

Gröbner bases on projective bimodules and the Hochschild cohomology *

Part IV. (Co)homology

YUJI KOBAYASHI

Department of Information Science, Toho University
Funabashi 274-8510, Japan

This is a continuation of the previous papers [3], [4] and [5]. We develop the theory of Gröbner bases on projective modules over an algebra based on a well-ordered semigroup. We construct resolutions of modules admitting Gröbner bases. This gives an effective way to compute the (co)homology of such modules.

14 Suitable orders

Let $S = B \cup \{0\}$ be a well-ordered reflexive semigroup with 0 and K be a commutative ring with 1. Let $F = K \cdot B$ be the K -algebra based on B and let I be a (two-sided) ideal of F . Let $A = F/I$ be the quotient algebra of F by I and $\rho : F \rightarrow A$ be the natural surjection. We fix a (reduced) Gröbner basis G of I .

Let X be a left edged set and $F \cdot X$ be the projective left F -module generated by X . Assume that a left compatible well-order $>$ on $B \cdot X$ is given and it is extended to a partial order $\succ$ on $F \cdot X$ in a natural way. The leading term of $f \in F \cdot X$ with respect to $\succ$ is denoted by $\text{lt}(f)$.

Let H be a set of monic left uniform elements of $F \cdot X$, which is considered to be a left edged set. Let $F \cdot H$ be the projective left F -module generated by H . For $h \in H$, $[h]$ denotes the formal generator of $F \cdot H$ corresponding to $h \in H$.

We define a (strict) partial order $>'$ on $B \cdot X$ as follows. For $x[h], x'[h'] \in B \cdot H$, such that $x \cdot \text{lt}(h) \neq 0$ and $x' \cdot \text{lt}(h') \neq 0$, define $x[h] >' x'[h']$ if and only if

- (i) $x \cdot \text{lt}(h) > x' \cdot \text{lt}(h')$, or
- (ii) $x \cdot \text{lt}(h) = x' \cdot \text{lt}(h')$ and $x > x'$.

Clearly, this partial order is well founded. Let $L'(H)$ (resp. $L''(H)$) be the K -subspace of $F \cdot H$ spanned by

$$\{x[h] \in B \cdot X \mid x \cdot \text{lt}(h) \neq 0\} \quad (\text{resp. } \{x[h] \in B \cdot X \mid x \cdot \text{lt}(h) = 0\}).$$

*This is a preliminary report and the details appear elsewhere.

Easily we see that $L''(H)$ is an F -submodule of $F \cdot H$ and

$$F \cdot H = L'(H) \oplus L''(H)$$

holds.

The partial order $>'$ is total on $\{x[h] \in B \cdot X \mid x \cdot \text{lt}(h) \neq 0\}$, and is extended to a partial order $\succ'$ on $L'(H)$ in the same way as we did on $F \cdot X$. The partial order $\succ'$ satisfies the following weak compatibility. For $f, g \in L'(H)$ and $a, b \in B$

$$(1) f \succ' g, axw \neq 0 \Rightarrow (af)' \succ' (ag)', \text{ and}$$

$$(2) a > b, axw \neq 0 \Rightarrow (af)' \succ' (bf)',$$

where $(af)', (ag)'$ and $(bf)'$ are the projections of af, ag and bf to $L'(H)$ respectively.

A left compatible well-order $>$ on $B \cdot H = \{x[h] \mid x \in B, h \in H\}$ is *suitable*, if

(i) it extends the partial order $>'$ on $L'(H)$, and

(ii) $x[h] \succ \int(xt)$ for any $x \in B$ and $h = w\xi - t \in H$,

where $\succ$ is the partial order on $F \cdot H$ naturally extended from $>$. So, a well-order $>$ on $F \cdot H$ is suitable, if for any $x, x', a, b \in B$ and $h = w\xi - t, h' = w'\xi' - t' \in H$ with $w, w' \in B, \xi, \xi' \in X$ and $t, t' \in F \cdot X$, the following conditions are satisfied:

$$(iii) x[h] > x'[h'], yx \neq 0, yx' \neq 0 \Rightarrow yx[h] > yx'[h'],$$

$$(iv) a > b, ax \neq 0, bx \neq 0 \Rightarrow ax[h] > bx[h].$$

$$(v) xw \neq 0, x'w' \neq 0, xw > x'w' \text{ or } (xw = x'w', x > x') \Rightarrow x[h] > x'[h'],$$

and (ii) above.

Remark that if $xw \neq 0$, the inequality $x[h] \succ \int(xt)$ in (ii) follows from (iii).

If the base semigroup S is coherent, that is, $xy \neq 0$ for any $x, y \in B$ with $\tau(x) = \sigma(y)$, then $F \cdot H = L'(H)$ and $>'$ is a total order on $B \cdot H$, and hence $>'$ itself is suitable. We do not know the general condition for the existence of a suitable well-order. In the next section we assume that $>$ is a suitable well-order on $B \cdot H$, and it is extended to a partial order $\succ$ on $F \cdot H$. For a nonzero $f \in LF \cdot H$, $\text{lt}(f)$ denotes the maximal term of f with respect to $\succ$, and set $\text{rt}(f) = f - \text{lt}(f)$.

15 Gröbner basis made from critical pairs and critical z-elements

Let $h = w\xi - t, h' = w'\xi - t' \in H$ and $x, x' \in B$ such that $xw = x'w' \neq 0$, the appearance (x, w) is at the immediate right of (x', w') in xw and x and x' are left coprime, then we have the critical pair of the first kind and the element

$$c_1 = x[h] - x'[h'] + \int(x \cdot t) - \int(x' t')$$

in (11.1) ([5]). Since $(x, w\xi) > (x', w'\xi)$, $x[h] > x'[h']$ by (iii) above. Moreover, $x[h] \succ \int(xt)$ and $x'[h'] \succ \int(x' t')$ by (ii) (or (iii)). Thus, $\text{lt}(c_1) = z[h]$.

Let $u - v \in G$ and $x, y, y' \in B$ such that $xw = yuy' \neq 0$, $(x, w\xi)$ is rightmost in $xw\xi$, (y, u, y') is rightmost in xw , and x and y are coprime, then we have the cortical pair of the second kind and the element

$$c_2 = x[h] + \int (x \cdot t) - \int (yvy'\xi)$$

in (11.2). We have $x[h] \succ \int (xt)$ and $x[h] \succ \int (yvy'\xi)$ because $xw\xi \succ xt$ and $xw\xi \succ yvy'\xi$. Thus, $\text{lt}(c_2) = z[h]$.

Let $z \in B$ be such that $xw = 0$, then we have a z -element zt . This situation is *critical*, if there is no nonidempotential left factor y of z ; $z = yz'$ such that $y'w = 0$. In this case we call zt a *critical z -element*, and we have the element

$$c_3 = z[h] + \int (z \cdot t)$$

in (11.3) made from a critical z -element. We see $\text{lt}(c_3) = z[h]$ by (ii).

Let C be the set of the elements c_1, c_2 made from critical pairs together with the elements c_3 made from critical z -elements.

Let $\delta : F \cdot H \rightarrow F \cdot X$ be the morphism of left F -modules defined by $\partial_1([h]) = h$ for $h \in H$, and let $\rho : F \cdot X \rightarrow A \cdot X$ be the canonical surjection. Let $\mathcal{K} = \text{Ker}(\delta \circ \rho)$.

Theorem 15.1. *If H is a Gröbner basis on $F \cdot X$ and $>$ is a suitable well-order on $B \cdot H$, then the set C is a Gröbner basis on $F \cdot H$ of the kernel $\mathcal{K}$ modulo G .*

Under the existence of a suitable order we can strengthen Theorem 11.3 in [5] as follows. Remark that the set C here excludes z -elements that are not critical.

Corollary 15.2. *If H is a Gröbner basis and $>$ is a suitable order on $F \cdot H$, then C generates $\mathcal{K}$ modulo G .*

16 Projective resolutions

Let M be a left A -module defined by a Gröbner basis H on the projective left A -module $A \cdot X$ generated by a left edged set X , that is, $M \cong F \cdot X / L^\ell(H, G)$, where $L^\ell(H, G)$ is the submodule of $F \cdot X$ generated by H modulo G . We assume that there is a suitable order $>$ on $B \cdot H$.

Let C be the Gröbner basis on $F \cdot H$ made from critical pairs and critical z -elements in the previous section. Considering C to be a left edged set, we have the projective left A -module $A \cdot C$. Let $\partial' : A \cdot C \rightarrow A \cdot H$ be the morphism of left A -modules defined by

$$\partial'([c]) = c$$

for $c \in C$. Let $\eta : A \cdot X \rightarrow M$ be the canonical surjection. Since H generates $L^\ell(H, G)$ and C generates the kernel $\text{Ker}(\rho \circ \delta)$ modulo G , we have

Theorem 16.1. *The sequence*

$$A \cdot C \xrightarrow{\partial'} A \cdot H \xrightarrow{\partial} A \cdot X \xrightarrow{\eta} M \rightarrow 0$$

is exact.

Suppose that a suitable well-order can be defined on the projective left F -module $F \cdot C$, then we have the Gröbner basis D on $F \cdot C$ made from critical pairs and critical z -elements with respect to C and G and a morphism $\partial'' : A \cdot D \rightarrow A \cdot C$ defined by $\partial''([d]) = d$. If we can repeat this construction (that is, if a suitable well-order exists at every step), then we can construct a projective resolution of M .

Corollary 16.2. *Let M be a left A -module defined by a Gröbner basis X_1 on the projective left A -module $A \cdot X_0$ generated by a left edged set X_0 . If at every step above, a suitable well-order exists, we have a projective resolution of M :*

$$\rightarrow A \cdot X_n \xrightarrow{\partial_n} A \cdot X_{n-1} \rightarrow \cdots \rightarrow A \cdot X_1 \xrightarrow{\partial_1} A \cdot X_0 \xrightarrow{\eta} M \rightarrow 0.$$

Suppose that F has an identity element 1 and A is supplemented with a morphism $\epsilon : A \rightarrow K$. Let X be a generating set of nonidenmotents of B , then $\{a - \epsilon(\rho(a)) \cdot 1 \mid a \in X\}$ forms a Gröbner basis for $\text{Ker}(\epsilon)$ modulo G . Starting with this Gröbner basis, we can construct a projective resolution of K and we can compute the (co) homology of the algebra A (or the semigroup S).

17 Bimodules and the Hochschild cohomology

The enveloping semigroup $S^e = (B \times B) \cup \{0\}$ of $S = B \cup \{0\}$ is a well-ordered reflexive semigroup, in which the product and the order are given as

$$(x, y) \cdot (x', y') = (xx', y'y),$$

and

$$(x, y) \succ (x', y') \Leftrightarrow x \succ x' \text{ or } (x = x' \text{ and } y \succ y')$$

for $x, y, x', y' \in B$, respectively. The enveloping algebra $A^e = A \otimes_K A^\circ$ of $A = F/I$ is isomorphic to the quotient F^e/I^e , where $I^e = I \otimes F + F \otimes I$, and the set

$$G^e = \{g \otimes 1, 1 \otimes g \mid g \in G\}.$$

is a Gröbner basis of the ideal I^e . An F -bimodule (resp. A -bimodule) is naturally a left F^e -module (resp. left A^e -module).

Let X be an edged set and

$$F \cdot X \cdot F = \bigoplus_{\xi \in X} F\sigma(\xi) \times \tau(\xi)F$$

be the projective F -bimodule generated by X and let H be a set of monic uniform elements of $F \cdot X \cdot F$. We have three kinds of critical pairs with respect

to H modulo G . Let $h = w\xi z - t, h' = w'\xi z' - t' \in H$, $u - v \in G$ and $x, y, x', y' \in B$.

First suppose that $xw = x'w' \neq 0$ and $zy = z'y' \neq 0$, x and x' are left coprime, y and y' are right coprime, and the appearance of $w\xi z$ in the context (x, y) is immediate right of the appearance of $w'\xi z'$ in the context (x', y') . Then we have a critical pair $(xty, x't'y')$ of the first kind and the element

$$c_1 = x[h]y - x'[h']y' + \int(xty) + \int(x't'y').$$

of the projective F -bimodule $F \cdot H \cdot F$ generated by H . Next suppose that $xw = yuy' \neq 0$, u is rightmost in xw , $w\xi$ is rightmost in $xw\xi$ and x and y are left coprime. Then, we have a critical pair $(xt, yvy'\xi w')$ of the second kind, and an element

$$c_2 = x[h] - \int(yvy'\xi z) + \int(xt)$$

of $F \cdot H \cdot F$. Dually suppose that $zx = y'uy \neq 0$, u is leftmost in zx , ξz is leftmost in $xw\xi$, and x and y are right coprime. Then, we have a critical pair $(tx, w\xi y'vy)$ of the third kind, and an element

$$c_3 = [h]x - \int(w\xi y'vy) + \int(tx)$$

of $F \cdot H \cdot F$. If $xw = 0$ but $xt \neq 0$ and there is no nonidempotential left factor y of x ; $x = yx'$ such that $x'w = 0$, we have a critical z -element xt and an element

$$c_4 = x[h] + \int(xt).$$

If $zx = 0$ but $tx \neq 0$ and there is no nonidempotential right factor y of x ; $x = x'y$ such that $zx' = 0$, we have a critical z -element tx and an element

$$c_5 = [h]x + \int(tx).$$

Let C be the collection of all elements c_1, c_2, c_3, c_4 and c_5 above, and let $A \cdot C \cdot A$ be the projective A -bimodule generated by C .

Let $\delta : F \cdot H \cdot F \rightarrow F \cdot X \cdot F$ be the morphisms of left F -bimodules defined by $\delta([h]) = h$ for $h \in H$, and let $\rho : F \cdot X \cdot F \rightarrow A \cdot X \cdot A$ be the canonical surjection. Let M be the A -bimodule defined by H modulo G , that is, $M = A \cdot X \cdot A / L(M, G)$, where $L(M, G)$ is the subbimodule of $A \cdot X \cdot A$ generated by $\rho(M)$. Let $\partial : A \cdot H \cdot A \rightarrow A \cdot X \cdot A$ and $\partial' : A \cdot C \cdot A \rightarrow A \cdot H \cdot A$ be the morphisms of A -bimodules defined by $\partial([h]) = h$ and $\partial'([c]) = c$.

Theorem 17.1. *If H is a Gröbner basis on $F \cdot X \cdot F$ and $>$ is a suitable well-order on $B \cdot H \cdot B$, then the set C is a Gröbner basis on $F \cdot H \cdot F$ of the kernel of $\rho \circ \delta$ modulo G . Moreover we have an exact sequence of A -bimodules:*

$$A \cdot C \cdot A \xrightarrow{\partial'} A \cdot H \cdot A \xrightarrow{\partial} A \cdot X \cdot A \xrightarrow{\eta} M \rightarrow 0$$

Corollary 17.2. *Let M be an A -bimodule defined by a Gröbner basis X_1 on the projective left F -bimodule $F \cdot X_0 \cdot F$ generated by a left edged set X_0 . If at every step above, a suitable well-order exists, we have a projective A -bimodule resolution of M :*

$$\rightarrow A \cdot X_n \cdot A \xrightarrow{\partial_n} A \cdot X_{n-1} \cdot A \rightarrow \cdots \rightarrow A \cdot X_1 \cdot A \xrightarrow{\partial_1} A \cdot X_0 \cdot A \xrightarrow{\eta} M \rightarrow 0.$$

Let E be the set of all idempotents in B , and let X be a generating set of nonidempotents of B . Considering them as edged sets we have projective F -bimodules $F \cdot E \cdot F$, $F \cdot X \cdot F$ and A -bimodules $A \cdot E \cdot A$, $A \cdot X \cdot A$ generated by them. We have an augmentation map $\epsilon : F \cdot E \cdot F \rightarrow F$ and $\bar{\epsilon} : A \cdot E \cdot A \rightarrow A$ defined by $\epsilon([e]) = e$ and $\bar{\epsilon}([e]) = e$ for $e \in E$.

Let

$$H = \{ a[\tau(a)] - [\sigma(a)]a \mid a \in X \}.$$

Then, H is a Gröbner basis on $F \cdot E \cdot F$ for $\text{Ker}(\epsilon)$. In this way we have an exact sequence

$$A \cdot X \cdot A \xrightarrow{\partial} A \cdot E \cdot A \xrightarrow{\epsilon} M \rightarrow 0,$$

where the morphism ∂ is defined by $\partial([a]) = a[\tau(a)] - [\sigma(a)]a$ ($a \in X$). Thus, if under the existence of suitable order in every step, we can construct a projective A -bimodule resolution of A . This gives a way to compute the Hochschild cohomology of the algebra A .

18 Examples

Since the free monoid Σ^* is well-ordered and coherent, its submonoids are well-ordered and coherent. So, the existence of suitable order is guaranteed in every step of construction. In this section we pick up some easy submonoids of Σ^* and compute the (co)homology (other examples can be found in [1], [2]).

Example 18.1. Let B be the submonoid of $\{a\}^*$ generated by $X = \{a^2, a^3\}$. B is isomorphic to the additive monoid $\mathbb{N} \setminus \{1\}$ of natural numbers excluding 1. Let $F = K \cdot B$ be the algebra based on $B \cup \{0\}$. We have an augmentation map $\epsilon : F \cdot [] \cdot F \rightarrow F$ given by $\epsilon([]) = 1$, and a Gröbner basis

$$\{ \alpha_1 = a^2[] - []a^2, \beta_1 = a^3[] - []a^3 \}$$

of $\text{Ker}(\epsilon)$. Let $X = \{\alpha, \beta\}$ and define a morphism $\partial_1 : F \cdot X \cdot F \rightarrow F \cdot [] \cdot F$ by $\partial_1([\alpha]) = \alpha_1$, and $\partial_1([\beta]) = \beta_1$.

From the equation $a^3 \cdot a^2 = a^2 \cdot a^3$ we have a critical pair of the first kind $(a^3[]a^2, a^2[]a^3)$ and an element

$$\alpha_2 = a^3[\alpha] - [\alpha]a^3 - a^2[\beta] + [\beta]a^2$$

of $F \cdot X \cdot F$. From the equation $(a^2)^2 \cdot a^2 = a^3 \cdot a^3$ we have another critical pair of first kind $(a^4[]a^2, a^3[]a^3)$ and an element

$$\beta_2 = a^4[\alpha] + a^2[\alpha]a^2 + [\alpha]a^4 - a^3[\beta] - [\beta]a^3$$

of $F \cdot X \cdot F$. There is no critical pairs of the other kinds because the Gröbner basis G on F is empty. There is no z -element either because S is coherent. Hence, these two elements form a Gröbner basis of $\text{Ker}(\partial_1)$. We have a morphism $\partial_2 : F \cdot X \cdot F \rightarrow F \cdot X \cdot F$ given by $\partial_2([\alpha]) = \alpha_2$ and $\partial_2([\beta]) = \beta_2$. Note that $\text{lt}(\alpha_2) = a^3[\alpha]$ and $\text{lt}(\beta_2) = a^4[\alpha]$.

From the equation $a^3 \cdot a^3 = a^2 \cdot a^4$ we have an element

$$\alpha_3 = a^3[\alpha] + [\alpha]a^3 - a^2[\beta] + [\beta]a^2,$$

and from the equation $(a^2)^2 \cdot a^3 = a^3 \cdot a^4$ we have an element

$$\beta_3 = a^4[\alpha] + a^2[\alpha]a^2 + [\alpha]a^4 - a^3[\beta] + [\beta]a^3.$$

They form a Gröbner basis of $\text{Ker}(\partial_2)$. continuing this calculation we can construction a free bimodule resolution of F :

$$\rightarrow A \cdot X \cdot A \xrightarrow{\partial_n} A \cdot X \cdot A \rightarrow \cdots \rightarrow A \cdot X \cdot A \xrightarrow{\partial_1} A \cdot [] \cdot A \xrightarrow{\eta} F,$$

where ∂_n is given by

$$\partial_1([\alpha]) = a^2[] - []a^2, \quad \partial_1([\beta]) = a^3[] - []a^3,$$

$$\partial_n([\alpha]) = a^3[\alpha] + (-1)^{n-1}[\alpha]a^3 - a^2[\beta] + [\beta]a^2$$

and

$$\partial_n([\beta]) = a^4[\alpha] + a^2[\alpha]a^2 + [\alpha]a^4 - a^3[\beta] + (-1)^{n-1}[\beta]a^3$$

for $n \geq 2$.

From this resolution we can compute the Hochschild cohomology of F as follows. Here, K is a field of characteristic p .

$$H^0(F) = F,$$

$$H^1(F) = \begin{cases} F & \text{if } p = 2 \text{ or } 3 \\ \oplus_{i \geq 2} K \cdot a^i & \text{otherwise.} \end{cases}$$

Let $n \geq 2$. If $p = 2$,

$$H^n(F) = K \oplus K \cdot a^2 \oplus K \cdot a^3 \oplus K \cdot a^5.$$

if $p = 3$,

$$H^n(F) = K \oplus K \cdot a^2 \oplus K \cdot a^4,$$

and if $p \neq 2, 3$,

$$H^n(F) = \begin{cases} K \oplus K \cdot a^2 & \text{if } n \text{ is even} \\ K \cdot (2a^2, 3a^3) \oplus K \cdot (2a^3, 3a^4) & \text{if } n \text{ is odd.} \end{cases}$$

Example 18.2. Let B be the submonoid of $\{a, b\}^*$ generated by $X = \{ab, ba, aba\}$, and let $S = B \cup \{0\}$ and $F = K \cdot B$ is the algebra based on S . We have an augmentation $\epsilon : F \cdot [] \cdot F \rightarrow F$ given by $\epsilon([]) = 1$. We have a Gröbner basis

$$\{ab[] - []ab, ba[] - []ba, aba[] - []aba\},$$

of $\text{Ker}(\epsilon)$ and a differential map

$$\partial_1 : F \cdot X \cdot F \rightarrow A \cdot [] \cdot A$$

with

$$\begin{aligned}\partial_1([ab]) &= ab[] - []ab, \quad \partial_1([ba]) = ba[] - []ba, \\ \partial_1([aba]) &= aba[] - []aba.\end{aligned}$$

X is not a code because we have a word equation $(aba)ba = ab(aba)$. From this equation we have a critical pair $(aba[]ba, ab[]aba)$, and we obtain a Gröbner basis of $\text{Ker}(\partial_1)$:

$$\{ aba[ba] + [aba]ba - ab[aba] - [ab]aba \}.$$

In this way we get a free bi-module resolution of F :

$$0 \rightarrow F \cdot \{ababa\} \cdot F \xrightarrow{\partial_2} F \cdot X \cdot F \xrightarrow{\partial_1} F \cdot [] \cdot F \xrightarrow{\epsilon} F,$$

where

$$\partial_2([ababa]) = aba[ba] + [aba]ba - ab[aba] - [ab]aba.$$

F is supplemented with $\epsilon : F \rightarrow K$ defined by $\epsilon(ab) = \epsilon(ba) = \epsilon(aba) = 0$. Tensoring with the F -module K on the right, we have a minimal free left resolution of K :

$$\begin{aligned}0 \rightarrow F \cdot \{ababa\} \xrightarrow{\bar{\partial}_2} F \cdot X \xrightarrow{\bar{\partial}_1} F \xrightarrow{\bar{\epsilon}} K, \\ \bar{\partial}_1([ab]) = ab, \quad \bar{\partial}_1([ba]) = ba, \quad \bar{\partial}_1([aba]) = aba, \\ \bar{\partial}_2([ababa]) = aba[ba] - ab[aba].\end{aligned}$$

The Betti number $b_2 = \dim_K(\text{Tor}_2^F(K, K)) = 1$ seems reflect the ambiguity of X ; how distant from codes.

References

- [1] Y. Kobayashi, *Gröbner bases of associative algebras and the Hochschild cohomology*, Trans. Amer. Math. Soc. **375** (2005), 1095–1124.
- [2] Y. Kobayashi, *Gröbner bases on path algebras and the Hochschild cohomology algebras*, Sci. Math. Japonicae **64** (2006), 411–437.
- [3] Y. Kobayashi, *Gröbner bases on projective bimodules and the Hochschild cohomology I. Rewriting on vector spaces*, Kokyuroku **1503**, RIMS (Kyoto University), 2006, 30–40.
- [4] Y. Kobayashi, *Gröbner bases on projective bimodules and the Hochschild cohomology II. Critical pairs*, Kokyuroku **1562**, RIMS (Kyoto University), 2007, 40–45.
- [5] Y. Kobayashi, *Gröbner bases on projective bimodules and the Hochschild cohomology III. Syzygies*, Kokyuroku **1604**, RIMS(Kyoto University), 2008, 142–150.