

Zero set theorem of a definable closed set

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1 Introduction

Let $\mathcal{M} = (\mathbb{R}, +, \cdot, <, \dots)$ be an o-minimal expansion of the standard structure $\mathcal{R} = (\mathbb{R}, +, \cdot, <)$ of $\mathbb{R}$. Note that if $\mathcal{M} = \mathcal{R}$, then a definable C^r manifold is a C^r Nash manifold. Definable C^r categories based on $\mathcal{M}$ are generalizations of the C^r Nash category.

For any definable closed subset A of $\mathbb{R}^n$ and $1 \leq r < \infty$, there exists a definable C^r function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $A = f^{-1}(0)$ ([2]). We consider the case where $r = \infty$ and its applications.

General references on o-minimal structures are [1], [2], see also [11]. The term “definable” means “definable with parameters in $\mathcal{M}$ ”.

Theorem 1.1. *Let X be an affine definable C^∞ manifold and V a definable subset closed in X . Then there exists a non-negative definable C^∞ function $f : X \rightarrow \mathbb{R}$ such that $f^{-1}(0) = V$.*

As applications of Theorem 1.1, we have the following results.

Theorem 1.2. *Let $\mathcal{M} = (\mathbb{R}, +, \cdot, <, e^x, \dots)$ be an exponential o-minimal expansion of the standard structure $\mathcal{R} = (\mathbb{R}, +, \cdot, <)$ of the field of real numbers with C^∞ cell decomposition. Then every n -dimensional definable C^∞ manifold X is definably C^∞ imbeddable into $\mathbb{R}^{2n+1}$.*

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Theorem 1.2 is proved in [3] and its definable C^r case ($1 \leq r < \infty$) is proved in [8]. We give another proof of it.

Theorem 1.2 is the definable version of Whitney's imbedding theorem (e.g. 2.14 [4]). Even in the Nash category (i.e. $\mathcal{M} = \mathcal{R}$), we cannot drop the assumption that $\mathcal{M}$ is exponential by Theorem 1.2 [10].

Theorem 1.3 ([6]). *If $0 \leq s < \infty$ and $\mathcal{M}$ is an exponential o-minimal expansion of $\mathcal{R} = (\mathbb{R}, +, \cdot, <)$ with C^∞ cell decomposition, then every definable C^s map between definable C^∞ manifolds is approximated in the definable C^s topology by definable C^∞ maps.*

Its equivariant version is proved in [6].

Using Theorem 1.3 and by a way similar to the proof of Theorem 1.2 and 1.3 [5], we have another proof of the following theorem ([3]).

Theorem 1.4 ([3]). *Let $1 \leq s < r \leq \infty$, then every definable C^s manifold admits a unique definable C^r manifold structure up to definable C^r diffeomorphism.*

2 Proof of our results.

Proof of Theorem 1.1. By definition of affineness and 3.2 [9], X is definably C^∞ diffeomorphic to a definable C^∞ submanifold of some $\mathbb{R}^l$ which is closed in $\mathbb{R}^l$. We identify X with its image. Thus V is closed in $\mathbb{R}^l$. Since $\mathcal{M}$ admits C^∞ cell decomposition, there exists a C^∞ cell decomposition $\mathcal{D}$ partitioning V . For every cell $C \in \mathcal{D}$, the closure $\overline{C}$ of C in X lies in V . Thus if $V = C_1 \cup \dots \cup C_m$, then $V = \overline{C_1} \cup \dots \cup \overline{C_m}$. If C_i is bounded and k -dimensional, then $\overline{C_i}$ is definably C^∞ diffeomorphic to $[-1, 1]^k$. Hence $\overline{C_i}$ is the zeros of a definable C^∞ function. Thus the case where V is compact is proved.

Let $\overline{C_i}$ be unbounded. Replacing $\mathbb{R}^l$ by $\mathbb{R}^{l+1}$, we may assume that $0 \notin \overline{C_i}$. Let $i : \mathbb{R}^{l+1} - \{0\} \rightarrow \mathbb{R}^{l+1} - \{0\}$, $i(x) = \frac{x}{\|x\|^2}$, where $\|x\|$ denotes the norm of x . Then $C'_i = i(\overline{C_i}) \cup \{0\}$ is the one point compactification of $\overline{C_i}$. Thus there exists a definable C^∞ function $\psi : \mathbb{R}^{l+1} \rightarrow \mathbb{R}$ with $C'_i = \psi^{-1}(0)$. Hence $\overline{C_i}$ is definably C^∞ diffeomorphic to the set $C''_i = \{(x, y) \in \mathbb{R}^{l+1} \times \mathbb{R} \mid \psi(x) = 0, \|x\|^2 y = 1\}$. Therefore $\overline{C_i}$ is the zeros of a definable C^∞ function. Since $V = \overline{C_1} \cup \dots \cup \overline{C_m}$, V is the zeros of a definable C^∞ function ϕ . Thus $f := \phi^2 : X \rightarrow \mathbb{R}$ is the required function. ■

The following is a definable C^∞ partition of unity.

Proposition 2.1. *Let $\{U_i\}_{i=1}^k$ be a definable open covering of a definable C^∞ manifold X . Then there exist definable C^∞ functions $\lambda_i : X \rightarrow \mathbb{R}$ ($1 \leq i \leq k$) such that $0 \leq \lambda_i \leq 1$, $\text{supp } \lambda_i \subset U_i$ and $\sum_{i=1}^k \lambda_i = 1$.*

If X is affine, then the definable C^r version of Proposition 2.1 is known in 4.8 [7].

Proof. We now prove that there exists a definable open covering $\{V_i\}_{i=1}^k$ of X such that $\overline{V_i} \subset U_i$, ($1 \leq i \leq k$), where $\overline{V_i}$ denotes the closure of V_i in X .

We proceed by induction on k . If $k = 1$, then there is nothing to prove. Assume that there exists a definable open covering $\{V_i\}_{i=1}^{k-1} \cup \{U_k\}$ of X such that $\overline{V_i} \subset U_i$, ($1 \leq i \leq k-1$).

Let $X_{k-1} := \bigcup_{i=1}^{k-1} V_i$. By the inductive hypothesis, there exists a definable open covering $\{W_i\}_{i=1}^{k-1}$ of X_{k-1} such that $\text{cl } W_i \subset V_i$, where $\text{cl } W_i$ means the closure of W_i in X_{k-1} .

We may assume that U_k is affine. Let $Z_k := U_k \cap \bigcup_{i=1}^{k-1} V_i$ and $\text{Cl } Z_k$ denote the closure of Z_k in U_k . By Theorem 1.1, there exists a non-negative definable C^∞ function $\phi_k : U_k \rightarrow \mathbb{R}$ such that $\phi_k^{-1}(0) = \text{Cl } Z_k$. Since $\text{cl } W_1 \subset V_1$, ϕ_k is extensible to a non-negative definable C^∞ function $\phi_k^1 : U_k \cup W_1 \rightarrow \mathbb{R}$ such that $\phi_k^{1-1}(0) = \text{Cl } Z_k \cup W_1$. Inductively, we have a non-negative definable C^∞ function $\phi : X \rightarrow \mathbb{R}$ such that $\phi^{-1}(0) = \text{Cl } Z_k \cup W_1 \cdots \cup W_{k-1}$. Let $V_k := \{x \in U_k \mid \phi(x) > 0\}$. Then $V_k = \{x \in X \mid \phi(x) > 0\}$, $\overline{V_k} \subset U_k$ and $\{V_i\}_{i=1}^k$ is the required definable open covering of X .

By Theorem 1.1, we have a non-negative definable C^∞ function $\mu_i : U_i \rightarrow \mathbb{R}$ such that $\mu_i^{-1}(0) = U_i - V_i$. Thus μ_i is extensible to a non-negative definable C^∞ function $\mu'_i : X \rightarrow \mathbb{R}$ such that $\mu_i'^{-1}(0) = X - V_i$. Therefore $\lambda_i := \mu'_i / \sum_{i=1}^k \mu'_i$ is the required definable C^r partition of unity. ■

Proof of Theorem 1.1. Let $\{\phi_i : U_i \rightarrow \mathbb{R}^n\}_{i=1}^k$ be a definable C^r atlas of X . By Proposition 2.1, we have definable C^∞ functions $\lambda_i : X \rightarrow \mathbb{R}$, ($1 \leq i \leq k$) such that $0 \leq \lambda_i \leq 1$, $\text{supp } \lambda_i \subset U_i$ and $\sum_{i=1}^k \lambda_i = 1$. Thus the map $F : X \rightarrow \mathbb{R}^{nk} \times \mathbb{R}^k$ defined by $F(x) = (\lambda_1(x)\phi_1(x), \dots, \lambda_k(x)\phi_k(x), \lambda_1(x), \dots, \lambda_k(x))$ is a definable C^∞ imbedding. Hence X is affine. Thus it is either compact or compactifiable by 1.2 [7]. Hence we may assume that X is affine and compact at the beginning. A similar argument of the proof of 1.4 [12], every definable C^∞ map $f : X \rightarrow \mathbb{R}^{2n+1}$ can be approximated in the C^r topology by an injective definable C^∞ immersion $h : X \rightarrow \mathbb{R}^{2n+1}$. Since X is compact, h is the required definable C^∞ imbedding. ■

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