of our result. We first recall the 2-dimensional case, and then we treat 3- and n -dimensional cases, where we focus our discussion on the 3-dimensional case which will illustrate the n -dimensional situation. We shall also present a characterization of the strict convexity of these norms, which extends our previous result in [4].

A norm $\|\cdot\|$ on $\mathbb{C}^n$ is called *absolute* if

$$(1) \quad \|(|x_1|, \dots, |x_n|)\| = \|(x_1, \dots, x_n)\| \quad \forall (x_1, \dots, x_n) \in \mathbb{C}^n,$$

and is called *normalized* if

$$(2) \quad \|(1, 0, \dots, 0)\| = \|(0, 1, 0, \dots, 0)\| = \dots = \|(0, \dots, 0, 1)\| = 1.$$

The ℓ_p -norms $\|\cdot\|_p$ are such examples:

$$\|(x_1, x_2, \dots, x_n)\|_p = \begin{cases} (|x_1|^p + \dots + |x_n|^p)^{1/p} & \text{if } 1 \leq p < \infty, \\ \max(|x_1|, \dots, |x_n|) & \text{if } p = \infty. \end{cases}$$

Let AN_n be the set of all absolute normalized norms on $\mathbb{C}^n$.

We recall the 2-dimensional case. For any $\|\cdot\| \in AN_2$ let

$$(3) \quad \psi(t) = \|(1-t, t)\| \quad (0 \leq t \leq 1).$$

Then ψ is convex continuous on $[0, 1]$ and

$$(4) \quad \psi(0) = \psi(1) = 1, \quad \max\{1-t, t\} \leq \psi(t) \leq 1.$$

Let Ψ_2 denote the set of all convex continuous functions on $[0, 1]$ which satisfies (4). Then the converse is valid.

Theorem 1 (Bonsall-Duncan [1]). *The sets AN_2 and Ψ_2 are in one-to-one correspondence under (3). That is, for any $\psi \in \Psi_2$ let*

$$(5) \quad \|(z, w)\|_\psi = \begin{cases} (|z| + |w|)\psi\left(\frac{|w|}{|z| + |w|}\right) & \text{if } (z, w) \neq (0, 0), \\ 0 & \text{if } (z, w) = (0, 0). \end{cases}$$

Then $\|\cdot\|_\psi \in N_2$ and

$$\psi(t) = \|(1 - t, t)\|_\psi \quad (0 \leq t \leq 1).$$

1. Absolute Norms on $\mathbb{C}^3$

Lemma 2. *Let $\|\cdot\| \in AN_3$. Then*

$$\begin{aligned} \|(0, y, z)\| &\leq \|(x, y, z)\|, \\ \|(x, 0, z)\| &\leq \|(x, y, z)\|, \\ \|(x, y, 0)\| &\leq \|(x, y, z)\|. \end{aligned}$$

In particular

$$\|\cdot\|_\infty \leq \|\cdot\| \leq \|\cdot\|_1.$$

Proof. For any $(x, y, z) \in \mathbb{C}^3$, we have

$$\begin{aligned} \|(x, y, 0)\| &= \frac{1}{2} \|(x, y, z) + (x, y, -z)\| \\ &\leq \frac{1}{2} (\|(x, y, z)\| + \|(x, y, -z)\|) = \|(x, y, z)\|. \end{aligned}$$

Similarly we have the other inequalities.

Lemma 3. *Let $\|\cdot\| \in AN_3$. If $|x_1| \leq |x_2|$, $|y_1| \leq |y_2|$ and $|z_1| \leq |z_2|$, then $\|(x_1, y_1, z_1)\| \leq \|(x_2, y_2, z_2)\|$.*

Let

$$\Delta_3 = \{(s, t) : 0 \leq s + t \leq 1, s, t \geq 0\}.$$

For any $\|\cdot\| \in AN_3$, we put

$$(6) \quad \psi(s, t) = \|(1 - s - t, s, t)\| \quad \text{for } (s, t) \in \Delta_3.$$

Then ψ is a continuous convex on Δ_3 and satisfies that

$$\psi(0, 0) = \psi(1, 0) = \psi(0, 1) = 1.$$

Further, by Lemma 2, we have

$$\begin{aligned}
\psi(s, t) &= \|(1-s-t, s, t)\| \geq \|(0, s, t)\| \\
&= (s+t) \left\| \left(0, \frac{s}{s+t}, \frac{t}{s+t} \right) \right\| = (s+t) \psi \left(\frac{s}{s+t}, \frac{t}{s+t} \right), \\
\psi(s, t) &= \|(1-s-t, s, t)\| \geq \|(1-s-t, 0, t)\| \\
&= (1-s) \left\| \left(1 - \frac{t}{1-s}, 0, \frac{t}{1-s} \right) \right\| = (1-s) \psi \left(0, \frac{t}{1-s} \right),
\end{aligned}$$

and similarly

$$\psi(s, t) \geq (1-t) \psi \left(\frac{s}{1-t}, 0 \right).$$

Let Ψ_3 denote the family of all continuous convex functions on Δ_3 satisfying the following conditions:

$$(7) \quad \psi(0, 0) = \psi(1, 0) = \psi(0, 1) = 1,$$

$$(8) \quad \psi(s, t) \geq (s+t) \psi \left(\frac{s}{s+t}, \frac{t}{s+t} \right),$$

$$(9) \quad \psi(s, t) \geq (1-s) \psi \left(0, \frac{t}{1-s} \right),$$

$$(10) \quad \psi(s, t) \geq (1-t) \psi \left(\frac{s}{1-t}, 0 \right).$$

Lemma 4. *Let $\psi \in \Psi_3$. Then*

$$1 \geq \psi(s, t) \geq \psi_\infty(s, t) \geq \frac{1}{3} \quad \text{for all } (s, t) \in \Delta_3.$$

Remark 5. We consider the following function ψ on Δ_3 :

$$\psi(s, t) = \max\{1 - 2s, 1 - 2t, 2s + 2t - 1\}.$$

Then ψ is continuous convex on Δ_3 and satisfies (7), but not in Ψ_3 . Indeed, suppose that ψ is in Ψ_3 . Then since $\psi \left(\frac{s}{s+t}, \frac{t}{s+t} \right) = \psi \left(0, \frac{t}{1-s} \right) = \psi \left(\frac{s}{1-t}, 0 \right) = 1$ for all $(s, t) \in \Delta_3$, we have

$$\psi(s, t) \geq \max\{s+t, 1-s, 1-t\} \geq \frac{2}{3}$$

by (8)-(10), whereas $\psi(\frac{1}{3}, \frac{1}{3}) = \frac{1}{3}$, which is a contradiction.

Theorem 6. For any $\|\cdot\| \in AN_3$, put

$$(11) \quad \psi(s, t) = \|(1 - s - t, s, t)\| \quad \text{for } (s, t) \in \Delta_3.$$

Then $\psi \in \Psi_3$. Conversely, for any $\psi \in \Psi_3$ define

$$(12) \quad \|(x, y, z)\|_\psi = \begin{cases} (|x| + |y| + |z|)\psi\left(\frac{|y|}{|x|+|y|+|z|}, \frac{|z|}{|x|+|y|+|z|}\right) & \text{if } (x, y, z) \neq (0, 0, 0), \\ 0 & \text{if } (x, y, z) = (0, 0, 0). \end{cases}$$

Then $\|\cdot\|_\psi$ is in AN_3 and $\|\cdot\|_\psi$ satisfies (11).

We see the converse assertion. Let $\psi \in \Psi_3$, and define $\|\cdot\|_\psi$ by (12). We only see the triangular inequality

$$\|(x_1, y_1, z_1) + (x_2, y_2, z_2)\|_\psi \leq \|(x_1, y_1, z_1)\|_\psi + \|(x_2, y_2, z_2)\|_\psi.$$

To do this, we show that, if $0 \leq p \leq a$, $0 \leq q \leq b$ and $0 \leq r \leq c$, then

$$\|(p, q, r)\|_\psi \leq \|(a, b, c)\|_\psi,$$

that is,

$$(13) \quad (p + q + r)\psi\left(\frac{q}{p + q + r}, \frac{r}{p + q + r}\right) \leq (a + b + c)\psi\left(\frac{b}{a + b + c}, \frac{c}{a + b + c}\right).$$

At first, we show that, if $0 \leq p < a$, then

$$(14) \quad (p + q + r)\psi\left(\frac{q}{p + q + r}, \frac{r}{p + q + r}\right) \leq (a + q + r)\psi\left(\frac{q}{a + q + r}, \frac{r}{a + q + r}\right).$$

Take any $(s, t) \in \Delta_3$ such that $0 < s + t < 1$. We consider the line segment $[(s, t), (\frac{s}{s+t}, \frac{t}{s+t})]$ in Δ_3 . For any real number λ such that $1 < \lambda \leq \frac{1}{s+t}$, we put $s' = \lambda s$ and $t' = \lambda t$. Then (s', t') is in $[(s, t), (\frac{s}{s+t}, \frac{t}{s+t})]$. Since

$$(s', t') = (\lambda s, \lambda t) = \frac{(s + t)(\lambda - 1)}{1 - s - t} \left(\frac{s}{s + t}, \frac{t}{s + t}\right) + \frac{1 - \lambda(s + t)}{1 - s - t} (s, t),$$

we have

$$\psi(s', t') \leq \frac{(s + t)(\lambda - 1)}{1 - s - t} \psi\left(\frac{s}{s + t}, \frac{t}{s + t}\right) + \frac{1 - \lambda(s + t)}{1 - s - t} \psi(s, t)$$

by the convexity of ψ . Therefore we have

$$\begin{aligned} & \frac{\psi(s, t)}{s} - \frac{\psi(s', t')}{s'} \\ & \geq \frac{\psi(s, t)}{s} - \frac{1}{\lambda s} \left\{ \frac{(s + t)(\lambda - 1)}{1 - s - t} \psi\left(\frac{s}{s + t}, \frac{t}{s + t}\right) + \frac{1 - \lambda(s + t)}{1 - s - t} \psi(s, t) \right\} \\ & = \frac{\lambda - 1}{\lambda s(1 - s - t)} \left\{ \psi(s, t) - (s + t)\psi\left(\frac{s}{s + t}, \frac{t}{s + t}\right) \right\} \geq 0. \end{aligned}$$

Thus, $\frac{\psi(s,t)}{s} \geq \frac{\psi(s',t')}{s'}$. Put

$$s' = \frac{q}{p+q+r}, \quad t' = \frac{r}{p+q+r}, \quad s = \frac{q}{a+q+r}, \quad \text{and} \quad t = \frac{r}{a+q+r},$$

respectively. Since $\frac{s'}{s} = \frac{t'}{t} = \frac{a+q+r}{p+q+r} > 1$, we have

$$\frac{\psi\left(\frac{q}{p+q+r}, \frac{r}{p+q+r}\right)}{\frac{q}{p+q+r}} \leq \frac{\psi\left(\frac{q}{a+q+r}, \frac{r}{a+q+r}\right)}{\frac{q}{a+q+r}}.$$

This implies (14). Repeating a similar discussion, we obtain (13).

Now, let $(x_1, y_1, z_1), (x_2, y_2, z_2) \in \mathbb{C}^3$. Then

$$\begin{aligned} (15) \quad & \| (x_1, y_1, z_1) + (x_2, y_2, z_2) \|_\psi \\ &= \| (x_1 + x_2, y_1 + y_2, z_1 + z_2) \|_\psi \\ &= \| (|x_1 + x_2|, |y_1 + y_2|, |z_1 + z_2|) \|_\psi \\ &\leq \| (|x_1| + |x_2|, |y_1| + |y_2|, |z_1| + |z_2|) \|_\psi \\ &= K\psi\left(\frac{|y_1| + |y_2|}{K}, \frac{|z_1| + |z_2|}{K}\right) \\ &\quad (K := |x_1| + |x_2| + |y_1| + |y_2| + |z_1| + |z_2|) \\ &\leq K \left\{ \frac{|x_1| + |y_1| + |z_1|}{K} \psi\left(\frac{|y_1|}{|x_1| + |y_1| + |z_1|}, \frac{|z_1|}{|x_1| + |y_1| + |z_1|}\right) \right. \\ &\quad \left. + \frac{|x_2| + |y_2| + |z_2|}{K} \psi\left(\frac{|y_2|}{|x_2| + |y_2| + |z_2|}, \frac{|z_2|}{|x_2| + |y_2| + |z_2|}\right) \right\} \\ &= \| (x_1, y_1, z_1) \|_\psi + \| (x_2, y_2, z_2) \|_\psi. \end{aligned}$$

2. Absolute Norms on $\mathbb{C}^n$

Lemma 7. *Let $\|\cdot\| \in AN_n$. Then*

$$\begin{aligned} (B_1) \quad & \| (0, x_2, x_3, \dots, x_n) \| \leq \| (x_1, \dots, x_n) \|, \\ (B_2) \quad & \| (x_1, 0, x_3, \dots, x_n) \| \leq \| (x_1, \dots, x_n) \|, \\ & \vdots \\ (B_n) \quad & \| (x_1, x_2, \dots, x_{n-1}, 0) \| \leq \| (x_1, \dots, x_n) \|. \end{aligned}$$

In particular,

$$\|\cdot\|_\infty \leq \|\cdot\| \leq \|\cdot\|_1.$$

Now let

$$\Delta_n = \{(s_1, s_2, \dots, s_{n-1}) : s_1 + s_2 + \dots + s_{n-1} \leq 1, s_i \geq 0 \ (\forall i)\}.$$

Take any $\|\cdot\| \in AN_n$ and let

$$(16) \quad \psi(s) = \|(1 - s_1 - s_2 - \dots - s_{n-1}, s_1, \dots, s_{n-1})\| \text{ for } s = (s_1, \dots, s_{n-1}) \in \Delta_n$$

Then ψ is continuous and convex on Δ_n , and

$$(A_0) \quad \psi(0, \dots, 0) = \psi(1, 0, \dots, 0) = \dots = \psi(0, \dots, 0, 1) = 1,$$

$$(A_1) \quad \psi(s_1, \dots, s_{n-1}) \geq (s_1 + \dots + s_{n-1})\psi\left(\frac{s_1}{s_1 + \dots + s_{n-1}}, \dots, \frac{s_{n-1}}{s_1 + \dots + s_{n-1}}\right).$$

$$(A_2) \quad \psi(s_1, \dots, s_{n-1}) \geq (1 - s_1)\psi\left(0, \frac{s_2}{1 - s_1}, \dots, \frac{s_{n-1}}{1 - s_1}\right),$$

.....

$$(A_n) \quad \psi(s_1, \dots, s_{n-1}) \geq (1 - s_{n-1})\psi\left(\frac{s_1}{1 - s_{n-1}}, \dots, \frac{s_{n-2}}{1 - s_{n-1}}, 0\right).$$

Let Ψ_n be the family of all continuous convex functions on Δ_n satisfying $(A_0), (A_1), \dots, (A_n)$. The functions corresponding to ℓ_p -norms are

$$\psi_p(s_1, s_2, \dots, s_{n-1}) = \begin{cases} ((1 - \sum_{i=1}^{n-1} s_i)^p + s_1^p + \dots + s_{n-1}^p)^{1/p} & \text{if } 1 \leq p < \infty, \\ \max(1 - \sum_{i=1}^{n-1} s_i, s_1, \dots, s_{n-1}) & \text{if } p = \infty. \end{cases}$$

Lemma 8. Let $\psi \in \Psi_n$. Then

$$\frac{1}{n} \leq \psi_\infty(s_1, s_2, \dots, s_{n-1}) \leq \psi(s_1, s_2, \dots, s_{n-1}) \leq 1$$

for all $(s_1, s_2, \dots, s_{n-1}) \in \Delta_n$.

Theorem 9. The sets AN_n and Ψ_n are in one-to-one correspondence with (16). That is, for any $\|\cdot\| \in AN_n$ the function ψ defined by

$$(16) \quad \psi(s_1, \dots, s_{n-1}) = \|(1 - s_1 - \dots - s_{n-1}, s_1, \dots, s_{n-1})\| \text{ for } (s_1, \dots, s_{n-1}) \in \Delta_n$$

is in Ψ_n . Conversely, for any $\psi \in \Psi_n$ define

$$\|(x_1, \dots, x_n)\|_\psi = \begin{cases} (|x_1| + \dots + |x_n|)\psi\left(\frac{|x_2|}{|x_1| + \dots + |x_n|}, \dots, \frac{|x_n|}{|x_1| + \dots + |x_n|}\right) & \text{if } (x_1, \dots, x_n) \neq (0, \dots, 0), \\ 0 & \text{if } (x_1, \dots, x_n) = (0, \dots, 0). \end{cases}$$

Then $\|\cdot\|_\psi \in AN_n$ and $\|\cdot\|_\psi$ satisfies (16).

3. Strict Convexity of Absolute Norms on $\mathbb{C}^n$

A Banach space X is called *strictly convex* if for all $x, y \in X$ ($x \neq y$, $\|x\| = \|y\| = 1$) we have $\|\frac{x+y}{2}\| < 1$. A function ψ on Δ_n is called *strictly convex* if for all $s, t \in \Delta_n$ ($s \neq t$) we have $\psi(\frac{1}{2}(s+t)) < \frac{1}{2}(\psi(s) + \psi(t))$.

For $x = (x_1, \dots, x_n) \in \mathbb{C}^n$, we define $|x|$ by $|x| = (|x_1|, \dots, |x_n|)$. We say that $|x| \leq |y|$ if $|x_j| \leq |y_j|$ for $1 \leq j \leq n$. Further, we say that $|x| < |y|$ if $|x| \leq |y|$ and $|x_j| < |y_j|$ for some j . Then we have the following lemma.

Lemma 10 ([3]). *Let $\psi \in \Psi_n$. Let $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n) \in \mathbb{C}^n$. Then*

- (i) *If $|x| \leq |y|$, then $\|x\|_\psi \leq \|y\|_\psi$.*
- (ii) *If ψ is strictly convex and $|x| < |y|$, then $\|x\|_\psi < \|y\|_\psi$.*

Theorem 11 ([3]) *Let $\psi \in \Psi_n$. Then $\|\cdot\|_\psi$ is strictly convex if and only if ψ is strictly convex on Δ_n .*

We see the proof in brief. Let $\|\cdot\|_\psi$ is strictly convex. Suppose that ψ is not strictly convex. Then there exist $s = (s_1, s_2, \dots, s_{n-1})$, $t = (t_1, t_2, \dots, t_{n-1}) \in \Delta_n$ ($s \neq t$) such that $\psi(\frac{1}{2}(s+t)) = \frac{1}{2}(\psi(s) + \psi(t))$. Put $x = (1 - s_1 - \dots - s_{n-1}, s_1, \dots, s_{n-1})$ and $y = (1 - t_1 - \dots - t_{n-1}, t_1, \dots, t_{n-1})$. Then a direct calculation shows that $\|x+y\|_\psi = \|x\|_\psi + \|y\|_\psi$. Since $\|\cdot\|_\psi$ is strictly convex, x and y are colinear, that is, there exists a $k > 0$ such that $x = ky$. This implies that $k = 1$, and so $x = y$. Hence we have $s = t$, a contradiction.

Conversely, suppose that ψ is strictly convex on Δ_n . Take any $x = (x_1, \dots, x_{n-1})$, $y = (y_1, \dots, y_{n-1}) \in \mathbb{C}^n$, $x \neq y$, such that $\|x\| = \|y\| = 1$. Put

$$s = \left(\frac{|x_2|}{|x_1| + \dots + |x_n|}, \dots, \frac{|x_n|}{|x_1| + \dots + |x_n|} \right) \text{ and } t = \left(\frac{|y_2|}{|y_1| + \dots + |y_n|}, \dots, \frac{|y_n|}{|y_1| + \dots + |y_n|} \right).$$

Then, $s, t \in \Delta_n$. If $s \neq t$, in the same way as (15), we have

$$\|x+y\|_\psi < \|x\|_\psi + \|y\|_\psi = 2$$

by Lemma 10 since ψ is strictly convex. In case of $s = t$, we have $|x_j| = |y_j|$ ($1 \leq j \leq n$). So, there exists a positive number θ_j ($0 \leq \theta_j < 2\pi$) such that $x_j = e^{i\theta_j} y_j$. Since $x \neq y$, there exists a j_0 such that $x_{j_0} \neq y_{j_0}$, whence $0 < \theta_{j_0} < 2\pi$. Put $c = |1 + e^{i\theta_{j_0}}|/2$. Then $0 < c < 1$, and we have

$$\begin{aligned} \|x+y\|_\psi &= \|(|x_1+y_1|, \dots, |x_n+y_n|)\|_\psi \\ &= \|(1+e^{i\theta_1}|y_1|, \dots, 1+e^{i\theta_{j_0}}|y_{j_0}|, \dots, 1+e^{i\theta_n}|y_n|)\|_\psi \\ &\leq \|(2|y_1|, \dots, 2c|y_{j_0}|, \dots, 2|y_n|)\|_\psi \\ &= 2\|(|y_1|, \dots, c|y_{j_0}|, \dots, |y_n|)\|_\psi \\ &< 2\|(|y_1|, \dots, |y_{j_0}|, \dots, |y_n|)\|_\psi = 2 \end{aligned}$$

by Lemma 10, as is desired.

References

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