

# Abstract approach to Dirac equation with time dependent potential

東京理科大学・理学部 吉井 健太郎 (Kentarou Yoshii)  
Department of Mathematics, Science University of Tokyo

## 1. Introduction and statement of the result

Let  $H^1(\mathbb{R}^3)$  be the usual Sobolev space and

$$H_1(\mathbb{R}^3) := \{u \in L^2(\mathbb{R}^3); (1 + |x|^2)^{1/2}u \in L^2(\mathbb{R}^3)\}, \quad \Sigma^1(\mathbb{R}^3) := H^1(\mathbb{R}^3) \cap H_1(\mathbb{R}^3).$$

In this paper we consider the Cauchy problem for Dirac equation in  $L^2(\mathbb{R}^3)^4$ :

$$(DE) \quad i \frac{\partial u}{\partial t} = H_0 u + V(t, x)u, \quad H_0 := \alpha \cdot D + m\beta$$

with  $u(\cdot, 0) = u_0 \in H^1(\mathbb{R}^3)^4 \cap H_1(\mathbb{R}^3)^4$ . Here  $u : [0, T] \times \mathbb{R}^3 \rightarrow \mathbb{C}^4$  is an unknown function and  $\alpha := (\alpha_1, \alpha_2, \alpha_3)$ ,  $\beta := \alpha_4$  are  $4 \times 4$  Hermitian matrices satisfying

$$\alpha_j \alpha_k + \alpha_k \alpha_j = 2\delta_{jk}I,$$

where  $\delta_{jk}$  is Kronecker symbol,  $I$  is the identity matrix and  $D = i^{-1}(\partial_1, \partial_2, \partial_3)$ ,  $m \geq 0$ .  $V(t, x)$  is a  $4 \times 4$  Hermitian matrix-valued potential.

We shall show the existence of a unique classical solution under some conditions on potential  $V$ . It seems that this problem is associated with Kato and Yajima [5].

Now we want to state our main theorem.

**Theorem 1.1.** *Set  $V = V_0(t, x) + q(t, x)I$ , where Hermitian matrix-valued function  $V_0$  and real valued function  $q$  satisfy the conditions:*

$$(V1) \quad |V_0(t, x)| \leq a|x|^{-1} + b,$$

$$(V2) \quad \left| \frac{\partial}{\partial t} V_0(t, x) \right| \leq (a|x|^{-1} + b)\sigma_1(t),$$

$$(q) \quad (1 + |x|^2)^{-1/2}q \in C(0, T; L^\infty(\mathbb{R}^3)) \cap L^\infty(0, T; W^{1,\infty}(\mathbb{R}^3)),$$

where  $|V|$  denotes the operator norm of  $V$ ,  $a, b$  are nonnegative constants with  $a < 1/2$  and  $\sigma \in L^1(0, T)$ . Then for every initial value  $u_0 \in \Sigma^1(\mathbb{R}^3)^4$  problem (DE) has a unique (classical) solution

$$u(\cdot) \in C^1([0, T]; L^2(\mathbb{R}^3)^4) \cap C([0, T]; \Sigma^1(\mathbb{R}^3)^4).$$

**Remark 1.** Conditions (V1) and (V2) imply  $(a|x|^{-1}+b)^{-1}V_0 \in W^{1,1}(0, T; L^\infty(\mathbb{R}^3)^{4 \times 4})$ .

To prove Theorem 1.1 we employ an abstract approach. This approach becomes a little simple than [10].

Let  $\{A(t); t \in [0, T]\}$  be a family of closed linear operators in a complex Hilbert space  $X$ . Then we consider the abstract Cauchy problem for linear evolution equations of the form

$$(ACP) \quad \begin{cases} \frac{d}{dt}u(t) + A(t)u(t) = f(t), & t \in [0, T], \\ u(0) = u_0. \end{cases}$$

Here the initial value  $u_0$  is selected as follows.

To introduce our assumption on  $\{A(t); t \in [0, T]\}$  we need one more family  $\{S(t); t \in [0, T]\}$  of auxiliary operators in  $X$ .

**Assumption on  $\{S(t)\}$ .** The family  $\{S(t)\}$  satisfies the following three conditions:

(S1) For every  $t \in [0, T]$ ,  $S(t)$  is positive selfadjoint in  $X$  and

$$(u, S(t)u) \geq \|u\|^2 \quad \text{for } u \in D(S(t)).$$

Let  $Y_t$  be the Hilbert space  $D(S(t)^{1/2})$  with new inner product  $(\cdot, \cdot)_{Y_t}$  and norm  $\|\cdot\|_{Y_t}$  for  $t \in [0, T]$  and  $u, v \in Y_t$ :

$$(u, v)_{Y_t} := (S(t)^{1/2}u, S(t)^{1/2}v), \quad \|u\|_{Y_t} := (u, u)_{Y_t}^{1/2},$$

embedded continuously and densely in  $X$ . In particular,  $Y := Y_0$  plays the roll of the space of initial values as in Theorem 1.3 (see below).

(S2) For  $t \in [0, T]$ ,  $Y_t = Y$ , and  $S(\cdot)^{1/2} \in C_*([0, T]; B(Y, X))$ , where  $B(Y, X)$  is the space of all bounded linear operators on  $Y$  to  $X$ , with norm  $\|\cdot\|_{B(Y, X)}$ , while the subscript  $*$  is used to refer the strong operator topology in  $B(Y, X)$  (for this notation see Kato [4]).

(S3) There exists a nonnegative function  $\sigma \in L^1(I)$  such that

$$\|S(t)^{1/2}v - S(s)^{1/2}v\| \leq \left| \int_s^t \sigma(r) dr \right| \max_{r \in \{s, t\}} \|S(r)^{1/2}v\|, \quad v \in Y, \quad t, s \in [0, T].$$

In connection with the symbol  $B(Y, X)$  we shall also use the abbreviation:  $B(X) := B(X, X)$ ,  $B(Y) := B(Y, Y)$ .

Let  $\{S(t)\}$  be as defined above. Then we may introduce the following

**Assumption on  $\{A(t)\}$ .** The family  $\{A(t)\}$  satisfies the following four conditions:

(A1) There exists a constant  $\alpha \geq 0$  such that

$$|\operatorname{Re} (A(t)v, v)| \leq \alpha \|v\|^2, \quad v \in D(A(t)), \quad t \in [0, T].$$

(A2)  $Y \subset D(A(t))$ ,  $t \in [0, T]$ .

(A3) There exists a constant  $\beta \geq \alpha$  such that

$$|\operatorname{Re}(A(t)u, S(t)u)| \leq \beta \|S(t)^{1/2}u\|^2, \quad u \in D(S(t)) \subset Y, \quad t \in [0, T].$$

(A4)  $A(\cdot) \in C_*([0, T]; B(Y, X))$ .

**Theorem 1.2.** *Suppose that Assumptions on  $\{A(t)\}$  and  $\{S(t)\}$  are satisfied. Then there exists a unique evolution operator  $\{U(t, s); (t, s) \in \Delta_+\}$  for (ACP), where  $\Delta_+ := \{(t, s); 0 \leq s \leq t \leq T\}$ , having the following properties:*

(i)  $U(\cdot, \cdot)$  is strongly continuous on  $\Delta_+$  to  $B(X)$ , with

$$\|U(t, s)\|_{B(X)} \leq e^{\alpha(t-s)}, \quad (t, s) \in \Delta_+.$$

(ii)  $U(t, r)U(r, s) = U(t, s)$  on  $\Delta_+$  and  $U(s, s) = 1$  (the identity).

(iii)  $U(t, s)Y \subset Y$  and  $U(\cdot, \cdot)$  is strongly continuous on  $\Delta_+$  to  $B(Y)$ , with

$$(1.1) \quad \|U(t, s)\|_{B(Y_s, Y_t)} \leq \exp\left(\int_s^t \tilde{\sigma}(r) dr\right), \quad (t, s) \in \Delta_+,$$

$$(1.2) \quad \|U(t, s)\|_{B(Y)} \leq \exp\left(\int_0^s \sigma(r) dr\right) \exp\left(2 \int_s^t \tilde{\sigma}(r) dr\right), \quad (t, s) \in \Delta_+,$$

where  $\tilde{\sigma}(t) := \beta + \sigma(t)$ .

Furthermore, let  $v \in Y$ . Then  $U(\cdot, \cdot)v \in C^1(\Delta_+; X)$ , with

(iv)  $(\partial/\partial t)U(t, s)v = -A(t)U(t, s)v$ ,  $(t, s) \in \Delta_+$ , and

(v)  $(\partial/\partial s)U(t, s)v = U(t, s)A(s)v$ ,  $(t, s) \in \Delta_+$ .

The equation in (ACP) is naturally interpreted if the solution has an additional property  $u(\cdot) \in C([0, T]; Y)$ . In fact, it is guaranteed by condition (A2) that  $u(t) \in Y \subset D(A(t))$  for every  $t \in [0, T]$ .

**Theorem 1.3.** *Let  $\{U(t, s)\}$  be the evolution operator for (ACP) as in Theorem 1.2 above. For  $u_0 \in Y$  and  $f(\cdot) \in C([0, T]; X) \cap L^1(0, T; Y)$  define  $u(\cdot)$  as*

$$u(t) := U(t, 0)u_0 + \int_0^t U(t, s)f(s) ds.$$

Then (ACP) has a unique (classical) solution

$$u(\cdot) \in C^1([0, T]; X) \cap C([0, T]; Y).$$

**Remark 2.** The assertion of Theorem 1.3 is same as in [10]. However, we can simplify assumption on  $\{S(t)\}$  and the proof of Theorem 1.2 a little than [10].

## 2. Preliminaries

Let  $X$  be a (complex) Hilbert space. In this section we prepare some useful lemmas.

### 2.1. Lemmas on time-independent operators

In this subsection we consider a pair  $\{A, S\}$  of closed linear operators in  $X$ . Let  $A$  be *quasi-accretive* in the sense of Kato [1, Section V.3.10]:

$$(2.1) \quad \operatorname{Re}(Av, v) \geq -\alpha \|v\|^2, \quad v \in D(A),$$

for some constant  $\alpha \geq 0$ ; in other words,  $\alpha + A$  is accretive. Let  $S$  be a positive-definite selfadjoint operator in  $X$ , with  $D(S) \subset D(A)$ . Now we can state a condition connecting  $A$  and  $S$ : assume that there exist a constant  $\beta \geq 0$  such that

$$(2.2) \quad \operatorname{Re}(Au, Su) \geq -\beta (u, Su), \quad u \in D(S) \subset D(A).$$

**Lemma 2.1.** *Let  $A$  and  $S$  be as in (2.1) and (2.2), respectively. Then*

(a)  $\alpha + A$  is  $m$ -accretive in  $X$ ; (b)  $D(S)$  is a core for  $A$ .

(a) was first proved by Okazawa [6], while (b) was later noted by Kato [3] (for a complete proof see Tanabe [13, Section 7.7] or Ouhabaz [11, Section 1.3.3]).

Given the pair  $\{A, S\}$  as in Lemma 2.1, let  $\{A_n; n > \alpha\}$  and  $\{S_\varepsilon; \varepsilon > 0\}$  be Yosida approximations of  $A$  and  $S$ , respectively:

$$(2.3) \quad A_n := AJ_n = n(1 - J_n), \quad J_n := (1 + n^{-1}A)^{-1},$$

$$(2.4) \quad S_\varepsilon := SJ_\varepsilon = \varepsilon^{-1}(1 - J_\varepsilon), \quad J_\varepsilon := (1 + \varepsilon S)^{-1}.$$

Then the pair  $\{A_n, S_\varepsilon\}$  satisfies conditions in Lemma 2.1 with  $\alpha$  and  $\beta$  replaced with  $\alpha(1 - n^{-1}\alpha)^{-1}$  and  $\beta(1 - n^{-1}\beta)^{-1}$ , respectively.

(2.1) and (2.2) are invariant under taking their Yosida approximation.

**Lemma 2.2** (Okazawa [7, Lemma 2.2]). *Given  $A$  as in Lemma 2.1, let  $\{A_n\}$  and  $\{J_n\}$  be as in (2.3). Then  $\|J_n\|_{B(X)} \leq (1 - n^{-1}\alpha)^{-1}$  ( $n > \alpha$ ) and  $\|A_n\|_{B(X)} \leq n$  ( $n \geq 2\alpha$ ), with*

$$(2.1)_n \quad \operatorname{Re}(A_n w, w) \geq -\alpha(1 - n^{-1}\alpha)^{-1} \|w\|^2, \quad w \in X, \quad n > \alpha.$$

**Lemma 2.3** ([9, Lemmas 2.7]). *Given the pair  $\{A, S\}$  as in Lemma 2.1, let  $\{A_n\}$  and  $\{S_\varepsilon\}$  be as in (2.3) and (2.4). Then*

$$(2.2)_{n,\varepsilon} \quad \operatorname{Re}(A_n w, S_\varepsilon w) \geq -\beta(1 - n^{-1}\beta)^{-1} \|S_\varepsilon^{1/2} w\|^2, \quad w \in X, \quad n > \beta \geq \alpha.$$

## 2.2. Lemmas on time-dependent operators

Since we need conditions (A1), (A3) and (S3) as a whole only in the last step of the proof of Theorem 1.2, we may introduce a set of weaker conditions:

(A1)<sub>+</sub> There exists a constant  $\alpha \geq 0$  such that

$$\operatorname{Re} (A(t)v, v) \geq -\alpha \|v\|^2, \quad v \in D(A(t)), \quad t \in [0, T].$$

(A3)<sub>+</sub> There exists a constant  $\beta \geq \alpha$  such that

$$\operatorname{Re} (A(t)v, S(t)v) \geq -\beta \|S(t)^{1/2}v\|^2, \quad v \in D(S(t)) \subset Y, \quad t \in [0, T].$$

(S3)<sub>+</sub> There exists a nonnegative function  $\sigma \in L^1(0, T)$  such that

$$\|S(t)^{1/2}u\| - \|S(s)^{1/2}u\| \leq \left( \int_s^t \sigma(r) dr \right) \|S(s)^{1/2}u\|, \quad u \in Y, \quad 0 \leq s \leq t \leq T.$$

By virtue of Lemma 2.1, it follows from conditions (A1)<sub>+</sub>, (A2) and (A3)<sub>+</sub> that  $\alpha + A(t)$  is  $m$ -accretive in  $X$ . We can obtain following

**Proposition 2.4.** *Let  $\{S(t)\}$  be a family of selfadjoint operators in  $X$ ,  $\{A(t)\}$  a family of closed linear quasi-accretive operators in  $X$ , satisfying conditions (S1), (S2), (S3)<sub>+</sub> for  $\{S(t)\}$ , and conditions (A1)<sub>+</sub>, (A2), (A3)<sub>+</sub>, (A4) for  $\{A(t)\}$ . Then there exist Yosida approximations  $\{A_n(t); n > \alpha\}$  and  $\{S_\varepsilon(t); \varepsilon > 0\}$  of  $\{A(t)\}$  and  $\{S(t)\}$ , respectively:*

$$(2.5) \quad A_n(t) := A(t)J_n(t) = n(1 - J_n(t)), \quad J_n(t) := (1 + n^{-1}A(t))^{-1},$$

$$(2.6) \quad S_\varepsilon(t) := S(t)J_\varepsilon(t) = \varepsilon^{-1}(1 - J_\varepsilon(t)), \quad J_\varepsilon(t) := (1 + \varepsilon S(t))^{-1}.$$

Put

$$(2.7) \quad \alpha_n := \alpha(1 - n^{-1}\alpha)^{-1}, \quad \beta_n := \beta(1 - n^{-1}\beta)^{-1} \quad (n > \beta \geq \alpha).$$

Then the pair of  $\{A_n(t)\}$  and  $\{S_\varepsilon(t)\}$  satisfies Assumptions on  $\{A(t)\}$  and  $\{S(t)\}$  with  $Y$ ,  $\alpha$  and  $\beta$  replaced with  $X$ ,  $\alpha_n$  and  $\beta_n$ , respectively:

(S<sub>ε</sub>1)  $D(S_\varepsilon(t)) = X$ ,  $t \in [0, T]$ .

(S<sub>ε</sub>2)  $S_\varepsilon(\cdot) \in C_*([0, T]; B(X))$ .

(S<sub>ε</sub>3)<sub>+</sub> For  $w \in X$ ,

$$\|S_\varepsilon(t)^{1/2}w\|^2 - \|S_\varepsilon(s)^{1/2}w\|^2 \leq \left[ \left( 1 + \int_s^t \sigma(r) dr \right)^2 - 1 \right] \|S_\varepsilon(s)^{1/2}w\|^2, \quad 0 \leq s \leq t \leq T.$$

(A<sub>n</sub>1)<sub>+</sub> For  $n > \alpha$ ,  $t \in [0, T]$ ,  $\|J_n(t)\|_{B(X)} \leq (1 - n^{-1}\alpha)^{-1}$  and

$$\operatorname{Re} (A_n(t)w, w) \geq -\alpha_n \|w\|^2, \quad w \in X.$$

(A<sub>n</sub>2)  $D(A_n(t)) = X$ ,  $t \in [0, T]$ .

(A<sub>n</sub>3)<sub>+</sub> For  $n > \beta$ ,  $t \in [0, T]$ ,  $\operatorname{Re} (A_n(t)w, S_\varepsilon(t)w) \geq -\beta_n \|S_\varepsilon(t)^{1/2}w\|^2$ ,  $w \in X$ .

(A<sub>n</sub>4)  $A_n(\cdot) \in C_*([0, T]; B(X))$  ( $n > \beta$ ), with  $\|A_n(t)\|_{B(X)} \leq n$  ( $n \geq 2\alpha$ ,  $t \in [0, T]$ ).

$(\mathbf{S}_\varepsilon \mathbf{1})$  and  $(\mathbf{A}_n \mathbf{2})$  are trivial.  $(\mathbf{A}_n \mathbf{1})_+$  and  $(\mathbf{A}_n \mathbf{3})_+$  follow from Lemmas 2.2 and 2.3, respectively.  $(\mathbf{A}_n \mathbf{4})$  and  $(\mathbf{S}_\varepsilon \mathbf{2})$  were proved in [7, Lemma 3.1 (a), (b)] and [10, Proposition 2.5], respectively. We show only the proof of  $(\mathbf{S}_\varepsilon \mathbf{3})_+$ . The first half of the proof is same as in [10, Proposition 2.5].

*Proof of  $(\mathbf{S}_\varepsilon \mathbf{3})_+$ .* First we remind of the definition of  $S_\varepsilon(\cdot)$ :

$$w = J_\varepsilon(t)w + \varepsilon S_\varepsilon(t)w, \quad w \in X.$$

Then the symmetry and positivity of  $J_\varepsilon(\cdot)$  yield that

$$\begin{aligned} (2.8) \quad & (w, S_\varepsilon(t)w - S_\varepsilon(s)w) \\ &= (w, S_\varepsilon(t)(J_\varepsilon(s) + \varepsilon S_\varepsilon(s))w - (J_\varepsilon(t) + \varepsilon S_\varepsilon(t))S_\varepsilon(s)w) \\ &= (J_\varepsilon(s)w, (S(t) - S(s))J_\varepsilon(s)w) + ((J_\varepsilon(t) - J_\varepsilon(s))w, (S(t) - S(s))J_\varepsilon(s)w) \\ &= (J_\varepsilon(s)w, (S(t) - S(s))J_\varepsilon(s)w) \\ &\quad - \varepsilon (J_\varepsilon(t)(S(t) - S(s))J_\varepsilon(s)w, (S(t) - S(s))J_\varepsilon(s)w) \\ &\leq (J_\varepsilon(s)w, (S(t) - S(s))J_\varepsilon(s)w) \\ &= \|S(t)^{1/2}J_\varepsilon(s)w\|^2 - \|S(s)^{1/2}J_\varepsilon(s)w\|^2. \end{aligned}$$

On the other hand, it follows from condition  $(\mathbf{S3})_+$  that

$$(2.9) \quad \|S(t)^{1/2}J_\varepsilon(s)w\| \leq \left(1 + \int_s^t \sigma(r) dr\right) \|S_\varepsilon(s)^{1/2}w\|,$$

where we have used  $\|J_\varepsilon(s)^{1/2}\|_{B(X)} \leq 1$  ( $\varepsilon > 0$ ). We see from (2.9) that

$$(2.10) \quad \|S(t)^{1/2}J_\varepsilon(s)w\|^2 - \|S(s)^{1/2}J_\varepsilon(s)w\|^2 \leq \left[\left(1 + \int_s^t \sigma(r) dr\right)^2 - 1\right] \|S_\varepsilon(s)^{1/2}w\|^2.$$

Combining (2.10) with (2.8), we obtain  $(\mathbf{S}_\varepsilon \mathbf{3})_+$ . □

The next proposition is essential in the proof of Theorem 1.2 in which it is required to differentiate  $S_\varepsilon(\cdot)$ . For that purpose  $\{S_\varepsilon(t); t \in [0, T]\}$  is replaced with a new family

$$(2.11) \quad S_\varepsilon^h(t) := \frac{1}{h} \int_t^{t+h} S_\varepsilon(s) ds, \quad h > 0, t \in [0, T],$$

where we define as  $S_\varepsilon(s) := S_\varepsilon(T)$  ( $s > T$ ). Then, in view of  $(\mathbf{S}_\varepsilon \mathbf{2})$ , we have

$$(2.12) \quad \frac{d}{dt} S_\varepsilon^h(t)w = \frac{1}{h} (S_\varepsilon(t+h)w - S_\varepsilon(t)w),$$

$$(2.13) \quad \text{s-}\lim_{h \downarrow 0} S_\varepsilon^h(t)w = S_\varepsilon(t)w, \quad w \in X.$$

**Proposition 2.5.** Assume that  $\{S(t)\}$  satisfies conditions (S1), (S2) and (S3)<sub>+</sub>. Let  $\{S_\varepsilon^h(t); t \in [0, T]\}$  be as in (2.11). Then  $S_\varepsilon^h(\cdot) \in C_*^1([0, T]; B(X))$ , with for  $w \in X$ ,

$$(\mathbf{S}_\varepsilon^h \mathbf{3})_+ \quad \left(w, \frac{d}{ds} S_\varepsilon^h(s)w\right) \leq \frac{1}{h} \left[ \left(1 + \int_s^{s+h} \sigma(r) dr\right)^2 - 1 \right] \|S_\varepsilon(s)^{1/2}w\|^2, \quad s \in [0, T],$$

where one defines as  $\sigma(s) := 0$  ( $s > T$ ). Moreover, for  $w \in X$  one has

$$(2.14) \quad \limsup_{h \downarrow 0} \int_{t_0}^t \left(w, \frac{d}{ds} S_\varepsilon^h(s)w\right) ds \leq 2 \int_{t_0}^t \sigma(s) \|S_\varepsilon(s)^{1/2}w\|^2 ds, \quad t_0 \leq t.$$

With regard to the proof of Proposition 2.6,  $(\mathbf{S}_\varepsilon^h \mathbf{3})_+$  follows from (2.12) and  $(\mathbf{S}_\varepsilon \mathbf{3})_+$ . On the other hand, (2.14) is a consequence of following:

**Lemma 2.6.** Let  $\varphi \in L^1(0, T)$  and  $\psi \in L^\infty(0, T)$ . Then

$$\lim_{h \downarrow 0} \frac{1}{h} \int_0^t \left[ \left(1 + \int_s^{s+h} \varphi(r) dr\right)^2 - 1 \right] \psi(s) ds = 2 \int_0^t \varphi(s) \psi(s) ds, \quad 0 \leq t \leq T,$$

where one define as  $\varphi(s) := 0$  ( $s > T$ ).

*Proof.* We also define that  $\varphi(s) := 0$  ( $s < 0$ ). Since

$$\left(1 + \int_s^{s+h} \varphi(r) dr\right)^2 - 1 = \left(\int_s^{s+h} \varphi(r) dr\right) \left(2 + \int_s^{s+h} \varphi(\tau) d\tau\right),$$

by integrating by parts, we obtain

$$\begin{aligned} & \int_0^t \left[ \left(1 + \int_s^{s+h} \varphi(r) dr\right)^2 - 1 \right] \psi(s) ds \\ &= \int_0^{t+h} \varphi(r) \left[ \int_{r-h}^r \left(2 + \int_s^{s+h} \varphi(\tau) d\tau\right) \tilde{\psi}(s) ds \right] dr \\ &= \int_0^{t+h} \varphi(r) \left[ \int_{r-h}^{r+h} \left(2\bar{\psi}(s) + \varphi(\tau) \int_{\tau-h}^\tau \bar{\psi}(s) ds\right) d\tau \right] dr, \end{aligned}$$

where  $\tilde{\psi}, \bar{\psi} \in L^1(\mathbb{R})$  are defined as

$$\tilde{\psi}(s) := \begin{cases} \psi(s), & 0 < s < t, \\ 0, & \text{otherwise,} \end{cases} \quad \bar{\psi}(s) := \begin{cases} \tilde{\psi}(s), & r-h < s < r, \\ 0, & \text{otherwise.} \end{cases}$$

Set

$$\Phi_h(r) := \frac{1}{h} \varphi(r) \left[ \int_{r-h}^{r+h} \left(2\bar{\psi}(s) + \varphi(\tau) \int_{\tau-h}^\tau \bar{\psi}(s) ds\right) d\tau \right].$$

Then we obtain  $\lim_{h \downarrow 0} \Phi_h(r) = 2\varphi(r)\tilde{\psi}(r)$  (a.a.  $r \in (0, t)$ ) and

$$\begin{aligned} |\Phi_h(r)| &\leq |\varphi(r)| \left(2 + \int_{r-h}^{r+h} |\varphi(\tau)| d\tau\right) \|\psi\|_{L^\infty(0, T)} \\ &\leq |\varphi(r)| \left(2 + \|\varphi\|_{L^1(0, T)}\right) \|\psi\|_{L^\infty(0, T)} \in L^1(0, T). \end{aligned}$$

Thus the assertion follows from the Lebesgue convergence theorem.  $\square$

### 3. Construction of evolution operators

In this section we shall prove Theorem 1.2. The major part of the assertions in Theorem 1.2 is contained in the following

**Theorem 3.1.** *Let  $\{S(t)\}$  be a family of selfadjoint operators in  $X$ ,  $\{A(t)\}$  a family of closed linear quasi-accretive operators in  $X$ , satisfying conditions (S1), (S2), (S3)<sub>+</sub> for  $\{S(t)\}$ , and conditions (A1)<sub>+</sub>, (A2), (A3)<sub>+</sub>, (A4) for  $\{A(t)\}$ . Then there exists a unique two-parameter family  $\{U(t, s); (t, s) \in \Delta_+\}$  in  $B(X)$ , having the properties:*

(iii)<sub>w</sub>  $U(t, s)Y \subset Y$ , with (1.1);

(iv)<sub>w</sub>  $U(\cdot, \cdot)v \in W^{1,1}(\Delta_+; X)$ , with

$$(\partial/\partial t)U(t, s)v = -A(t)U(t, s)v, \quad \text{a.a. } t \in (s, T), \quad v \in Y;$$

in addition to properties (i), (ii) and (v) in Theorem 1.2.

Therefore the first purpose of this section is to prove Theorem 3.1. To replace (iii)<sub>w</sub> and (iv)<sub>w</sub> with (iii) and (iv) in the final step we need the whole conditions (S1)–(S3) and (A1)–(A4) in Theorem 1.2.

To prove Theorem 3.1 we consider the approximate problem:

$$(ACP)_n \quad \begin{cases} (d/dt)u_n(t) + A_n(t)u_n(t) = 0, & t \in [s, T], \\ u_n(s) = w \in X, \end{cases}$$

where  $\{A_n(t); n > \alpha\}$  is the Yosida approximation as in (2.5).

According to Pazy [12, Theorems 5.1.1 and 5.1.2] (in which  $A_n(\cdot) \in C([0, T]; B(X))$  is assumed, however, it can be replaced by  $A_n(\cdot) \in C_*([0, T]; B(X))$  (condition (A<sub>n</sub>4)) with appropriate modification of the conclusion), we obtain the following

**Proposition 3.2.** *Let  $s \in [0, T)$  and  $n > 2\beta$ , where  $\beta$  is defined in (A3)<sub>+</sub>. Then the approximate problem  $(ACP)_n$  has a unique classical solution  $u_n(\cdot) \in C^1([s, T]; X)$ . Accordingly, there exists a unique evolution operator  $\{U_n(t, s); (t, s) \in \Delta_+\}$  for  $(ACP)_n$  having the following properties:*

(i)<sub>n</sub>  $U_n(\cdot, \cdot)$  is strongly continuous on  $\Delta_+$  to  $B(X)$ , with

$$\|U_n(t, s)\|_{B(X)} \leq e^{n(t-s)}, \quad (t, s) \in \Delta_+.$$

(ii)<sub>n</sub>  $U_n(t, r)U_n(r, s) = U_n(t, s)$  on  $\Delta_+$  and  $U_n(s, s) = 1$  (the identity).

(iii)<sub>n</sub>  $U_n(t, s)$  is uniformly continuous on  $\Delta_+$ .

(iv)<sub>n</sub>  $(\partial/\partial t)U_n(t, s)w = -A_n(t)U_n(t, s)w, \quad w \in X, \quad (t, s) \in \Delta_+.$

(v)<sub>n</sub>  $(\partial/\partial s)U_n(t, s)v = U_n(t, s)A_n(s)v, \quad w \in X, \quad (t, s) \in \Delta_+.$

For the limiting procedure for  $\{U_n(t, s)\}$  we need several estimates of  $\{U_n(t, s)\}$  which are independent of  $n$ .

**Lemma 3.3.** Let  $\{U_n(t, s)\}$  be as in Proposition 3.2,  $\alpha_n, \beta_n$  as in (2.7) and  $\sigma$  as in condition **(S3)**<sub>+</sub>. If  $n > 2\beta$ , then for  $(t, s) \in \Delta_+$ ,

(a)  $\|U_n(t, s)\|_{B(X)} \leq e^{\alpha_n(t-s)}.$

(b)  $U_n(t, s)Y \subset Y$  and with

$$\|S(t)^{1/2}U_n(t, s)v\| \leq e^{\beta_n(t-s)} \exp\left(\int_s^t \sigma(r) dr\right) \|S(s)^{1/2}v\|, \quad v \in Y.$$

(c) There exists a constant  $c \geq 0$  such that

$$(3.1) \quad \|A(t)v\| \leq c\|S(t)^{1/2}v\|, \quad v \in Y,$$

and hence

$$(3.2) \quad \|A_n(t)U_n(t, s)v\| \leq M\|v\|_Y, \quad v \in Y,$$

where  $M := 2c \exp(2\beta T + \|\sigma\|_{L^1(0,T)})$ .

*Proof.* (a) We see from property **(iv)**<sub>n</sub> and **(A<sub>n</sub>1)**<sub>+</sub> that for  $w \in X$ ,

$$\begin{aligned} (\partial/\partial r)\|U_n(r, s)w\|^2 &= -2 \operatorname{Re} (A_n(r)U_n(r, s)w, U_n(r, s)w) \\ &\leq 2\alpha_n\|U_n(r, s)w\|^2, \quad s \leq r \leq t. \end{aligned}$$

Integrating this inequality, we obtain the assertion.

(b) Let  $\{S_\varepsilon(t)\}$  and  $\{S_\varepsilon^h(t)\}$  be as in (2.6) and (2.11). Since  $S_\varepsilon^h(t)^{1/2}$  is bounded and symmetric on  $X$ , it follows from property **(iv)**<sub>n</sub> that for  $v \in Y$ ,

$$(3.3) \quad \begin{aligned} (\partial/\partial r)\|S_\varepsilon^h(r)^{1/2}U_n(r, s)v\|^2 &= -2 \operatorname{Re} (A_n(r)U_n(r, s)v, S_\varepsilon^h(r)U_n(r, s)v) \\ &\quad + (U_n(r, s)v, ((d/dr)S_\varepsilon^h(r))U_n(r, s)v). \end{aligned}$$

Integrating this equality on  $[s, t]$ , we see from **(S<sub>ε</sub><sup>h</sup>3)**<sub>+</sub> that

$$\begin{aligned} \|S_\varepsilon^h(t)^{1/2}U_n(t, s)v\|^2 &\leq \|S_\varepsilon^h(s)^{1/2}v\|^2 - 2 \int_s^t \operatorname{Re} (A_n(r)U_n(r, s)v, S_\varepsilon^h(r)U_n(r, s)v) dr \\ &\quad + \int_s^t \frac{1}{h} \left[ \left( 1 + \int_r^{r+h} \sigma(\tau) d\tau \right)^2 - 1 \right] \|S_\varepsilon(r)^{1/2}U_n(r, s)v\|^2 dr. \end{aligned}$$

Passing to the limit as  $h \downarrow 0$ , we see from (2.13), (2.14) and **(A<sub>n</sub>3)**<sub>+</sub> that

$$\begin{aligned} \|S_\varepsilon(t)^{1/2}U_n(t, s)v\|^2 &\leq \|S_\varepsilon(s)^{1/2}v\|^2 - 2 \int_s^t \operatorname{Re} (A_n(r)U_n(r, s)v, S_\varepsilon(r)U_n(r, s)v) dr \\ &\quad + 2 \int_s^t \sigma(r) \|S_\varepsilon(r)^{1/2}U_n(r, s)v\|^2 dr \\ &\leq \|S(s)^{1/2}v\|^2 + 2 \int_s^t (\beta_n + K\sigma(r)) \|S_\varepsilon(r)^{1/2}U_n(r, s)v\|^2 dr. \end{aligned}$$

Applying the Gronwall lemma, we obtain

$$\|S_\varepsilon(t)^{1/2}U_n(t, s)v\|^2 \leq \exp\left(2\int_s^t (\beta_n + \sigma(r)) dr\right) \|S(s)^{1/2}v\|^2.$$

Passing to the limit as  $\varepsilon \rightarrow 0$ , we obtain the assertion.

(c) The existence of such a constant  $c$  is guaranteed by conditions **(A2)** and **(A4)**. Therefore we see from **(A<sub>n</sub>1)<sub>+</sub>** and (3.1) that

$$\|A_n(t)v\| \leq (1 - n^{-1}\alpha)^{-1} \|A(t)v\| \leq 2c\|S(t)^{1/2}v\|, \quad v \in Y.$$

Noting that

$$(3.4) \quad \|S(s)^{1/2}v\| \leq \left(1 + \int_0^s \sigma(r) dr\right) \|S(0)^{1/2}v\| \leq \exp\left(\int_0^s \sigma(r) dr\right) \|v\|_Y,$$

we obtain (3.2) as a consequence of (b).  $\square$

Next two lemmas guarantee the existence and uniqueness of evolution operator.

**Lemma 3.4** ([10, Lemma 3.4]). *Let  $\{U_n(t, s)\}$  be the evolution operator for  $(ACP)_n$ . Then there exists a new family  $\{U(t, s); (t, s) \in \Delta_+\}$  such that  $U(t, s) := \text{s-lim}_{n \rightarrow \infty} U_n(t, s)$ , where the convergence is uniform on  $\Delta_+$ , and has properties (i) and (ii) in Theorem 1.2, with*

$$\|U(t, s)v - U_n(t, s)v\| \leq \sqrt{\frac{2T}{n - 2\alpha}} M e^{2\alpha T} \|v\|_Y, \quad v \in Y, \quad n > 2\beta.$$

**Lemma 3.5** ([10, Lemma 3.4]). *Let  $\{U(t, s)\}$  be as in Lemma 3.4 and  $v \in Y$ . Then*

(a)  $U(t, s)Y \subset Y$  and

$$(3.5) \quad \|S(t)^{1/2}U(t, s)v\| \leq \exp\left(\int_s^t \tilde{\sigma}(r) dr\right) \|S(s)^{1/2}v\|, \quad (t, s) \in \Delta_+, \quad v \in Y,$$

where  $\tilde{\sigma}$  is defined as in Theorem 1.2 (iii).

(b)  $U(\cdot, \cdot)v \in W^{1,\infty}(\Delta_+; X)$ , with properties (iv)<sub>w</sub> and (v).

(c)  $\{U(t, s); (t, s) \in \Delta_+\}$  is unique:  $U(t, s) \equiv V(t, s)$  on  $\Delta_+$  if  $\{V(t, s); (t, s) \in \Delta_+\}$  is another family in  $B(X)$  with properties (i), (ii) and (v).

This completes the proof of Theorem 3.1. The purpose of the second half in this section is to prove properties (iii) and (iv) in Theorem 1.2.

**Lemma 3.6.** *Let  $\{U(t, s)\}$  be as in Lemma 3.4 and  $v \in Y$ . Assume that  $S(t)$  satisfies conditions **(S1)**–**(S3)**. Then*

(a)  $S(t)^{1/2}U(t, s)v$  is weakly continuous on  $\Delta_+$ .

(b)  $S(t)^{1/2}U(t, s)v \rightarrow S(t_0)^{1/2}v$  as  $(t, s) \rightarrow (t_0, t_0)$ .

(c) For  $t \in (0, T]$ ,  $S(t)^{1/2}U(t, \cdot)v \in C([0, t]; X)$ .

*Proof.* (a) Since  $S_\varepsilon(\cdot)^{1/2} \in C_*([0, T]; B(X))$  (see **(S<sub>ε</sub>2)**),  $S_\varepsilon(t)^{1/2}U(t, s)v$  is continuous on  $\Delta_+$ . For  $w \in Y$ , we see that

$$\begin{aligned} & |(S(t)^{1/2}U(t, s)v, w) - (S_\varepsilon(t)^{1/2}U(t, s)v, w)| \\ &= |(S(t)^{1/2}U(t, s)v, [1 - (1 + \varepsilon S(t))^{-1/2}]w)| \\ &\leq \exp(\|\tilde{\sigma}\|_{L^1(0, T)}) \|S(0)^{1/2}v\| \cdot \|w - (1 + \varepsilon S(t))^{-1/2}w\| \\ &\leq \varepsilon^{1/2} \exp(\|\tilde{\sigma}\|_{L^1(0, T)}) \|v\|_Y \|w\|_Y, \end{aligned}$$

where we have used (1.2) and (3.4). Passing to  $\varepsilon \downarrow 0$ , since  $Y$  is dense in  $X$ , we can conclude (a).

(b) It follows from (a) that

$$\|S(t_0)^{1/2}v\| \leq \liminf_{(t, s) \rightarrow (t_0, t_0)} \|S(t)^{1/2}U(t, s)v\|.$$

On the other hand, it follows from (3.5) that

$$\limsup_{(t, s) \rightarrow (t_0, t_0)} \|S(t)^{1/2}U(t, s)v\| \leq \|S(t_0)^{1/2}v\|.$$

Combining these estimates and (a), we obtain the assertion.

(c) Let  $r \in [s, t]$ . Then we see from (3.5) that

$$\|S(t)^{1/2}U(t, r)v - S(t)^{1/2}U(t, s)v\| \leq \exp(\|\tilde{\sigma}\|_{L^1(0, T)}) \|S(r)^{1/2}(1 - U(r, s))v\|.$$

Therefore the assertion follows from (b).  $\square$

Now we are in a position to prove **(iii)** and **(iv)** of Theorem 1.2. As is easily seen, the proof of **(iii)** and **(iv)** is based on Lemmas 3.7 and 3.8 below. In other words, we need the whole assumptions on  $\{A(t)\}$  and  $\{S(t)\}$ .

**Lemma 3.7** ([7, Lemma 3.9]). *Let  $\{A(t)\}$  and  $\{S(t)\}$  be as in Theorem 1.2. Assume that conditions **(A1)**, **(A2)** and **(A3)** are satisfied. Then*

$$(3.6) \quad |\operatorname{Re}(A(t)v, S_\varepsilon(t)v)| \leq \beta \|S_\varepsilon(t)^{1/2}v\|^2, \quad v \in Y, \quad t \in [0, T].$$

Under conditions **(S1)**–**(S3)** Proposition 2.5 is modified as follows.

**Lemma 3.8.** *Assume that  $\{S(t)\}$  satisfies conditions **(S1)**–**(S3)**. Let  $\{S_\varepsilon^h(t)\}$  be as in Proposition 2.5. Then condition **(S<sub>ε</sub><sup>h</sup>3)<sub>+</sub>** is replaced with*

$$(S_\varepsilon^h 3) \quad \left| \left( w, \frac{d}{ds} S_\varepsilon^h(s)w \right) \right| \leq \frac{1}{h} \left[ \left( 1 + \int_s^{s+h} \sigma(r) dr \right)^2 - 1 \right] \max_{r \in \{s, s+h\}} \|S_\varepsilon(r)^{1/2}w\|^2$$

for  $w \in X$ ,  $h > 0$  and  $s \in [0, T]$ . Consequently, for  $t, t_0 \in [0, T]$  one has

$$(3.7) \quad \limsup_{h \downarrow 0} \left| \int_{t_0}^t \left( w, \frac{d}{ds} S_\varepsilon^h(s)w \right) ds \right| \leq 2 \left| \int_{t_0}^t \sigma(s) \|S_\varepsilon(s)^{1/2}w\|^2 ds \right|.$$

The next lemma completes the proof of Theorem 1.2.

**Lemma 3.9.** *For  $\{A(t)\}$  and  $\{S(t)\}$  as in Theorem 1.2 let  $\{U(t, s)\}$  be as defined in Lemma 3.4 and  $v \in Y$ . Then*

- (a)  $U(t, \cdot)v \in C([0, t]; Y)$ ,  $t \in (0, T]$ .
- (b)  $S(\cdot)^{1/2}U(\cdot, s)v \in C([s, T]; X)$ ,  $s \in [0, T]$ .
- (c)  $U(\cdot, s)v \in C([s, T]; Y)$ ,  $s \in [0, T]$ .
- (d)  $U(\cdot, \cdot)v \in C(\Delta_+; Y)$ ; this establishes property (iii) in Theorem 1.2.
- (e)  $U(\cdot, \cdot)v \in C^1(\Delta_+; X)$ , with property (iv) in Theorem 1.2.

*Proof.* We follow the idea in [7, Lemma 3.10].

- (a) We see from condition (S3) that for  $(t, s) \in \Delta_+$ ,

$$(3.8) \quad \|S(s)^{1/2}w\| \leq \exp\left(\int_s^t \sigma(r) dr\right) \|S(t)^{1/2}w\|, \quad w \in Y.$$

Hence we obtain

$$\|U(t, r)v - U(t, s)v\|_Y \leq \exp(\|\sigma(r)\|_{L^1(0, T)}) \|S(t)^{1/2}U(t, r)v - S(t)^{1/2}U(t, s)v\|.$$

Therefore the assertion follows from Lemma 3.6 (c).

- (b) By virtue of Lemma 3.6 (a), it suffices to show that

$$(3.9) \quad \|S(\cdot)^{1/2}U(\cdot, s)v\| \in C[s, T].$$

We trace the proof of Lemma 3.3 (b). Let us  $t, t_0 \in [s, T]$ . Integrating the inequality (3.3) from  $r = t_0$  to  $r = t$  and passing to the limit as  $n \rightarrow \infty$ , we have

$$\begin{aligned} & \|S_\varepsilon^h(t)^{1/2}U(t, s)v\|^2 - \|S_\varepsilon^h(t_0)^{1/2}U(t_0, s)v\|^2 \\ &= -2 \int_{t_0}^t \operatorname{Re} (A(r)U(r, s)v, S_\varepsilon^h(r)U(r, s)v) dr \\ &+ \int_{t_0}^t \left( U(r, s)v, \left( \frac{d}{dr} S_\varepsilon^h(r) \right) U(r, s)v \right) dr. \end{aligned}$$

Passing to the limit as  $h \downarrow 0$ , we see from (3.7) that

$$\begin{aligned} & \left| \|S_\varepsilon(t)^{1/2}U(t, s)v\|^2 - \|S_\varepsilon(t_0)^{1/2}U(t_0, s)v\|^2 \right| \\ & \leq 2 \left| \int_{t_0}^t \operatorname{Re} (A(r)U(r, s)v, S_\varepsilon(r)U(r, s)v) dr \right| \\ & + 2 \left| \int_{t_0}^t \sigma(r) \|S_\varepsilon(r)^{1/2}U(r, s)v\|^2 dr \right|. \end{aligned}$$

Therefore (3.6) yields that

$$\left| \|S_\varepsilon(t)^{1/2}U(t, s)v\|^2 - \|S_\varepsilon(t_0)^{1/2}U(t_0, s)v\|^2 \right| \leq 2 \left| \int_{t_0}^t \tilde{\sigma}(r) \|S(r)^{1/2}U(r, s)v\|^2 dr \right|.$$

By virtue of (3.5) and (3.4), we have

$$\left| \|S_\varepsilon(t)^{1/2}U(t, s)v\|^2 - \|S_\varepsilon(t_0)^{1/2}U(t_0, s)v\|^2 \right| \leq 2 \exp(2\|\tilde{\sigma}\|_{L^1(0,T)}) \left| \int_{t_0}^t \tilde{\sigma}(\tau) d\tau \right| \|v\|_Y^2.$$

Passing to the limit as  $\varepsilon \downarrow 0$ , we obtain (3.9).

(c) We see from (3.8) that

$$\|U(t, s)v - U(t_0, s)v\|_Y \leq \exp(\|\tilde{\sigma}\|_{L^1(0,T)}) \|S(t)^{1/2}U(t, s)v - S(t)^{1/2}U(t_0, s)v\|.$$

The assertion is a consequence of (b) and condition **(S2)**.

(d) The assertion follows from (c) and Lemma 3.6 (c) (see also [9, Lemma 3.11 (b)] and Kato [2, Remark 5.4]). Since (3.4), (3.8) and Lemma 3.5 (a) yield (1.2), this completes the proof of property **(iii)**.

(e) By virtue of Lemma 3.5 (b), it suffices to show that  $A(\cdot)U(\cdot, s)v \in C([s, T]; X)$ . Let  $t, t_0 \in [s, T]$ . Then we see from (3.1) that the desired continuity is reduced to those of  $S_0^{1/2}U(\cdot, s)v$  and  $A(\cdot)U(t_0, s)v$ :

$$\begin{aligned} & \|A(t)U(t, s)v - A(t_0)U(t_0, s)v\| \\ & \leq \|A(t)U(t, s)v - A(t)U(t_0, s)v\| + \|A(t)U(t_0, s)v - A(t_0)U(t_0, s)v\| \\ & \leq c\|S(t)^{1/2}U(t, s)v - S(t)^{1/2}U(t_0, s)v\| + \|A(t)U(t_0, s)v - A(t_0)U(t_0, s)v\|. \end{aligned}$$

Therefore the conclusion follows from (c) and condition **(A4)**. Finally, property **(iv)** is a consequence of property **(iv)<sub>w</sub>**.  $\square$

## 4. Applications to the Dirac equation

In this section we consider, as an application of Theorem 1.3, the Cauchy problem for the Dirac equation:

$$(DE) \quad \begin{cases} i \frac{d}{dt} u = H(t)u & \text{for } t \in (0, T), \\ u(0) = u_0 \end{cases}$$

in the Hilbert Space  $X = L^2(\mathbb{R}^3)^4$ , where  $u_0 \in Y := \Sigma^1(\mathbb{R}^3)^4$ .

First we define an operator  $H(t)$ . Let

$$\mathcal{H}(t) := H_0 + V(t, x) = H_0 + V_0(t, x) + q(t)I$$

with domain  $D(\mathcal{H}(t)) = C_0^\infty(\mathbb{R}^3)^4$ , where  $H_0$  is the free Dirac operator,  $V_0(t, x)$  and  $q(t, x)$  are satisfying condition (V1), (V2) and (q). Since  $\mathcal{H}(t)$  is symmetric,  $\mathcal{H}(t)$  is closable. Then we take as  $H(t)$  the closure  $\tilde{\mathcal{H}}(t)$  of  $\mathcal{H}(t)$ , i.e.,  $H(t) = \tilde{\mathcal{H}}(t)$ . Set

$$\begin{aligned} S(t) &:= (H_0 + V_0(t))^2 + (1 + |x|^2)I, \\ D(S(t)) &:= \{u \in L^2(\mathbb{R}^3)^4; S(t)u \in L^2(\mathbb{R}^3)^4\}. \end{aligned}$$

Then we can show that  $S(t)$  is positive selfadjoint on  $D(S(t))$  (see [8, Lemma 5.4]) and  $Y_t := D(S(t)^{1/2})$  is regarded as a Hilbert space, embedded continuously and densely in  $L^2(\mathbb{R}^3)^4$ , with inner product

$$(u, v)_{Y_t} = (S(t)^{1/2}u, S(t)^{1/2}v), \quad u, v \in Y_t.$$

We can show that  $S(t)$  satisfies condition (S1) and the first half of condition (S2):

**Lemma 4.1** (cf. [8, Lemma 6.1]). *Let  $S(t)$  be as above. Then for  $t \in [0, T]$ ,*

$$Y_t = Y = \Sigma^1(\mathbb{R}^3)^4$$

*and there exist time independent positive constants  $c_1, c_2$  such that*

$$(4.1) \quad c_1 \|S(t)^{1/2}u\|^2 \leq \|u\|^2 + \|\nabla u\|^2 + \| |x|u \|^2 \leq c_2 \|S(t)^{1/2}u\|^2, \quad u \in Y.$$

Moreover the second half of condition (S2) follows from (V2), because (V2) implies that  $V_0(\cdot, x)$  is continuous on  $[0, T]$ .

The following lemma guarantees that  $S(t)$  satisfies condition (S3):

**Lemma 4.2.** *Let  $S(t)$  be as above. Then*

$$\|S(t)^{1/2}v\| - \|S(s)^{1/2}v\| \leq \left| \int_s^t \sigma_1(r) dr \right| \max_{r \in \{s, t\}} \|S(r)^{1/2}v\|, \quad v \in Y, \quad t, s \in [0, T].$$

*Proof.* Since

$$\|S(t)^{1/2}v\|^2 = \|(H_0 + V_0(t))v\|^2 + \| |x|v \|^2 + \|v\|^2,$$

we have

$$\frac{d}{dt} \|S(t)^{1/2}v\|^2 = 2 \left( \frac{\partial}{\partial t} V_0(t)v, (H_0 + V_0(t))v \right).$$

Therefore we obtain from the Schwartz inequality and Hardy inequality that

$$\left| \frac{d}{dt} \|S(t)^{1/2}v\|^2 \right| = 2\sigma(t)(\|2a\nabla v\| + \|v\|)\|(H_0 + V_0(t))v\|.$$

It follows from (4.1) that

$$\left| \frac{d}{dt} \|S(t)^{1/2}v\| \right| = (2a + b)c_2\sigma(t)\|S(t)^{1/2}v\|.$$

Thus the assertion follows by  $\sigma_1(t) = (2a + b)c_2\sigma(t)$ . □

Now we shall verify conditions (A1)–(A4).

**Lemma 4.3.** *Let  $A(t) = iH(t)$  and  $S(t)$  be as above. Then for each  $T > 0$*

(A1)  $\operatorname{Re}(A(t)v, v) = 0$ ,  $v \in D(A(t))$ , a.a.  $t \in (0, T)$ .

(A2)  $Y = H^1(\mathbb{R}^3)^4 \cap H_1(\mathbb{R}^3)^4 \subset D(A(t))$ , a.a.  $t \in (0, T)$ .

(A3) *There exists a constant  $\beta \geq 0$  such that*

$$|\operatorname{Re}(A(t)u, S(t)u)| \leq \beta \|S(t)^{1/2}u\|^2, \quad u \in D(S(t)), \text{ a.a. } t \in (0, T).$$

(A4)  $A(\cdot) \in C_*(0, T; B(\Sigma^1(\mathbb{R}^3)^4, L^2(\mathbb{R}^3)^4))$ .

*Proof.* Noting that  $\operatorname{Re}(A(t)u, u) = -\operatorname{Im}(H(t)u, u)$  the assertion follows from symmetry of  $H(t)$ . Moreover the continuity of  $A(t)$  follows from (V2) and (q). Therefore, it is sufficient to show that there exist  $\beta > 0$  such that

$$|\operatorname{Im}(H(t)u, S(t)u)| \leq \beta(t) \|S(t)^{1/2}u\|^2, \quad u \in D(S), \text{ a.a. } t \in (0, T).$$

By integration by parts we have

$$\begin{aligned} \operatorname{Im}(H(t), S(t)u) &= \operatorname{Im}((H_0 + V_0(t))u, |x|^2u) + \operatorname{Im}(q(t)u, (H_0 + V_0(t))^2u) \\ &= \operatorname{Re}((\alpha \cdot x)u, u) - \operatorname{Re}((\alpha \cdot \nabla q(t))u, (H_0 + V_0(t))u). \end{aligned}$$

We see from condition (q) that there exists a constant  $c_q > 0$  such that

$$\|(1 + |x|^2)^{-1/2} \nabla q(t)\|_{L^\infty} \leq c_q.$$

Hence it follows from the Schwarz inequality and that

$$\begin{aligned} |\operatorname{Im}(H(t), S(t)u)| &\leq \| |x|u \| \cdot \|u\| + \| |\nabla q(t)|u \| \cdot \|(H_0 + V_0(t))u\| \\ &\leq \| |x|u \| \cdot \|u\| + c_q \|(1 + |x|^2)^{1/2}u\| \cdot \|(H_0 + V_0(t))u\|, \end{aligned}$$

Therefore we obtain the desired inequality.  $\square$

According to the above lemmas, we can obtain Theorem 1.1.

## References

- [1] T. Kato, "Perturbation Theory for Linear Operators", Grundlehren der mathematischen Wissenschaften **132**, Springer-Verlag, Berlin and New York, 1966; 2nd ed., 1976.
- [2] T. Kato, *Linear evolution equations of "hyperbolic" type*, J. Fac. Sci. Univ. Tokyo, Sec. I. **17** (1970), 241–258.
- [3] T. Kato, *Singular perturbation and semigroup theory*, in Lecture Notes in Math. **565**, Springer-Verlag, Berlin and New York, 1976, 104–112.
- [4] T. Kato, *Abstract Differential Equations and Nonlinear Mixed Problems*, Lezioni Fermiane [Fermi Lectures], Accad. Naz. Lincei, Scuola Normale Superiore, Pisa, 1985.

- [5] T. Kato and K. Yajima, *Dirac equations with moving nuclei*, Ann. Inst. H. Poincaré, Phys. Théor. **54** (1991), 209–221.
- [6] N. Okazawa, *Remarks on linear  $m$ -accretive operators in a Hilbert space*, J. Math. Soc. Japan **27** (1975), 160–165.
- [7] N. Okazawa, *Remarks on linear evolution equations of hyperbolic type in Hilbert space*, Adv. Math. Sci. Appl. **8** (1998), 399–423.
- [8] N. Okazawa and K. Yoshii, *Abstract approach to the Dirac equation*, RIMS Kôkyûroku **1640** (2009), 67–84.
- [9] N. Okazawa and K. Yoshii, *Linear evolution equations with strongly measurable families and application to the Dirac equation*, Discrete Contin. Dyn. Syst. Series S **4** (2011), 723–744.
- [10] N. Okazawa and K. Yoshii, *Linear Schrödinger evolution equations with moving Coulomb singularities*, J. Differential Equations **254** (2013), 2964–2999.
- [11] E. M. Ouhabaz, “Analysis of Heat Equations on Domains”, London Math. Soc. Monograph, Princeton Univ. Press, Princeton and Oxford, 2005.
- [12] A. Pazy, “Semigroups of Linear Operators and Applications to Partial Differential Equations”, Applied Math. Sci. **44**, Springer-Verlag, Berlin and New York, 1983.
- [13] H. Tanabe, “Functional Analytic Methods for Partial Differential Equations”, Pure and Applied Mathematics **204**, Marcel Dekker, New York, 1997.