

p -TORAL GROUPS WITHOUT THE WEAK BORSUK-ULAM PROPERTY

ABSTRACT. In our recent research, we constructed certain $S^1 \times C_p$ -maps that serve as counterexamples to the weak Borsuk-Ulam property in the class of p -toral groups. In this note, we present our main results together with an outline of the proof. We also discuss some related results on the right Borsuk-Ulam invariant.

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1. INTRODUCTION

Let G be a compact Lie group. In this note, a G -representation V is assumed to be orthogonal and fixed-point-free, that is, $V^G = 0$, unless otherwise stated. $S(V)$ denotes the unit sphere of V , called a G -representation sphere. All maps between G -spaces are assumed to be continuous.

Borsuk [2] proved the celebrated Borsuk-Ulam theorem, which was later extended in several directions. One extension provides the non-existence of equivariant maps. The following result is known as a generalization of the Borsuk-Ulam theorem.

Theorem 1.1 (Borsuk-Ulam theorem for $G = C_p^k, T^k$). *Let G be an elementary abelian group C_p^k of rank k or a k -dimensional torus T^k . Then if there exists a G -map $f : S(V) \rightarrow S(W)$, then the inequality $\dim V \leq \dim W$ holds.*

A natural question is: for which groups does the Borsuk-Ulam theorem hold? We say that G has the *Borsuk-Ulam property* if the Borsuk-Ulam theorem holds for G . After many contributions by various authors, we finally proved the following:

Theorem 1.2 (see [6, 7]). *G has the Borsuk-Ulam property if and only if G is C_p^k or T^k .*

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On the other hand, Bartsch [1] studied a weaker version of the Borsuk-Ulam theorem, referred to here as the *weak Borsuk-Ulam theorem*. We say that G has the *weak Borsuk-Ulam property* if the weak Borsuk-Ulam theorem holds for G . Bartsch [1] showed that finite p -groups have the weak Borsuk-Ulam theorem. It is reasonable to consider extending this result to p -toral groups, where we say that G is a p -toral group if G has an extension $1 \rightarrow T^k \rightarrow G \rightarrow P \rightarrow 1$, P a finite p -group. However, this expectation turned out to be incorrect. We have the following result.

Theorem 1.3. *The group $S^1 \times C_p$ does not have the weak Borsuk-Ulam property.*

By lifting the group action, we immediately obtain the following.

Corollary 1.4. *If G has a quotient group isomorphic to $S^1 \times C_p$, then G does not have the weak Borsuk-Ulam property.*

In the following sections, we first recall the weak Borsuk-Ulam property and known results, and we then proceed to outline the proof of the theorem. Further details can be found in [8].

2. THE WEAK BORSUK-ULAM PROPERTY

We recall the Borsuk-Ulam function $b_G : \mathbb{N} \rightarrow \mathbb{N}$, introduced by Bartsch [1].

Definition. The value $b_G(n)$ is defined as the minimum of $\dim W$ such that there exists a G -map $f : S(V) \rightarrow S(W)$ for some V with $\dim V \geq n$.

Clearly b_G is non-decreasing. Moreover, if there exists a G -map $f : S(V) \rightarrow S(W)$, then $b_G(\dim V) \leq \dim W$.

Definition. We say that G has the weak Borsuk-Ulam property if $\lim_{n \rightarrow \infty} b_G(n) = \infty$.

Clearly, the Borsuk-Ulam property implies the weak Borsuk-Ulam property. Hence C_p^k and T^k have the weak Borsuk-Ulam property. Furthermore, the following result is known.

Proposition 2.1 ([1]). *Let G be a compact Lie group.*

- (1) *If G has the weak Borsuk-Ulam property, then G is a p -toral group.*
- (2) *Weak Borsuk-Ulam theorem. If G is a finite p -group, then G has the weak Borsuk-Ulam property.*

Therefore, in the case of finite groups, a group G has the weak Borsuk-Ulam property if and only if G is a p -group. These results might lead one to expect that all p -toral groups have the weak Borsuk-Ulam property. However, this is not the case. In fact, as mentioned in the Introduction, the group $S^1 \times C_p$ does not have the weak Borsuk-Ulam property.

3. OUTLINE OF THE PROOF

A full proof of the main result can be found in [8]. Here, we provide an outline of the argument. Set

$$G := S^1 \times C_p = \{ta^i \mid t \in S^1, i \in \mathbb{Z}\}.$$

We denote by $V_{k,l}, (= \mathbb{C})$ the complex irreducible G -representations, where $k \in \mathbb{Z}$ and $l \in \mathbb{Z}/p$. The action of G on $V_{k,l}$ is defined by

$$t \cdot z = t^k z, \quad a \cdot z = \xi_p^l z \quad (\xi_p = e^{2\pi i/p}, z \in V_{k,l}).$$

For $n \geq 1$ and $m \geq 1$, set

$$V_n := V_{1,1} \oplus V_{p,1} \oplus \cdots \oplus V_{p^{n-1},1}, \text{ and}$$

$$W_m := V_{p^m,0} \oplus V_{0,1}.$$

Note $\dim S(V_n) = 2n - 1$, and $\dim S(W_m) = 3$. To prove the main result, we suffice to show the following.

Theorem 3.1. *For every $n \geq 1$, there exist $m \geq n$ and a G -map $f_n : S(V_n) \rightarrow S(W_m)$.*

Outline of proof. (i) The Case of $n = 1$. Let $m = 1$. There exists a G -map

$$f_1 : S(V_{1,1}) \rightarrow S(V_{p,1} \oplus V_{0,1})$$

given by

$$f_1(z) = (z^p, 0) \quad \text{for } z \in S(V_1).$$

(ii) The Case of $n = 2$. Let $m = 1$. Inspired by Crabb [4], we can define a G -map

$$\bar{f}_2 : V_2 = V_{1,1} \oplus V_{p,1} \rightarrow W_2 = V_{p^2,0} \oplus V_{0,1}$$

by

$$\bar{f}_2(z, w) = (z^{p^2} - w^p, \bar{z}^p w).$$

Since $\bar{f}_2^{-1}(0, 0) = (0, 0)$, we obtain a G -map

$$f_2 : S(V_2) \rightarrow S(W_2), \quad f_2(z, w) = \frac{\bar{f}_2(z, w)}{\|\bar{f}_2(z, w)\|}.$$

Remark. The degree of f_2 is zero, since f_2 is not surjective.

(iii) The Case of $n = 3$. We show that there exists a G -map $f_3 : S(V_3) \rightarrow S(W_3)$.

Decompose $S(V)$ into

$$S(V_3) \cong X_0 \cup X \times I \cup X_1,$$

where $X_0 = S(V_{1,1}) \times D(V_{p,1} \oplus V_{p^2,1})$, $X = S(V_{1,1}) \times S(V_{p,1} \oplus V_{p^2,1})$, $X_1 = D(V_{1,1}) \times S(V_{p,1} \oplus V_{p^2,1})$. Define a G -map $g_0 : X_0 \rightarrow S(W_3)$ by $g_0(u, z, w) = (u^{p^3}, 0)$. Lifting f_2 , we have a G -map

$$\tilde{f}_2 : S(V_{p,1} \oplus V_{p^2,1}) \rightarrow S(W_3) = S(V_{p^3,0} \oplus V_{0,1}).$$

Set a G -map $g_1 : X_1 \rightarrow S(W_3) \not\sim g_1(u, z, w) = \tilde{f}_2(z, w)$, $h_0 = g_0|_{\partial X_0} : \partial X_0 = X \times \{0\} \rightarrow S(W_3)$ and $h_1 = g_1|_{\partial X_1} : \partial X_1 = X \times \{1\} \rightarrow S(W_3)$. We construct a G -homotopy $H : X \times I \rightarrow S(W_3)$ between h_0 and h_1 . This implies that there exists a G -map f_3 .

Let $Y = S(V_{p,1} \oplus V_{p^2,1})$. Then the map

$$\bar{F} : \text{Map}_G(X, S(W_3)) \rightarrow \text{Map}_{C_p}(Y, S(W_3)),$$

$$\bar{F}(\alpha)(z_1, z_2) = \alpha(1, (z_1, z_2))$$

is bijective for $\alpha : X \rightarrow S(W_3)$, $X = S(V_{1,1}) \times S(V_{p,1} \oplus V_{p^2,1})$, where $1 \in S(V_{1,1}) = S^1 \subset \mathbb{C}$. Since C_p acts freely on Y , the equivariant Hopf theorem [3, 5] shows that

$$\deg : [Y, S(W_3)]_{C_p} \rightarrow [Y, S(W_3)]_{\{1\}} = [S^3, S^3] \cong \mathbb{Z}$$

is injective. Since $\deg F([h_0]) = \deg F([h_1]) = 0$, it is seen that $\bar{F}(h_1)$ and $\bar{F}(h_1)$ is C_p -homotopic and so h_0 and h_1 is G -homotopic. Therefore there exists a G -homotopy H between h_0 and h_1 . $\square$

Next consider the case of $n \geq 4$. We show the existence f_n by induction on n . Namely we show that if there exists a G -map $f_{n-1} : S(V_{n-1}) \rightarrow S(W_{m'})$ for some $m' \geq n-1 \geq 3$, then there exists a G -map $f_n : S(V_n) \rightarrow S(W_m)$ for some $m \geq n$.

As before, decompose $S(V_n)$ into

$$S(V_n) = X_0 \cup X \times I \cup X_1,$$

where $X_0 = S(V_{1,1}) \times D(V_{p,1} \oplus \cdots \oplus V_{p^{n-1},1})$, $X = S(V_{1,1}) \times S(V_{p,1} \oplus \cdots \oplus V_{p^{n-1},1})$, $X_1 = D(V_{1,1}) \times S(V_{p,1} \oplus \cdots \oplus V_{p^{n-1},1})$. Define a G -map $g_0 : X_0 \rightarrow S(W_{m'+1})$: $g_0(u, \mathbf{x}) = (u^{p^{m'+1}}, 0)$ for $\mathbf{x} \in Y = S(V_{p,1} \oplus \cdots \oplus V_{p^{n-1},1})$ and a G -map $g_1 : X_1 \rightarrow S(W_{m'+1})$, $g_1(u, \mathbf{x}) = \tilde{f}_{n-1}(\mathbf{x})$, where $\tilde{f}_{n-1}$ is the lift of f_{n-1} via $\pi : G \rightarrow G$, $ta^i \mapsto t^p a^i$. Set $h_0 = g_0|_{\partial X_0} : X \times \{0\} \rightarrow S(W_{m'+1})$ and $h_1 = g_1|_{\partial X_1} : X \times \{1\} \rightarrow S(W_{m'+1})$. In this case, there might be an obstruction to the existence of a G -homotopy. However, the following result ensures that the construction can proceed.

Lemma 3.2. *There exists a G -map $R_m : S(W_{m'+1}) \rightarrow S(W_m)$ for some $m \geq m'+1$ such that $R_m \circ h_0$ and $R_m \circ h_1$ are G -homotopic.*

If there exists such a G -homotopy H between $R_m \circ h_0$ and $R_m \circ h_1$, then we obtain a G -map

$$f_n = R_m \circ g_0 \cup H \cup R_m \circ g_1 : S(V_n) \rightarrow S(W_m).$$

$\square$

Sketch of the proof of Lemma 3.2. This lemma is proved by using equivariant obstruction theory. The key facts are as follows:

First, the equivariant obstruction groups

$$\mathfrak{H}_G^k(S(V_n), \pi_k) \cong H^k(L^3(p); \pi_k),$$

where $L^3(p)$ is a lens space and $\pi_k = \pi_k(S^3)$, are torsion groups.

Second, for any G -map $R_m : S(W_{m'}) \rightarrow S(W_m)$, the obstruction class of the composition $R_m \circ f_{n-1}$ on the k -skeleton of X satisfies

$$o_G(R_m \circ f_{n-1}) = (\deg R_m) o_G(f_{n-1}).$$

Therefore by choosing a suitable m such that $\deg(R_m)$ annihilates the class $o_G(f_{n-1})$, we see that the obstruction $o_G(R_m \circ f_{n-1})$ vanishes. By repeating this process inductively, we obtain the desired G -map f_n . $\square$

4. RELATED RESULTS

We give some remarks on the Borsuk-Ulam function.

Proposition 4.1. *Let $G = S^1 \times C_p$, where p is an odd prime. Then $b_G(n) = 4$ for any $n \geq 4$.*

Proof. By Theorem 3.1, we have $b_G(n) \leq 4$. It remains to show that $b_G(n) = 4$.

Suppose, for contradiction, that $b_G(n) < 4$. Then $b_G(n) \leq 2$, since p is odd and $\dim V$ is even. Now assume $\dim V \geq 4$ and $\dim W = 2$. We claim that there is no G -maps $f : S(V) \rightarrow S(W)$. Indeed, if $W = V_{k,l}$ for some $(k, l) \neq (0, 0) \in \mathbb{Z} \times \mathbb{Z}/p$, then we have $W^{S^1} = 0$ or $W^{C_p} = 0$. Accordingly, $V^{S^1} = 0$ or $V^{C_p} = 0$. By the Borsuk-Ulam theorem for S^1 or C_p , it follows that $\dim V \leq \dim W$, which contradicts $\dim V \geq 4$. $\square$

Next, let G be a compact Lie group. We define

$$r_G(W) = \sup\{\dim V \mid \exists f : S(V) \rightarrow S(W) \text{ } G\text{-map}\} \leq \infty,$$

which is called the *right Borsuk-Ulam invariant* of W . As before, we assume that V, W are fixed-point-free representations. We observe the following:

Lemma 4.2. *Let $H \leq G$, W be a G -representation W with $W^H = 0$. Then $r_G(W) \leq r_H(W)$.*

Determining the right Borsuk-Ulam invariant is a difficult problem. Nevertheless, the following finiteness result holds, which may be regarded as a Borsuk-Ulam type theorem.

Proposition 4.3. *If G is a p -toral group, then $r_G(W) < \infty$ for every G -representation W with $W^G = 0$.*

Proof. Consider the short exact sequence

$$1 \rightarrow T^k \rightarrow G \rightarrow P \rightarrow 1,$$

where P is a finite p -group. By [5, Chapter IV (3.8), 3)], there exists a sequence $\{P_n\}$ of finite p -subgroups of G such that $\lim_{n \rightarrow \infty} P_n = G$ with respect to the

Hausdorff metric. For sufficiently large n , we have $W^{P_n} = W^G = 0$. By Lemma 4.2, we have $r_G(W) \leq r_{P_n}(W)$. Since finite p -groups have the weak Borsuk-Ulam property, we conclude that $r_{P_n}(W) < \infty$, and thus $r_G(W) < \infty$. $\square$

Conversely, if G is not a p -toral group, then, as shown by Bartsch [1], there exists a G -representation W such that $r_G(W) = \infty$. Note also that, by Theorem 3.1, we have $\lim_{m \rightarrow \infty} r_G(W_m) = \infty$, where $G = S^1 \times C_p$ and W_m is the representation given in Theorem 3.1.

Finally, we propose the following conjecture:

Conjecture. Positive-dimensional p -toral groups other than T^k do not have the weak Borsuk-Ulam property.

REFERENCES

- [1] T. Bartsch, *On the existence of Borsuk-Ulam theorems*, Topology **31** (1992), 533–543.
- [2] K. Borsuk, *Drei Sätze über die n -dimensionale Sphäre*, Fund. Math, **20** (1933), 177–190.
- [3] C. Borszyc, *On the winding number and equivariant homotopy classes of maps of manifolds with some finite group actions*, Fund. Math. **115** (1983), 235–247.
- [4] M. C. Crabb, *Equivariant maps between spheres: Nagasaki's examples*, J. Fixed Point Theory Appl. **24** (2022), article number 78.
- [5] T. tom Dieck, *Transformation groups*, Walter de Gruyter, Berlin, New York 1987.
- [6] I. Nagasaki, *Elementary abelian p -groups are the only finite groups with the Borsuk-Ulam property*, J. Fixed Point Theory Appl. **21** (2019), article number 16.
- [7] I. Nagasaki, *Determination of compact Lie groups with the Borsuk-Ulam property*, op. Appl. **319** (2022), article number 108216.
- [8] I. Nagasaki, *The existence of $S^1 \times C_p$ -maps between representation spheres and its applications*, J. Fixed Point Theory Appl. **27** (2025), article number 25.

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