

ON CLASS NUMBER of a GALOIS EXTENSION

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Let K be a finite Galois extension of an algebraic number field k . The central extension $\hat{K}$ and the genus field K^* over k are defined in [1], [3]. The central class number $\bar{z}_{K/k} = (\hat{K} : K)$ and the genus number $g_{K/k} = (K^* : K)$ of K with respect to k are given in [1], [3].

Let $a_{K/k}$ be the ambiguous ideal class number of K respect to k . If K/k is a cyclic extension, or a Galois extension of a prime power degree, we have some relations between the class number h_K of K and $a_{K/k}$ ([2], [3], [5]).

In [4], it is proved:

"Let K/k be a cyclic extension of a prime power degree l^v , and suppose K and the absolute class field $\bar{k}$ of k are disjoint over k , i.e. $K \cap \bar{k} = k$. Then h_K is prime to l if and only if $a_{K/k} = h_K$ and h_K is prime to l ."

We shall generalize this criterion to a Galois extension of a prime power degree. In this note, h_K and $\bar{k}$ for a finite algebraic number field k mean as above.

PROPOSITION. Let k be an algebraic number field and K/k be a Galois extension of degree n . Suppose $K \cap \bar{k} = k$.

- (1) If h_K is prime to n , then $\bar{z}_{K/k} = g_{K/k} = a_{K/k} = h_K$, and h_K is prime to n .
- (2) Let $n = l^v$ be a prime power. Then h_K is prime to l if and only if $\bar{z}_{K/k} = g_{K/k} = a_{K/k} = h_K$ and h_K is prime to l .

Proof. (1) Since $\bar{z}_{K/k} = (\hat{K} : K^*) \cdot g_{K/k}$, $g_{K/k} = (K^* : K \cdot \bar{K}) h_K$, h_K , $\bar{z}_{K/k}$ and $g_{K/k}$ are divisible by h_K . In the decomposition

$$\frac{h_K}{h_K} = \frac{g_{K/k}}{h_K} \frac{h_K}{g_{K/k}} = \frac{\bar{z}_{K/k}}{h_K} \frac{h_K}{\bar{z}_{K/k}}$$

if h_K is prime to n , then $\frac{g_{K/k}}{h_K}$, $\frac{\bar{z}_{K/k}}{h_K}$ are prime to n .

From the genus number formula and the central class number formula, we have

$$\frac{g_{K/k}}{h_K} = \frac{T_g T_{e_g}}{(K_0 : k) (E_K : E_K \cap N_{K/k} U_K)},$$

$$\frac{\bar{z}_{K/k}}{h_K} = \frac{T_g T_{e_g} \cdot (K^* \cap N_{K/k} J_K : N_{K/k} K^*)}{(K_0 : k) (E_K : E_K \cap N_{K/k} K^*)}$$

(notations in these formulas are defined in [1], [3]).

$$E_K \cap N_{K/k} U_K \supset E_K^n = \{ \varepsilon^n = N_{K/k} \varepsilon \mid \varepsilon \in E_K \},$$

$$E_K \cap N_{K/k} K^* \supset E_K^n$$

and it is well-known

$$(K^* \cap N_{K/k} J_K) / N_{K/k} K^* \cong H^{-3}(G, \mathbb{Z}) / F$$

where G is a Galois group of K over k and F is the subgroup of TATE cohomology group $H^{-3}(G, \mathbb{Z})$, generated by $\text{Inf}_{G_g/G} H^{-3}(G_g, \mathbb{Z})$ for all the infinite and the finite prime divisors g and its decomposition group G_g .

Therefore, the prime factors of $\frac{g_{K/k}}{h_K}$ and $\frac{\bar{z}_{K/k}}{h_K}$ are those of n , also

$$\frac{g_{K/k}}{h_K} = \frac{\bar{z}_{K/k}}{h_K} = 1; \bar{z}_{K/k} = g_{K/k} = h_K.$$

Let I_K be the ideal group of K and P_K be the principal ideal group of K . Then

$$A = \{ \alpha \in I_K \mid \alpha^{1-\sigma} \in P_K \text{ for any } \sigma \in G \},$$

$$H = \{ \alpha \in I_K \mid N_{K/k} \alpha \in P_K \}$$

are the subgroups of I_K , and

$$A \cap H \subset \{ \alpha \in I_K \mid \alpha^n \in P_K \}.$$

It follows for the ambiguous ideal class number $a_{K/k}$ of K respect to k

$$a_{K/k} = (A : P_K) = (A : A \cap H) (A \cap H : P_K),$$

where the prime factors of $(A \cap H : P_K)$ are also those of n . If

f_K is prime to n , then $a_{K/k}$ is prime to n , hence

$$(A \cap H : P_K) = 1; \quad A \cap H = P_K.$$

By $g_{K/k} = f_K$, $K^* = K \cdot \bar{k}$ is the genus field of K over k , i.e. the class field corresponding to H over K . So we have

$$\begin{aligned} g_{K/k} &= (K^* : K) = (I_K : H) = (I_K : AH) (AH : H) \\ &= (I_K : AH) (A : A \cap H) = (I_K : AH) \cdot a_{K/k}. \end{aligned}$$

On the other hand, the number $a_{K/k}^{(v)}$ of ideal classes represented by an ambiguous ideal in K/k is given by [5]:

$$a_{K/k}^{(v)} = \frac{f_K \cdot T_g \cdot e_g}{|H^1(G, E_K)|}.$$

If f_K is prime to n , then $a_{K/k}$, $a_{K/k}^{(v)}$ and f_K are prime to n , that is, $a_{K/k}^{(v)} = f_K$. Since $g_{K/k} = f_K \equiv 0 \pmod{a_{K/k}}$, and $a_{K/k} \equiv 0 \pmod{a_{K/k}^{(v)}}$, hence $g_{K/k} = f_K = a_{K/k}$.

(2) In case of $n = l^v$, we know in [3]

$$f_K \equiv \mathbb{Z}_{K/k} \pmod{l}.$$

References

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