

Some mixed norm estimates of free Schrödinger waves

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Abstract

In this paper we consider mixed norm estimates of linear Schrödinger waves. In [13] Shao obtained some estimates under spherical symmetry condition. We generalize them and show that the symmetry condition can be substituted by angular regularity.

2000 Mathematics Subject Classification. Primary 42B37; Secondary 35Q40

Keywords and phrases. mixed norm estimates, free Schrödinger waves, angular regularity

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[†]Supported by the Korea Research Foundation Grant funded by the Korean Government(KRF-2008-313-C00065)

1 Introduction

The free Schrödinger wave is the solution to the Cauchy problem

$$iu_t - \Delta u = 0 \text{ in } \mathbb{R}^{1+n}, \quad u(0) = \varphi \text{ in } \mathbb{R}^n, n \geq 2. \quad (1.1)$$

It can be written as

$$u(t, x) = (e^{-it\Delta}\varphi)(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(x \cdot \xi + t|\xi|^2)} \widehat{\varphi}(\xi) d\xi.$$

Here $\mathcal{F}(\cdot) = \widehat{(\cdot)}$ is the Fourier transform defined by

$$\mathcal{F}(\varphi)(\xi) = \widehat{\varphi}(\xi) = \int_{\mathbb{R}^n} e^{-ix \cdot \xi} \varphi(x) dx$$

and its inverse is given by

$$\mathcal{F}^{-1}(\varphi)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \varphi(\xi) d\xi.$$

We are concerned with mixed norm estimates of free Schrödinger waves, especially with ones in Sobolev type spaces, which are defined in spherical coordinates as follows:

$$\begin{aligned} \dot{H}_r^{s,p} H_\sigma^{\alpha,\ell} &= \text{the closure of } C_0^\infty \text{ w.r.t. the norm} \\ \|f\|_{\dot{H}_r^{s,p} H_\sigma^{\alpha,\ell}} &= \| |\nabla|^s D_\sigma^\alpha f \|_{L_r^p L_\sigma^\ell}, |s| < n/p, \alpha \in \mathbb{R}, \end{aligned} \quad (1.2)$$

where $|\nabla| = (-\Delta)^{\frac{1}{2}}$, $D_\sigma = \sqrt{1 - \Delta_\sigma}$, Δ_σ is the Laplace-Beltrami operator defined on the unit sphere and

$$\|f\|_{L_r^p L_\sigma^\ell} = \left(\int_0^\infty \left(\int_{S^{n-1}} |f(r\sigma)|^\ell d\sigma \right)^{\frac{p}{\ell}} r^{n-1} dr \right)^{\frac{1}{p}}, \quad 1 \leq p, \ell < \infty.$$

Here we denoted $L^p(r^{n-1}dr)$ by L_r^p . We also use the time-space normed spaces,

$$\|v\|_{L_t^q \dot{H}_r^{s,p} H_\sigma^{\alpha,\ell}} = \left(\int_{\mathbb{R}} \|v(\cdot, t)\|_{\dot{H}_r^{s,p} H_\sigma^{\alpha,\ell}}^q dt \right)^{\frac{1}{q}}, \quad 1 \leq q \leq \infty.$$

If $p = 2$ and $\ell = 2$, then we will use $\dot{H}_r^s H_\sigma^\alpha$ for $\dot{H}_r^{s,2} H_\sigma^{\alpha,2}$. We remark here that if $\alpha = 0$ and $p = \ell$, then the mixed norm is just Schtrichartz one (see Remark 1 and 2 for definition).

If $\ell = 2$, then one can expand any $v \in L_t^q \dot{H}_r^{p,s} H_\sigma^\alpha$ by the spherical harmonics of orthonormal basis $\{Y_k^l\}$, $k \geq 0, 1 \leq l \leq d(k)$ ($d(k)$ is the dimension of spherical

harmonics of order k) such that there exists a unique sequence of measurable functions $a_k^l(t, r) \in L_t^q L_r^p$ satisfying that

$$|\nabla|^s v(t, r, \sigma) = \sum_{k \geq 0, 1 \leq l \leq d(k)} a_k^l(t, r) Y_k^l(\sigma) \text{ in } L_t^q L_r^p H_\sigma^\alpha$$

and

$$\|v\|_{L_t^q \dot{H}_r^{p, s} H_\sigma^\alpha} = \left\| \left(\sum_{k, l} (1 + k(k + n - 2))^\alpha |a_k^l|^2 \right)^{\frac{1}{2}} \right\|_{L_t^q L_r^p}.$$

Here, we used the identity $-\Delta_\sigma Y_k^l = k(k + n - 2) Y_k^l$.

Now let us introduce our main theorem.

Theorem 1.1. *Let $1/2 \leq \alpha \leq n/2$ and $n \geq 2$. Suppose that q and α are numbers such that*

$$(2n + 6\alpha - 2)/(n - 1 + \alpha) < q \leq 6, \\ \tilde{\alpha} = (3\alpha + 2)/q - (\alpha + 3)/2.$$

Then for any $\varphi \in L^2$ the solution u of (1.1) satisfies

$$\|u\|_{L_t^q \dot{H}_q^s H_\sigma^{\tilde{\alpha}}} \lesssim \|\varphi\|_{L^2}, \quad (1.3)$$

where $s = \frac{n+2}{q} - \frac{n}{2}$.

Theorem 1.1 is a generalization of Shao's results [13] for spherically symmetric data. In particular, if we take $\alpha = 1/2$, then we can recover his result for $q > \frac{4n+2}{2n-1}$. If $\alpha > (3q - 4)/(6 - q)$ for $q \neq 6$, then we can take a positive $\tilde{\alpha}$. Hence we get a slight spatial and angular regularity gain for $\tilde{\alpha} > 0$, $(2n + 6\alpha - 2)/(n - 1 + \alpha) < q < (2n + 4)/n$ and $n \geq 6$. For another angular smoothing effects of Strichartz estimate see [10, 19], in which two dimensional endpoint case was treated.

Remark 1. Applying the Christ-Kiselev lemma (see [4, 19, 1]) and Strichartz estimate (see [7] for instance), it is possible to consider an inhomogeneous estimate. Let $q, \tilde{\alpha}, s$ be as above. Then we have

$$\left\| \int_0^t e^{-(t-t')\Delta} F(t') dt' \right\|_{L_t^q \dot{H}_q^s H_\sigma^{\tilde{\alpha}}} \lesssim \|F\|_{L_t^{\tilde{q}} L_x^{\tilde{r}}} \quad (1.4)$$

for any pairs $(\tilde{q}, \tilde{r})$ such that $2/\tilde{q} + n/\tilde{r} = n/2$, $2 \leq \tilde{q} \leq \infty$ and $(\tilde{q}, \tilde{r}) \neq (2, \infty)$ if $n = 2$. We call such pair admissible one.

Remark 2. We now apply the estimates (1.3) and (1.4) to the mass critical Schrödinger equations;

$$iu_t - \Delta u = V(u)u, u(0) = \varphi \in L^2, V(u) = \pm |u|^{\frac{4}{n}} \text{ or } \pm |x|^{-2} * |u|^2.$$

The existence of local or small data or spherical symmetric global solutions is well-known (see [8, 11, 20]). The Strichartz estimate is main tool for that problem. Actually one can find a solution u in $S_T \equiv \sup_{(\tilde{q}, \tilde{r}): \text{admissible}} L_{[0, T]}^{\tilde{q}} L_x^{\tilde{r}}$. Then the estimate (1.4) and standard nonlinear estimate for critical nonlinearity give us that the solution u is in $L_t^q \dot{H}_q^s H_\sigma^{\tilde{\alpha}}$. Hence if $n \geq 6$, $\tilde{\alpha} > 0$ and $(2n + 6\alpha - 2)/(n - 1 + \alpha) < q < (2n + 4)/n$, then the solution u obtain a spatial and angular regularity.

Theorem 1.1 follows directly from dyadic decomposition and interpolation between the estimates of linear operator T_R defined as

$$T_R f(t, x) = \chi_R(x) \int e^{i(x \cdot \xi + t|\xi|^2)} f(\xi) d\xi,$$

where $R > 0$ is dyadic number and χ_R is characteristic function on the annulus $\{R \leq |x| \leq 2R\}$.

Proposition 1.2. *Let f be supported in the annulus $\{1 \leq |\xi| \leq 2\}$. Then we have*

(1) *for $f \in L^2$ and $1/2 \leq \alpha \leq n/2$*

$$\|T_R f\|_{L_t^2 L_r^2 H_\sigma^{\alpha-1/2}} \lesssim \min(R^\alpha, R^{\frac{n}{2}}) \|f\|_{L^2}. \quad (1.5)$$

(2) *for $f \in L^p$ for $1 < p \leq 2$ and $q = 3p'$*

$$\|T_R f\|_{L_t^q L_r^q L_\sigma^2} \lesssim \min(R^{-(n-1)(1/2-1/q)}, R^{n/q}) \|f\|_{L_r^p H_\sigma^{1+1/q}}. \quad (1.6)$$

The estimates (1.5) and (1.6) can be used to get local or weighted global smoothing estimates which were obtained by many authors [3, 6, 5, 12, 14, 17, 18, 21]. It is also possible to consider the end point cases $(q, p) = (4, 4)$ and $(\infty, 1)$. But we will append them in the last section since they are not essential in proving the main theorem (see Proposition 3.1 below).

2 Proof of Proposition 1.2

2.1 Case $R \gtrsim 1$; Proof of (1)

We first expand f as $f(\xi) = f(\rho\sigma) = \sum_{k \geq 0, 1 \leq l \leq d(k)} a_k^l(\rho) Y_k^l(\sigma)$. Then for the proof of (1.5) we may assume that $f \in L_r^2 H_\sigma^{1/2-\alpha}$ and a_k^l are supported in

$\{1 \leq \rho \leq 2\}$ for all k, l . Using the Fourier transform of spherical harmonic functions (see [16])

$$\widehat{Y}_k^l(\rho\sigma) = c_{n,k}\rho^{-\frac{n-2}{2}}J_\nu(\rho)Y_k^l(-\sigma), \nu = \nu(k) = \frac{n-2+2k}{2},$$

we have

$$T_R f(t, r\sigma) = \sum_{k,l} c_{n,k}\chi_R(r)r^{-\frac{n-2}{2}} \int e^{it\rho^2} a_k^l(\rho)\rho^{\frac{n}{2}} J_\nu(r\rho) d\rho Y_k^l(-\sigma).$$

It should be noticed that $|c_{n,k}| \leq C$ for all k and C does not depend on k . By the change of variables $T_R f$ is converted into

$$\frac{1}{2} \sum_{k,l} c_{n,k}\chi_R(r)r^{-\frac{n-2}{2}} \int e^{it\rho} a_k^l(\sqrt{\rho})\rho^{\frac{n}{4}-\frac{1}{2}} J_\nu(r\sqrt{\rho}) d\rho Y_k^l(-\sigma).$$

Hence taking $L_t^2 L_\sigma^2$ norm, the Plancherel's theorem and the reverse change variables give

$$\|T_R f(\cdot, r)\|_{L_t^2 L_\sigma^2} \lesssim \left(\sum_{k,l} \chi_R(r)r^{-(n-2)} \int |a_k^l(\rho)|^2 \rho^{n-3} |J_\nu(r\rho)|^2 d\rho \right)^{\frac{1}{2}}. \quad (2.1)$$

To estimate L_r^2 norm of RHS we are going to use some asymptotic behavior of Bessel function. For this purpose we choose smooth cut-off functions ψ_1, ψ_2 and ψ_3 so that $\psi_1(r) = 1$ on $\{|r| < \frac{1}{4}\}$, $\psi_1(r) = 0$ on $\{|r| > \frac{1}{2}\}$, $\psi_2(r) = 1$ on $\{1/2 < |r| < 1\}$, $\psi_2(r) = 0$ on $\{|r| \leq 1/4 \text{ or } |r| \geq 2\}$, $\psi_3 = 0$ on $\{|s| < 2\}$, $\psi_3 = 1$ on $\{|s| > 3\}$, and $\psi_1 + \psi_2 + \psi_3 = 1$. Now we introduce four types of asymptotic behavior of Bessel function as follows: for $\nu \geq 1$

$$|J_\nu(r)| \leq C \exp(-C\nu), \quad \text{if } 0 \leq r \leq \frac{\nu}{2}, \quad (2.2)$$

$$|J_\nu(r)| \leq C\nu^{-\frac{1}{3}}(1 + \nu^{-\frac{1}{3}}|r - \nu|)^{-\frac{1}{4}} \quad \text{for all } \frac{\nu}{2} < r < 2\nu, \quad (2.3)$$

$$|J_\nu(r)| \leq Cr^{-\frac{1}{2}} \quad \text{for all } r \geq 2\nu, \quad (2.4)$$

$$J_\nu(r)\psi_3\left(\frac{r}{\nu}\right) = r^{-\frac{1}{2}}(b_+e^{ir} + b_-e^{-ir})\psi_3\left(\frac{r}{\nu}\right) + \Phi_\nu(r)\psi_3\left(\frac{r}{\nu}\right), \quad (2.5)$$

where $|\Phi_\nu(r)| \leq C\frac{\nu^{\frac{3}{2}}}{r}$, $|b_\pm| \leq C$ and the constant C is independent of ν . For the proof of (2.2), one can use the Poisson representation formula (2.7) and Stirling's formula below. Invoking the Schl\"afli's integral representation (see p.176 in [22])

$$J_\nu(r) = \frac{1}{\pi} \int_0^\pi e^{i(r \sin \theta - \nu \theta)} d\theta - \frac{\sin(\nu\pi)}{\pi} \int_0^\infty e^{-\nu\tau - r \sinh \tau} d\tau,$$

the two asymptotic behaviors (2.3) and (2.4) follow from the easy estimate

$$\left| \frac{\sin(\nu\pi)}{\pi} \int_0^\infty e^{-\nu\tau - r \sinh \tau} d\tau \right| \leq \frac{C}{\nu + r}$$

and the method of stationary phase for $\frac{1}{\pi} \int_0^\pi e^{i(r \sin \theta - \nu \theta)} d\theta$ when $\nu/2 < r < 2\nu$ or $r \geq 2\nu$. For instance see the page 1478 of [19] and Lemma 3 of [2]. For (2.5), see 5.2 on the page 356 of [15].

Now taking L_r^2 norm on both sides of (2.1) and then changing variables $r \mapsto r\rho$, we get

$$\|T_R f\|_{L_t^2 L_r^2 L_\sigma^2}^2 \lesssim \sum_{k,l} \int |a_k^l(\rho)|^2 \rho^{n-5}(\star) d\rho,$$

where

$$\begin{aligned} (\star) &= \int \chi_{R\rho}(r) r |J_\nu(r)|^2 dr \\ &= \int \chi_{R\rho}(r) r |J_\nu(r)|^2 (\psi_1(r/\nu) + \psi_2(r/\nu) + \psi_3(r/\nu)) dr \equiv I_1 + I_2 + I_3. \end{aligned}$$

For the Bessel function estimates we may assume that $\nu \geq 1$. By (2.2),

$$\begin{aligned} I_1 &\lesssim \int_R^{\min(4R, \nu/2)} e^{-C\nu} r dr \\ &\lesssim R e^{-CR} \int_R^{\min(4R, \nu/2)} e^{-C\nu/2} dr \\ &\lesssim R e^{-C\nu/2} \lesssim R^{2\alpha} e^{-C\nu/2} \text{ because } \alpha \geq 1/2. \end{aligned}$$

By (2.3),

$$\begin{aligned} I_2 &\lesssim \int_{\max(R, \nu/2)}^{\min(4R, 2\nu)} \nu^{-2/3} \nu^{1/6} |r - \nu|^{-1/2} r dr \\ &\lesssim R^{2\alpha} \nu^{1/2-2\alpha} \int_{\nu/2}^{2\nu} |r - \nu|^{-1/2} dr \\ &\lesssim R^{2\alpha} \nu^{1-2\alpha}. \end{aligned}$$

By (2.4),

$$I_3 \lesssim \int_{\max(R, 2\nu)}^{4R} 1 dr \lesssim R^{2\alpha} \nu^{1-2\alpha}.$$

Substituting these estimates into $(\star)$, we get the desired estimate.

2.2 Case $R \gtrsim 1$; Proof of (2)

Similarly to the case (1), we expand f as $f(\rho\sigma) = \sum_{k,l} a_k^l(\rho) Y_k^l(\sigma)$ and assume that $f \in L_r^2 H_s^{1+1/q}$ and a_k^l are supported in $\{1 \leq \rho \leq 2\}$. Then we need only to show that

$$\begin{aligned} & \|\chi_R r^{-(n-2)/2} \int e^{it\rho^2} \rho^{\frac{n}{2}} a_k^l(\rho) J_\nu(r\rho) d\rho\|_{L_t^q L_r^q} \\ & \lesssim R^{-(n-1)(1/2-1/q)} \nu^{1+1/q} \|a_k^l\|_{L_r^p} \end{aligned}$$

Using cut-off functions and asymptotic behaviors of Bessel functions, LHS is bounded by

$$\begin{aligned} & \|\chi_R r^{-(n-2)/2} \int e^{it\rho^2} \rho^{\frac{n}{2}} a_k^l(\rho) J_\nu(r\rho) \psi_1(r\rho/\nu) d\rho\|_{L_t^q L_r^q} \\ & + \|\chi_R r^{-(n-2)/2} \int e^{it\rho^2} \rho^{\frac{n}{2}} a_k^l(\rho) J_\nu(r\rho) \psi_2(r\rho/\nu) d\rho\|_{L_t^q L_r^q} \\ & + \|\chi_R r^{-(n-2)/2} \int e^{it\rho^2} \rho^{\frac{n}{2}} a_k^l(\rho) (r\rho)^{-1/2} (b_+ e^{ir\rho} + b_- e^{-ir\rho}) d\rho\|_{L_t^q L_r^q} \\ & + \|\chi_R r^{-(n-2)/2} \int e^{it\rho^2} \rho^{\frac{n}{2}} a_k^l(\rho) (r\rho)^{-1/2} (b_+ e^{ir\rho} + b_- e^{-ir\rho}) (1 - \psi_3(r\rho/\nu)) d\rho\|_{L_t^q L_r^q} \\ & + \|\chi_R r^{-(n-2)/2} \int e^{it\rho^2} \rho^{\frac{n}{2}} a_k^l(\rho) \Psi_\nu(r\rho) \psi_3(r\rho/\nu) d\rho\|_{L_t^q L_r^q} \\ & = \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4 + \mathcal{I}_5. \end{aligned}$$

Taking L_t^q norm first and then L_r^q norm, we get the bounds

$$\begin{aligned} \mathcal{I}_1 & \lesssim \|a_k^l(\rho)\| \|\chi_R \rho r^{-(n-2)/2} J_\nu(r) \psi_1(r/\nu)\|_{L_r^q} \|a_k^l\|_{L_\rho^{q'}} \\ & \lesssim e^{-C\nu} R^{-(n-2)/2+(n-1)/q} (\min(4R, \nu/2) - R)^{\frac{1}{q}} \|a_k^l\|_{L_\rho^{q'}} \\ & \lesssim \nu^{\frac{1}{q}} e^{-C\nu} R^{-(n-1)(1/2-1/q)} \|a_k^l\|_{L_\rho^{q'}}, \end{aligned}$$

for some $0 < \delta < 4/q$

$$\begin{aligned} \mathcal{I}_2 & \lesssim \|a_k^l(\rho)\| \|\chi_R \rho r^{-(n-2)/2} J_\nu(r) \psi_2(r/\nu)\|_{L_r^q} \|a_k^l\|_{L_\rho^{q'}} \\ & \lesssim \|a_k^l\|_{L_\rho^{q'}} R^{-(n-1)(1/2-1/q)} \nu^{\frac{1}{2}} \left(\int_{\max(R, \nu/2)}^{\min(4R, 2\nu)} \nu^{-q/3+\delta q/12} |r - \nu|^{-\delta q/4} dr \right)^{\frac{1}{q}} \\ & \lesssim R^{-(n-1)(1/2-1/q)} \nu^{\frac{1-\delta}{6} + \frac{1}{q}} \|a_k^l\|_{L_\rho^{q'}}, \end{aligned}$$

$$\begin{aligned}
\mathcal{I}_4 &\lesssim \|a_k^l(\rho)\| \chi_{R\rho} r^{-(n-1)/2} (1 - \psi_3(r/\nu)) \|_{L_r^q} \|_{L_\rho^{q'}} \\
&\lesssim \|a_k^l\|_{L_\rho^{q'}} R^{-(n-1)(1/2-1/q)} \left(\int_R^{\min(4R, 2\nu)} 1 \, dr \right)^{\frac{1}{q}} \\
&\lesssim R^{-(n-1)(1/2-1/q)} \nu^{\frac{1}{q}} \|a_k^l\|_{L_\rho^{q'}},
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}_5 &\lesssim \|a_k^l(\rho)\| \chi_{R\rho} r^{-n/2} \psi_3(r/\nu) \|_{L_r^q} \|_{L_\rho^{q'}} \\
&\lesssim \|a_k^l\|_{L_\rho^{q'}} R^{-(n-1)(1/2-1/q)} \nu^{3/2} R^{-1/2} \left(\int_{\max(R, 2\nu)}^{4R} 1 \, dr \right)^{\frac{1}{q}} \\
&\lesssim R^{-(n-1)(1/2-1/q)} \nu^{1+\frac{1}{q}} \|a_k^l\|_{L_\rho^{q'}}.
\end{aligned}$$

It should be noted that $q' = (3p')' < p$ for $1 < p \leq 2$.

For $\mathcal{I}_3$ one can use 2-d oscillatory integral estimate of [15] or Proposition 3.6 of [13] to get

$$\begin{aligned}
\mathcal{I}_3 &\lesssim R^{-(n-2)(1/2-1/q)} \left\| \int e^{it\rho^2} \rho^{\frac{n-1}{2}} a_k^l(\rho) (b_+ e^{ir\rho} + b_- e^{-ir\rho}) \, d\rho \right\|_{L_t^q L_r^q} \\
&\lesssim R^{-(n-2)(1/2-1/q)} \|a_k^l\|_{L_\rho^p}.
\end{aligned}$$

This completes the proof of the case $R \gtrsim 1$ of (2).

2.3 Case $R \ll 1$

Now we consider the case when $R \ll 1$. More generally we will get for all $2 \leq q \leq \infty, \alpha > 0$

$$\|T_R f\|_{L_t^q L_r^q H_\sigma^\alpha} \lesssim R^{\frac{n}{q}} \|f\|_{L^{q'}}. \quad (2.6)$$

Using spherical harmonic expansion as above, we have only to show that

$$\|\chi_{Rr} r^{-(n-2)/2} \int e^{it\rho^2} \rho^{\frac{n}{2}} a_k^l(\rho) J_\nu(r\rho) \, d\rho\|_{L_t^q L_r^q} \lesssim R^{\frac{n}{q}} e^{-\nu} \|a_k^l\|_{L_\rho^{q'}}.$$

In fact, by Hausdorff-Young's inequality we have

$$\begin{aligned}
LHS &\lesssim \|\chi_{Rr} r^{-(n-2)/2} \|a_k^l(\rho) J_\nu(r\rho)\|_{L_\rho^{q'}} \|_{L_r^q} \\
&\lesssim \|a_k^l(\rho)\| \chi_{R\rho} r^{-(n-2)/2} J_\nu(r) \|_{L_r^q} \|_{L_\rho^{q'}}.
\end{aligned}$$

And since $\nu \geq (n-2)/2$, the inner integral is bounded by

$$\begin{aligned} & \left(\int_R^{4R} r^{-q(n-2)/2+n-1} |J_\nu(r)|^q dr \right)^{\frac{1}{q}} \\ & \lesssim R^{-(n-2)/2+n/q} \frac{(2R)^\nu}{\Gamma(\nu+1/2)} \\ & \lesssim R^{n/q} R^{-(n-2)/2+\nu} \left(\frac{2e}{\nu+\frac{1}{2}} \right)^\nu \\ & \lesssim R^{n/q} e^{-\nu}. \end{aligned}$$

Here we used the Poisson representation formula [15, 22]

$$J_\nu(r) = \frac{\left(\frac{r}{2}\right)^\nu}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})} \int_{-1}^1 e^{irs} (1-s^2)^{\nu-\frac{1}{2}} ds \quad (2.7)$$

and the Stirling's formula [9] $\Gamma(t) \sim \sqrt{2\pi} t^{-\frac{1}{2}} e^{-t}$ for large t .

2.4 Proof of Theorem 1.1

Interpolating (1.5) and (1.6) with $(q, p) = (6, 2)$, we get for $2 \leq q \leq 6$

$$\|T_R f\|_{L_t^q L_r^q H_\sigma^{\tilde{\alpha}}} \lesssim \min(R^{-\frac{n-1}{2}+\alpha+\frac{3\alpha+n-1}{q}}, R^{\frac{n}{q}}) \|f\|_{L^2},$$

where $\tilde{\alpha} = 3(\alpha+2/3)/q - (\alpha+3)/2$. Thus $\tilde{\alpha}$ satisfies the hypothesis and for $q > \frac{2n+6\alpha-2}{n-1+\alpha}$ the dyadic sum $\sum_{R:dyadic} \|T_R f\|_{L_t^q L_r^q H_\sigma^{\tilde{\alpha}}}$ is bounded by $\|f\|_{L^2}$.

If $\widehat{\varphi}$ is supported in $\{1 \leq |\xi| \leq 2\}$, then

$$\|u\|_{L_t^q L_r^q H_\sigma^{\tilde{\alpha}}} \lesssim \sum_{R:dyadic} \|T_R(\widehat{\varphi})\|_{L_t^q L_r^q H_\sigma^{\tilde{\alpha}}} \lesssim \|\varphi\|_{L^2}.$$

If $\text{supp}(\widehat{\varphi}) \subset \{N \leq |\xi| \leq 2N\}$, by rescaling we get

$$\|u\|_{L_t^q L_r^q H_\sigma^{\tilde{\alpha}}} \lesssim N^{-(n+2)/q+n/2} \|\varphi\|_{L^2}.$$

Now we decompose the solution u dyadically in frequency space. For this we use frequency projection operator P_N whose symbol is supported in $\{N \leq |\xi| \leq 2N\}$ and then $P_N u = e^{-it\Delta} P_N \varphi$. Thus by summing w.r.t. dyadic frequency we get the desired estimate.

3 Endpoint cases

Proposition 3.1. *Let f be supported in the annulus $\{1 \leq |\xi| \leq 2\}$. Then we have*

(1) *for $f \in L^4$ and $0 < \varepsilon \ll 1$*

$$\|T_R f\|_{L_t^4 L_r^4 L_\sigma^2} \lesssim \min(R^{-(n-1)/4+\varepsilon}, R^{\frac{n}{4}}) \|f\|_{L_r^2 H_\sigma^{\frac{1}{4}}}. \quad (3.1)$$

(2) *for $f \in L^1$*

$$\|T_R f\|_{L_t^\infty L_r^\infty L_\sigma^2} \lesssim \min(R^{-(n-1)/2}, 1) \|f\|_{L_r^1 H_\sigma^\alpha}, \quad (3.2)$$

where $\alpha \geq 1/6$.

Proof. Write f as $\sum_{k,l} a_k^l Y_k^l$. Then in view of Section 2.3 we have only to consider the case $R \gtrsim 1$ and to show that

$$\|\chi_{Rr^{-(n-2)/2}} \int e^{it\rho^2} \rho^{\frac{n}{2}} a_k^l(\rho) J_\nu(r\rho) d\rho\|_{L_t^4 L_r^4} \lesssim R^{-(n-1)/4+\varepsilon} \nu^{\frac{7}{4}} \|a_k^l\|_{L_\rho^2}.$$

Using cut-off functions and asymptotic behaviors of Bessel functions, LHS is bounded by

$$\begin{aligned} & \|\chi_{Rr^{-(n-2)/2}} \int e^{it\rho^2} \rho^{\frac{n}{2}} a_k^l(\rho) J_\nu(r\rho) \psi_1(r\rho/\nu) d\rho\|_{L_t^4 L_r^4} \\ & + \|\chi_{Rr^{-(n-2)/2}} \int e^{it\rho^2} \rho^{\frac{n}{2}} a_k^l(\rho) J_\nu(r\rho) \psi_2(r\rho/\nu) d\rho\|_{L_t^4 L_r^4} \\ & + \|\chi_{Rr^{-(n-2)/2}} \int e^{it\rho^2} \rho^{\frac{n}{2}} a_k^l(\rho) (r\rho)^{-1/2} (b_+ e^{ir\rho} + b_- e^{-ir\rho}) d\rho\|_{L_t^4 L_r^4} \\ & + \|\chi_{Rr^{-(n-2)/2}} \int e^{it\rho^2} \rho^{\frac{n}{2}} a_k^l(\rho) (r\rho)^{-1/2} (b_+ e^{ir\rho} + b_- e^{-ir\rho}) (1 - \psi_3(r\rho/\nu)) d\rho\|_{L_t^4 L_r^4} \\ & + \|\chi_{Rr^{-(n-2)/2}} \int e^{it\rho^2} \rho^{\frac{n}{2}} a_k^l(\rho) \Psi_\nu(r\rho) \psi_3(r\rho/\nu) d\rho\|_{L_t^4 L_r^4} \\ & = \mathbb{I}_1 + \mathbb{I}_2 + \mathbb{I}_3 + \mathbb{I}_4 + \mathbb{I}_5. \end{aligned}$$

The terms $\mathbb{I}_i, i = 1, 2, 4, 5$ are treated similarly to $\mathbb{I}_i, i = 1, 2, 4, 5$ and their sum actually has the bound

$$R^{-(n-1)/4} \nu^{\frac{7}{4}} \|a_k^l\|_{L_\rho^{\frac{4}{3}}}.$$

As for $\mathbb{I}_3$ one can follow the proof of Proposition 3.5 in [13] and can get

$$\mathbb{I}_3 \lesssim R^{-(n-1)/4+\varepsilon} \|a_k^l\|_{L_\rho^2}.$$

This proves (1).

For the proof of (2) we show that

$$\|\chi_R r^{-(n-2)/2} \int e^{it\rho^2} \rho^{\frac{n}{2}} a_k^l(\rho) J_\nu(r\rho) d\rho\|_{L_t^\infty L_r^\infty} \lesssim R^{-(n-1)/2} \nu^\alpha \|a_k^l\|_{L_\rho^1}.$$

We bound LHS by

$$\begin{aligned} & \|\chi_R r^{-(n-2)/2} \int \rho^{\frac{n}{2}} |a_k^l(\rho)| e^{-C\nu} |\psi_1(r\rho/\nu)| d\rho\|_{L_r^\infty} \\ & + \|\chi_R r^{-(n-2)/2} \int \rho^{\frac{n}{2}} |a_k^l(\rho)| \nu^{-1/3} \psi_2(r\rho/\nu) d\rho\|_{L_r^\infty} \\ & + \|\chi_R r^{-(n-2)/2} \int \rho^{\frac{n}{2}} |a_k^l(\rho)| (r\rho)^{-1/2} d\rho\|_{L_r^\infty}. \end{aligned}$$

The first term is bounded by

$$\|a_k^l\|_{L_\rho^1} e^{-C\nu} \|\chi(R, 4R)(r) r^{-(n-2)/2} \psi_1(r/\nu)\|_{L_r^\infty} \lesssim R^{-(n-1)/2} \nu^{\frac{1}{2}} e^{-C\nu} \|a_k^l\|_{L_\rho^1},$$

the second one by

$$R^{-(n-1)/2} \nu^{\frac{1}{6}} \|a_k^l\|_{L_\rho^1} \text{ because } R \sim \nu$$

and the last one by $R^{-(n-1)/2} \|a_k^l\|_{L_\rho^1}$. This completes the proof of Proposition 3.1. $\square$

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